Considerations on certain series *

Leonhard Euler

§1 After I had discovered that the sums of the series of reciprocals contained in this form

$$1 \pm \frac{1}{3^n} + \frac{1}{5^n} \pm \frac{1}{7^n} + \frac{1}{9^n} \pm \frac{1}{11^n} + \text{etc.},$$

where the upper signs hold, if n is an even number, the lower on the other hand, if n is an odd number, depend on the quadrature of the circle and involve the n-th power of the circumference of the circle, π, I made several observations concerning both these series themselves and their utility for the summation of other series. Because these observations are not obvious and might be useful for other similar tasks, I think it will not be out of place to explain them here.

§2 Having constantly put the ratio of the diameter to the circumference of the circle to be 1 to π I will consider the circle, whose radius is $= 1$, and π will denote the half of its circumference or the arc of 180°. Therefore, if one now takes the arc $= s$ on this circle and the sine of this arc is $= y$, the cosine is $= y$ and the tangent is $= t$, it will be

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$$y = s - \frac{s^3}{1\cdot 2\cdot 3} + \frac{s^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{s^7}{1\cdot 2\cdots 7} + \text{etc.},$$
$$x = 1 - \frac{s^2}{1\cdot 2} + \frac{s^4}{1\cdot 2\cdot 3\cdot 4} - \frac{s^6}{1\cdot 2\cdots 6} + \text{etc.}$$

and therefore

$$0 = t - s - \frac{s^2 t}{1\cdot 2} + \frac{s^3}{1\cdot 2\cdot 3} + \frac{s^4 t}{1\cdot 2\cdot 3\cdot 4} - \frac{s^5}{1\cdot 2\cdots 5} - \frac{s^6 t}{1\cdot 2\cdots 6} + \text{etc.}$$

or

$$0 = 1 - \frac{s}{t} - \frac{s^2}{1\cdot 2} + \frac{s^3}{1\cdot 2\cdot 3t} + \frac{s^4}{1\cdot 2\cdot 3\cdot 4} - \frac{s^5}{1\cdot 2\cdots 5t} - \frac{s^6}{1\cdot 2\cdots 6} + \text{etc.}$$

§3 Hence first let us consider the equation containing the relation among the sine y and the arc s; and it is manifest that the value s is not constant for a given y, but denotes all the arcs with the same sine y. So let the smallest of these arcs be $= \frac{m}{n}\pi$; all the following arcs

$$\begin{array}{cccccc} \frac{m}{n}\pi, & \frac{n-m}{n}\pi, & \frac{2n+m}{n}\pi, & \frac{3n-m}{n}\pi, & \frac{4n+m}{n}\pi & \text{etc.} \\ \frac{-n-m}{n}\pi, & \frac{-2n+m}{n}\pi, & \frac{-3n-m}{n}\pi, & \frac{-4n+m}{n}\pi, & \frac{-5n-m}{n}\pi & \text{etc.} \end{array}$$

will have the common sine y. Therefore, this equation

$$0 - \frac{s}{1y} + \frac{s^3}{1\cdot 2\cdot 3y} - \frac{s^5}{1\cdot 2\cdot 3\cdot 4\cdot 5y} + \frac{s^7}{1\cdot 2\cdots 7y} - \text{etc.}$$

will the following and infinitely many factors

$$\left(1 - \frac{ns}{m\pi}\right)\left(1 + \frac{ns}{(n+m)\pi}\right)\left(1 - \frac{ns}{(n-m)\pi}\right)\left(1 + \frac{ns}{(2n-m)\pi}\right)\left(1 - \frac{ns}{(2n+m)\pi}\right)\text{etc.}$$

§4 Therefore, the values of $\frac{1}{s}$ will constitute the following series

$$\frac{n}{m\pi}+\frac{n}{(n-m)\pi}-\frac{n}{(n+m)\pi}-\frac{n}{(2n-m)\pi}+\frac{n}{(2n+m)\pi}+\frac{n}{(3n-m)\pi}-\text{etc.}$$

Their sum will be equal to the coefficient of $-s$ in the equation, which coefficient is

$$=\frac{1}{1y}.$$

The sum of the products of two factors in each term will be $=0$, the sum of three

$$=-\frac{1}{1\cdot 2\cdot 3y}\quad \text{etc.;}$$

hence it will be as follows

$$\begin{aligned}
&\text{sum of the terms only} && =-\frac{1}{1y},\\
&\text{sum of the products of two terms} && =0,\\
&\text{sum of the products of three terms} && =\frac{-1}{1\cdot 2\cdot 3y},\\
&\text{sum of the products of four terms} && =0,\\
&\text{sum of the products of five terms} && =\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5y},\\
&\text{sum of the products of six terms} && =0,\\
&\text{sum of the products of seven terms} && =\frac{-1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7y},\\
&\text{sum of the products of eight terms} && =0
\end{aligned}$$

etc.

§5 But if, considering this series in general

$$a + b + c + d + e + \text{etc.},$$

we call the

$$\begin{aligned}
&\text{sum of the terms itself} &&= \alpha,\\
&\text{sum of the products of two terms} &&= \beta,\\
&\text{sum of the products of three terms} &&= \gamma,\\
&\text{sum of the products of four terms} &&= \delta,\\
&\text{sum of the products of five terms} &&= \varepsilon,\\
&\text{sum of the products of six terms} &&= \zeta,
\end{aligned}$$

etc.,

one will be able to assign the sums of the squares, cubes, fourth powers and of any powers of the terms of this series. Therefore, if it is

$$\begin{aligned}
a + b + c + d + \text{etc.} &= A,\\
a^2 + b^2 + c^2 + d^2 + \text{etc.} &= B,\\
a^3 + b^3 + c^3 + d^3 + \text{etc.} &= C,\\
a^4 + b^4 + c^4 + d^4 + \text{etc.} &= D,\\
a^5 + b^5 + c^5 + d^5 + \text{etc.} &= E,\\
a^6 + b^6 + c^6 + d^6 + \text{etc.} &= F
\end{aligned}$$

etc.,

the values of these sums will be determined in the following way

$$\begin{aligned}
A &= \alpha,\\
B &= \alpha A - 2\beta,\\
C &= \alpha B - \beta A + 3\gamma,\\
D &= \alpha C - \beta B + \gamma A - 4\delta,\\
E &= \alpha D - \beta C + \gamma B - \delta A + 5\varepsilon,\\
F &= \alpha E - \beta D + \gamma C - \delta B + \varepsilon A - 6\zeta
\end{aligned}$$

etc.

Because this progression follows has a simple structure and each term is conveniently determined from the preceding terms, we will be able to define the values of $\frac{1}{s}$ of the upper series exhibiting the sum of any powers of the terms.

§6 But before we leave this general progression, it will be convenient to note a remarkable relation among the values of the letters A, B, C, D etc. These letters result from the expansion of the expression

$$\frac{\alpha - 2\beta z + 3\gamma z^2 - 4\delta z^3 + 5\varepsilon z^4 - 6\zeta z^5 + 7\eta z^6 - \text{etc.}}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4 - \varepsilon z^5 + \zeta z^6 - \text{etc.}}$$

if the quotient is expanded into a power series in z by actual division, of course. For, from that the division the following quotient will result

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \text{etc.},$$

so that this series is equal to that fraction we considered at the beginning. Furthermore, it is to be noted, if the sum of the series

$$1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4 - \text{etc.}$$

is put $= Z$ and so Z is the denominator of that fraction, that then the numerator will be $-\frac{-dZ}{dz}$. Using this result the sum of the series

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.}$$

will be

$$= \frac{-dZ}{Zdz}.$$

Therefore, not only the sums of the powers of the propounded series $a + b + c + d + \text{etc.}$, namely the values of the letters A, B, C, D etc. can be found from the given products of two, three, four factors, but one will also be able to assign the sum of the series which these powers, multiplied respectively by the terms of another new geometric progression, constitute[1]; for, the sum of the following series can be assigned

[1]By this Euler means the power series which has those letters A, B, C, D etc. as coefficients.

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.}$$

And having noted this property will be helpful in the following, where we will investigate new series.

§7 Therefore, since both the sum of the terms of this series itself, namely

$$\frac{n}{\pi}\left(\frac{1}{m} + \frac{1}{n-m} - \frac{1}{m+n} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \frac{1}{3n+m} - \text{etc.}\right)$$

and the sums of the products of two, three, four terms of the series and so forth are given, namely

$$
\begin{aligned}
A &= \frac{1}{1y},\\
B &= \frac{A}{1y},\\
C &= \frac{B}{1y} - \frac{1}{1\cdot 2y},\\
D &= \frac{C}{1y} - \frac{A}{1\cdot 2\cdot 3y},\\
E &= \frac{D}{1y} - \frac{B}{1\cdot 2\cdot 3y} + \frac{1}{1\cdot 2\cdot 3\cdot 4y},\\
F &= \frac{E}{1y} - \frac{C}{1\cdot 2\cdot 3y} + \frac{A}{1\cdot 2\cdot 3\cdot 4\cdot 5y},\\
G &= \frac{F}{1y} - \frac{D}{1\cdot 2\cdot 3y} + \frac{B}{1\cdot 2\cdot 3\cdot 4\cdot 5y} - \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6y}
\end{aligned}
$$

etc.,

it will be as follows

$$\frac{1}{m}+\frac{1}{n-m}-\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}+\text{etc.}=\frac{A\pi}{n},$$

$$\frac{1}{m^2}+\frac{1}{(n-m)^2}+\frac{1}{(n+m)^2}+\frac{1}{(2n-m)^2}+\frac{1}{(2n+m)^2}+\text{etc.}=\frac{B\pi^2}{n^2},$$

$$\frac{1}{m^3}+\frac{1}{(n-m)^3}-\frac{1}{(n+m)^3}-\frac{1}{(2n-m)^3}+\frac{1}{(2n+m)^3}+\text{etc.}=\frac{C\pi^3}{n^3},$$

$$\frac{1}{m^4}+\frac{1}{(n-m)^4}+\frac{1}{(n+m)^4}+\frac{1}{(2n-m)^4}+\frac{1}{(2n+m)^4}+\text{etc.}=\frac{D\pi^4}{n^4},$$

$$\frac{1}{m^5}+\frac{1}{(n-m)^5}-\frac{1}{(n+m)^5}-\frac{1}{(2n-m)^5}+\frac{1}{(2n+m)^5}+\text{etc.}=\frac{E\pi^5}{n^5},$$

$$\frac{1}{m^6}+\frac{1}{(n-m)^6}+\frac{1}{(n+m)^6}+\frac{1}{(2n-m)^6}+\frac{1}{(2n+m)^6}+\text{etc.}=\frac{F\pi^6}{n^6}$$

etc.,

where for the even powers all terms have the sign $+$, for the odd powers on the other hand the signs agree with the signs of the first series.

§8 Let the letters A, B, C, D, E denote the same values as above, and let this series be propounded

$$A+Bz+Cz^2+Dz^3+Ez^4+\text{etc.},$$

whose sum we found using the rule given in § 6. For, the sum of this series is

$$=\frac{-dZ}{Zdz},$$

while it is

$$Z=1-\frac{z}{1y}+\frac{z^3}{1\cdot 2\cdot 3y}-\frac{z^5}{1\cdot 2\cdots 5y}+\frac{z^7}{1\cdot 2\cdots 7y}-\text{etc.}=1-\frac{1}{y}\sin z.$$

Since here y has to be put to be constant, it will be

$$dZ=\frac{-dz\cos z}{y}$$

and therefore the sum of the propounded series

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.}$$

will be

$$= \frac{\cos z}{y - \sin z}.$$

Therefore, the sum of this series will be

$$Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \text{etc.} = \frac{z\cos z}{y - \sin z}.$$

§9 Let $z = \frac{p\pi}{n}$; then this series will express the sum of all these series

$$\begin{aligned}
&+\frac{p}{m} + \frac{p}{n-m} - \frac{p}{n+m} - \frac{p}{2n-m} + \frac{p}{2n+m} + \text{etc.},\\
&+\frac{p^2}{m^2} + \frac{p^2}{(n-m)^2} + \frac{p^2}{(n+m)^2} + \frac{p^2}{(2n-m)^2} + \frac{p^2}{(2n+m)^2} + \text{etc.},\\
&+\frac{p^3}{m^3} + \frac{p^3}{(n-m)^3} - \frac{p^3}{(n+m)^3} - \frac{p^3}{(2n-m)^3} + \frac{p^3}{(2n+m)^3} + \text{etc.}
\end{aligned}$$

But adding these series vertically one will find

$$\frac{p}{m-p} + \frac{p}{n-m-p} - \frac{p}{n+m+p} - \frac{p}{2n-m+p} + \frac{p}{2n+m-p} + \text{etc.},$$

whose sum therefore is

$$= \frac{p\pi\cos\frac{p\pi}{n}}{ny - n\sin\frac{p\pi}{n}};$$

or, because y is the sine of the arc $\frac{m\pi}{n}$, one also finds the sum of this series to be

$$= \frac{p\pi \cos \frac{p\pi}{n}}{n \sin \frac{m\pi}{n} - n \sin \frac{p\pi}{n}}.$$

Therefore, if one puts

$$m - p = a \quad \text{and} \quad m + p = b,$$

so that it is

$$m = \frac{a+b}{2} \quad \text{and} \quad p = \frac{b-a}{2},$$

the sum of this series

$$\frac{1}{a} + \frac{1}{n-b} - \frac{1}{n+b} - \frac{1}{2n-a} + \frac{1}{2n+a} + \frac{1}{3n-b} - \frac{1}{3n+b} - \text{etc.}$$

or of this

$$\frac{1}{a} + \frac{2b}{n^2-b^2} - \frac{2a}{4n^2-a^2} + \frac{2b}{9n^2-b^2} - \frac{2a}{16n^2-a^2} + \frac{2b}{25n^2-b^2} - \text{etc.}$$

will result as

$$= \frac{\pi \cos \frac{(b-a)\pi}{2n}}{n \sin \frac{(b+a)\pi}{2n} - n \sin \frac{(b-a)\pi}{2n}}.$$

§10 But these considerations are too general and so it is difficult to understand everything what can actually derived from them. Therefore, let us consider some special cases and let us put the sine $y = 1$; it will be $m = 1$ and $n = 2$. Therefore, we obtain the following series

$$\frac{1}{1} + \frac{1}{1} - \frac{1}{3} - \frac{1}{3} + \frac{1}{5} + \frac{1}{5} - \frac{1}{7} - \text{etc.} = \frac{A\pi}{2},$$

$$\frac{1}{1^2} + \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.} = \frac{B\pi^2}{2^2},$$

$$\frac{1}{1^3} + \frac{1}{1^3} - \frac{1}{3^3} - \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{5^3} - \frac{1}{7^3} - \text{etc.} = \frac{C\pi^3}{2^3},$$

$$\frac{1}{1^4} + \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{5^4} + \frac{1}{7^4} + \text{etc.} = \frac{D\pi^4}{2^4}$$

etc.

or these

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \text{etc.} = \frac{A\pi}{2^2},$$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \text{etc.} = \frac{B\pi^2}{2^3},$$

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \text{etc.} = \frac{C\pi^3}{2^4},$$

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \text{etc.} = \frac{D\pi^4}{2^5},$$

$$1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \text{etc.} = \frac{E\pi^5}{2^6},$$

$$1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \text{etc.} = \frac{F\pi^6}{2^7},$$

$$1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \frac{1}{9^7} - \text{etc.} = \frac{G\pi^7}{2^8},$$

$$1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} + \text{etc.} = \frac{H\pi^8}{2^9}$$

etc.

But the values of the letters A, B, C, D etc. are found from the following relations

$$
\begin{aligned}
A &= 1, \\
B &= \frac{A}{1}, \\
C &= \frac{B}{1} - \frac{1}{1\cdot 2}, \\
D &= \frac{C}{1} - \frac{A}{1\cdot 2\cdot 3}, \\
E &= \frac{D}{1} - \frac{B}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4}, \\
F &= \frac{E}{1} - \frac{C}{1\cdot 2\cdot 3} + \frac{A}{1\cdot 2\cdot 3\cdot 4\cdot 5}, \\
G &= \frac{F}{1} - \frac{D}{1\cdot 2\cdot 3} + \frac{B}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{1}{1\cdot 2\cdots 6}, \\
H &= \frac{G}{1} - \frac{E}{1\cdot 2\cdot 3} + \frac{C}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{A}{1\cdot 2\cdots 7}
\end{aligned}
$$

etc.,

whence one finds the following values

$$
\begin{aligned}
A &= 1 \cdot \frac{\pi}{2^2} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{etc.}, \\
B &= \frac{1}{1} \cdot \frac{\pi^2}{2^3} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.}, \\
C &= \frac{1}{1\cdot 2} \cdot \frac{\pi^3}{2^4} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \text{etc.}, \\
D &= \frac{2}{1\cdot 2\cdot 3} \cdot \frac{\pi^4}{2^5} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \text{etc.},
\end{aligned}
$$

$$
\begin{aligned}
E &= \frac{5}{1\cdot 2\cdot 3\cdot 4} \cdot \frac{\pi^5}{2^6} = 1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \text{etc.},\\
F &= \frac{16}{1\cdot 2\cdot 3\cdots 5} \cdot \frac{\pi^6}{2^7} = 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \text{etc.},\\
G &= \frac{61}{1\cdot 2\cdot 3\cdots 6} \cdot \frac{\pi^7}{2^8} = 1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \text{etc.},\\
H &= \frac{272}{1\cdot 2\cdot 3\cdots 7} \cdot \frac{\pi^8}{2^9} = 1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \text{etc.},\\
I &= \frac{1385}{1\cdot 2\cdot 3\cdots 8} \cdot \frac{\pi^9}{2^{10}} = 1 - \frac{1}{3^9} + \frac{1}{5^9} - \frac{1}{7^9} + \text{etc.},\\
K &= \frac{7936}{1\cdot 2\cdot 3\cdots 9} \cdot \frac{\pi^{10}}{2^{11}} = 1 + \frac{1}{3^{10}} + \frac{1}{5^{10}} + \frac{1}{7^{10}} + \text{etc.},\\
L &= \frac{50521}{1\cdot 2\cdot 3\cdots 10} \cdot \frac{\pi^{11}}{2^{12}} = 1 - \frac{1}{3^{11}} + \frac{1}{5^{11}} - \frac{1}{7^{11}} + \text{etc.},\\
M &= \frac{353792}{1\cdot 2\cdot 3\cdots 11} \cdot \frac{\pi^{12}}{2^{13}} = 1 + \frac{1}{3^{12}} + \frac{1}{5^{12}} + \frac{1}{7^{12}} + \text{etc.},\\
N &= \frac{2702765}{1\cdot 2\cdot 3\cdots 12} \cdot \frac{\pi^{13}}{2^{14}} = 1 - \frac{1}{3^{13}} + \frac{1}{5^{13}} - \frac{1}{7^{13}} + \text{etc.},\\
O &= \frac{22368256}{1\cdot 2\cdot 3\cdots 13} \cdot \frac{\pi^{14}}{2^{15}} = 1 + \frac{1}{3^{14}} + \frac{1}{5^{14}} + \frac{1}{7^{14}} + \text{etc.}
\end{aligned}
$$

etc.

§11 Here, the letters A, B, C etc. denote only the numerical coefficients of the powers of π divided by the powers of two; even though the values can be defined conveniently by using the given rule, one can nevertheless formulate another rule, which seems to be even more appropriate for calculations. To do so, I will consider the series

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.},$$

whose sum, which I will denote by the letter s, by § 8 is

$$= \frac{\cos z}{1 - \sin z}$$

- because of $y = 1$. Therefore, if using this equation

$$s = \frac{\cos z}{1 - \sin z}$$

the value of s is expressed in the series as a power series in z, the assumed series will have to result, namely

$$A + Bz + Cz^2 + Dz^3 + \text{etc.}$$

For, one cannot assign another series of the same form, say

$$P + Qz + Rz^2 + Sz^3 + \text{etc.},$$

that is equal to that one we started from,

$$A + Bz + Cz^2 + Dz^3 + \text{etc.},$$

that at the same time the coefficients of the powers of z are identical and it is[2]

$$P = A, \quad Q = B, \quad R = C, \quad S = D \quad \text{etc.}$$

But $\frac{\cos z}{1-\sin z}$ on the other hand expresses the tangent of the arc $\left(\frac{\pi}{4} + \frac{z}{2}\right)$, or it will be

$$s = \tan\left(\frac{\pi}{4} + \frac{z}{2}\right)$$

and hence by transforming this equation into

$$\frac{\pi}{4} + \frac{z}{2} = \arctan s = \int \frac{ds}{1 + ss}$$

and having taken differentials, because of the constant $\frac{\pi}{4}$ or arc of 45 degrees, one will have

$$\frac{dz}{2} = \frac{ds}{1 + ss}$$

or

$$dz + ssdz = 2ds.$$

Now let us put

[2]Euler basically that the identity theorem for power series here.

$$S = A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.};$$

it will be

$$\begin{aligned}
\frac{2ds}{dz} &= 2B \;\; + 4C \;\; z + 6D \;\; z^2 + 8E \;\; z^3 + 10F \; z^4 + \text{etc.},\\
ss \;\; &= A^2 \;\; + 2ABz + 2ACz^2 + 2ADz^3 + 2AEz^4 + \text{etc.},\\
& \qquad\qquad\qquad + B^2 \;\; z^2 + 2BC\,z^3 + 2BDz^4\\
& \qquad\qquad\qquad\qquad\qquad\qquad + C \;\; z^4\\
1 \;\; &= +1.
\end{aligned}$$

Now having compared the homogeneous terms[3] the values of the letters one will discover that the coefficients of the single powers of z vanish; and one will obtain the following relations among the letters A, B, C, D,E etc., while, as we just found, $A = 1$:

$$\begin{aligned}
A &= 1,\\
B &= \frac{A^2+1}{2},\\
C &= \frac{2AB}{4},\\
D &= \frac{2AC+B^2}{6},\\
E &= \frac{2AD+2BC}{8},
\end{aligned}$$

[3]By this Euler means the coefficients of the same power z^n

$$F = \frac{2AE + 2BD + C^2}{10},$$

$$G = \frac{2AF + 2BE + 2CD}{12},$$

$$H = \frac{2AG + 2BF + 2CE + D^2}{14}$$

etc.

Therefore, the same equations which followed from the formulas given in §10 will result for the letters A, B, C, D etc.

§12 Because the denominators of the fractions, to which the letters A, B, C, D etc. were found to be equal, proceed regularly enough, it is possible to derive a rule to find the numerators. To to so, let us put

$$A = \alpha, \qquad F = \frac{\zeta}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5},$$

$$B = \frac{\beta}{1}, \qquad G = \frac{\eta}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6},$$

$$C = \frac{\gamma}{1 \cdot 2}, \qquad H = \frac{\theta}{1 \cdot 2 \cdot 3 \cdots 7},$$

$$D = \frac{\delta}{1 \cdot 2 \cdot 3}, \qquad I = \frac{\iota}{1 \cdot 2 \cdot 3 \cdots 8},$$

$$E = \frac{\gamma}{1 \cdot 2 \cdot 3 \cdot 4}, \qquad K = \frac{\varkappa}{1 \cdot 2 \cdot 3 \cdots 9}$$

etc.

and having substituted the values the rule will turn out to be:

$$\alpha = 1,$$

$$\beta = \frac{\alpha^2 + 1}{2},$$

$$\gamma = \alpha\beta,$$

$$\begin{aligned}
\delta &= \alpha\gamma + \beta^2, \\
\varepsilon &= \alpha\delta + 3\beta\gamma, \\
\zeta &= \alpha\varepsilon + 4\beta\delta \quad + 3\gamma^2, \\
\eta &= \alpha\zeta + 5\beta\varepsilon \quad + \frac{5\cdot 4}{1\cdot 2}\gamma\delta, \\
\theta &= \alpha\eta + 6\beta\zeta \quad + \frac{6\cdot 5}{1\cdot 2}\gamma\varepsilon \quad + \frac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3}\cdot\frac{\delta^2}{2}, \\
\iota &= \alpha\theta + 7\beta\eta \quad + \frac{7\cdot 6}{1\cdot 2}\gamma\zeta \quad + \frac{7\cdot 6\cdot 5}{1\cdot 2\cdot 3}\delta\varepsilon, \\
\varkappa &= \alpha\iota + 8\beta\theta \quad + \frac{8\cdot 7}{1\cdot 2}\gamma\varepsilon \quad + \frac{8\cdot 7\cdot 6}{1\cdot 2\cdot 3}\delta\zeta + \frac{8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{\varepsilon^2}{2}
\end{aligned}$$

etc.

The structure of these formulas is perspicuous; one only has to note, if the last term is a square, this square has additionally to be divided by two.

§13 Now let us consider this series

$$Az + Bz^2 + C^3 + Dz^4 + Ez^5 + \text{etc.},$$

whose sum is known to be

$$= \frac{z\cos z}{1-\sin z},$$

and put

$$z = \frac{p\pi}{2};$$

it will be

$$\frac{p\pi\cos\frac{p\pi}{2}}{2-2\sin\frac{p\pi}{2}} = \frac{A\pi}{2^2}\cdot 2p + \frac{B\pi^2}{2^3}\cdot 2p^2 + \frac{C\pi^3}{2^4}\cdot 2p^3 + \frac{D\pi^4}{2^5}\cdot 2p^4 + \text{etc.}$$

or

$$\frac{\pi \cos \frac{p\pi}{2}}{4 - 4\sin \frac{p\pi}{2}} = \frac{A\pi}{2^2} + \frac{pB\pi^2}{2^3} + \frac{p^2C\pi^3}{2^4} + \frac{p^3D\pi^4}{2^5} + \text{etc.}$$

Therefore if the series from § 10 are substituted for the single terms, it will result

$$\begin{aligned}
\frac{\pi \cos \frac{p\pi}{2}}{4 - 4\sin \frac{p\pi}{2}} = &+1 \;\; - \frac{1}{3} \;\; + \frac{1}{5} \;\; - \frac{1}{7} \;\; + \frac{1}{9} \;\; - \text{etc.} \\
&+ p \;\; + \frac{p}{3^2} + \frac{p}{5^2} + \frac{p}{7^2} + \frac{p}{9^2} + \text{etc.} \\
&+ p^2 - \frac{p^2}{3^3} + \frac{p^2}{5^3} - \frac{p^2}{7^3} + \frac{p^2}{9^3} - \text{etc.} \\
&+ p^3 + \frac{p^3}{3^4} + \frac{p^3}{5^4} + \frac{p^3}{7^4} + \frac{p^3}{9^4} + \text{etc.} \\
&\text{etc.}
\end{aligned}$$

But all these series added column by column give this one

$$\frac{1}{1-p} - \frac{1}{3+p} + \frac{1}{5-p} - \frac{1}{7+p} + \frac{1}{9-p} - \text{etc.}$$

whose sum therefore is

$$\frac{\pi \cos \frac{p\pi}{2}}{4 - 4\sin \frac{p\pi}{2}}.$$

§14 It will be possible to derive many summable series of this kind from the series given at the end of § 9. Let us put $a = b = m$ and we will have this series

$$\frac{2m}{n^2 - m^2} - \frac{2m}{4n^2 - m^2} + \frac{2m}{9n^2 - m^2} - \frac{2m}{16n^2 - m^2} + \frac{2m}{25n^2 - m^2} - \text{etc.},$$

whose sum will be

$$= \frac{\pi}{n \sin \dfrac{m\pi}{n}} - \frac{1}{m}$$

- because of $\cos 0\pi = 1$ and $\sin 0\pi = 0$. Hence, if the series is divided by $2m$, one will find

$$\frac{1}{n^2 - m^2} - \frac{1}{4n^2 - m^2} + \frac{1}{9n^2 - m^2} - \frac{1}{16n^2 - m^2} + \frac{1}{25n^2 - m^2} - \text{etc.}$$
$$= \frac{\pi}{2mn \sin \dfrac{m\pi}{n}} - \frac{1}{2mm}.$$

Further, let us put $a = -m$ and $b = +b$ and it will be

$$-\frac{\pi \cos \dfrac{m\pi}{n}}{n \sin \dfrac{m\pi}{n}} + \frac{1}{m}$$
$$= \frac{2m}{n^2 - m^2} + \frac{2m}{4n^2 - m^2} + \frac{2m}{9n^2 - m^2} + \frac{2m}{16n^2 - m^2} + \frac{2m}{25n^2 - m^2} + \text{etc.}$$

Therefore, if it happens that $\cos \frac{m\pi}{n}$ vanishes, the sum will be assignable algebraically; for, it is $= \frac{1}{2m^2}$, of course. But this happens, if it was $\frac{m}{n} = \frac{2i+1}{2}$ or $m = 2i + 1$ and $n = 2$, whence it will be

$$\frac{1}{2(2i+1)^2} = \frac{1}{4 - (2i+1)^2} + \frac{1}{16 - (2i+1)^2} + \frac{1}{36 - (2i+1)^2} + \frac{1}{64 - (2i+1)^2} + \text{etc.}$$

Therefore, the following paradoxical proposition arises that it is

$$\frac{1}{4 - p} + \frac{1}{16 - p} + \frac{1}{36 - p} + \frac{1}{64 - p} + \frac{1}{100 - p} + \text{etc.} = \frac{1}{2p},$$

as often as p was an integer square and odd number.

§15 Let us put $n = 1$ and $m^2 = p$; it will be

$$\frac{1}{1-p} - \frac{1}{4-p} + \frac{1}{9-p} - \frac{1}{16-p} + \frac{1}{25-p} - \text{etc.} = \frac{\pi\sqrt{p}}{2p\sin\pi\sqrt{p}} - \frac{1}{2p},$$

$$\frac{1}{1-p} + \frac{1}{4-p} + \frac{1}{9-p} + \frac{1}{16-p} + \frac{1}{25-p} + \text{etc.} = \frac{1}{2p} - \frac{\pi\sqrt{p}\cos\pi\sqrt{p}}{2p\sin\pi\sqrt{p}};$$

if these series are added, it follows that it will be

$$\frac{1}{1-p} + \frac{1}{-p} + \frac{1}{25-p} + \text{etc.} = \frac{\pi\sqrt{p}\ \text{versin}\ \pi\sqrt{p}}{4p\sin\pi\sqrt{p}};$$

but if the same series are subtracted from each other, it will be

$$\frac{1}{4-p} + \frac{1}{16-p} + \frac{1}{36-p} + \text{etc.} = \frac{1}{2p} - \frac{\pi\sqrt{p}(1+\cos\pi\sqrt{p})}{4p\sin\pi\sqrt{p}}.$$

But it is

$$\frac{\text{versin}\ \pi\sqrt{p}}{\sin\pi\sqrt{p}} = \tan\frac{\pi\sqrt{p}}{2} \quad \text{und} \quad \frac{1+\cos\pi\sqrt{p}}{\sin\pi\sqrt{p}} = \cot\frac{\pi\sqrt{p}}{2},$$

whence the last sums are simplified.

§16 Therefore, we are able to sum the following series

$$\frac{1}{1-p} \pm \frac{1}{4-p} + \frac{1}{9-p} \pm \frac{1}{16-p} + \text{etc.},$$

if p denotes a positive number, of course. But if a negative number is substituted for p, say $-q$, then so the sine and the cosine as the arcs $\pi\sqrt{p}$ or $\pi\sqrt{-q}$ become imaginary quantities. But because the sums of the series nevertheless are real and finite, the imaginary quantities cancel each other. Therefore, it will be convenient to investigate, real quantities of which kind are contained in these forms

$$\frac{\pi\sqrt{-q}}{\sin\pi\sqrt{-q}} \quad \text{and} \quad \frac{\pi\sqrt{-q}}{\tan\pi\sqrt{-q}}.$$

Therefore, let us put

$$u = \frac{\pi\sqrt{-q}}{\sin\pi\sqrt{-q}}$$

and it will be

$$\sin\pi\sqrt{-q} = \frac{\pi\sqrt{-q}}{u} \quad \text{und} \quad \pi\sqrt{-q} = \arcsin\frac{\pi\sqrt{-q}}{u};$$

now take the differentials with respect to the variables π and u; one will find

$$d\pi = \frac{ud\pi - \pi du}{u\sqrt{uu + q\pi^2}}.$$

Put $u = \pi v$; and hence this equation will result

$$d\pi = \frac{-dv}{v\sqrt{q+vv}} \quad \text{and} \quad \pi = \frac{1}{\sqrt{q}}\log\frac{\sqrt{q}+\sqrt{q+vv}}{cv}.$$

Therefore, it will be

$$e^{\pi\sqrt{p}}cv = \sqrt{q} + \sqrt{q+vv} \quad \text{and} \quad v = \frac{2e^{\pi\sqrt{q}}c\sqrt{q}}{e^{2\pi\sqrt{q}} - 1}$$

and

$$u = \frac{2\pi e^{\pi\sqrt{q}}c\sqrt{q}}{e^{2\pi\sqrt{q}}c^2 - 1}.$$

But the constant c has to be taken in such a way that for $\pi = 0$ u becomes $= 1$, whence it is $c = 1$. Therefore, in total it will be

$$\frac{\pi\sqrt{-q}}{\sin\pi\sqrt{-q}} = \frac{2e^{\pi\sqrt{q}}\pi\sqrt{q}}{e^{2\pi\sqrt{q}} - 1}.$$

In like manner it will be

$$\frac{\pi\sqrt{-q}}{\tan\pi\sqrt{-q}} = \frac{\pi}{v};$$

further, it will be

$$\pi\sqrt{-v} = \tan\pi\sqrt{-q} \quad \text{und} \quad \pi\sqrt{-q} = \arctan v\sqrt{-q}$$

and by differentiating

$$d\pi = \frac{dv}{1-qvv}.$$

Now integrate again; it will be

$$\pi = \frac{1}{2\sqrt{q}} \log \frac{1+v\sqrt{q}}{1-v\sqrt{q}} \quad \text{and} \quad e^{2\pi\sqrt{q}} - e^{2\pi\sqrt{q}} v\sqrt{q} = 1 + v\sqrt{q},$$

whence it follows

$$v = \frac{e^{2\pi\sqrt{q}} - 1}{(e^{2\pi\sqrt{q}} + 1)\sqrt{q}}$$

and

$$\frac{\pi\sqrt{-q}}{\tan \pi\sqrt{-q}} = \frac{(e^{2\pi\sqrt{q}} + 1)\pi\sqrt{q}}{e^{2\pi\sqrt{q}} - 1}.$$

§17 Hence we obtained the following eight series, whose sums can be assigned, which we want to list up altogether:

$$\frac{1}{1-p} - \frac{1}{4-p} + \frac{1}{9-p} - \frac{1}{16-p} + \frac{1}{25-p} - \text{etc.} = \frac{\pi\sqrt{p}}{2p\sin\pi\sqrt{p}} - \frac{1}{2p},$$

$$\frac{1}{1-p} + \frac{1}{4-p} + \frac{1}{9-p} + \frac{1}{16-p} + \frac{1}{25-p} + \text{etc.} = \frac{1}{2p} - \frac{\pi\sqrt{p}}{2p\tan\pi\sqrt{p}},$$

$$\frac{1}{1-p} + \frac{1}{9-p} + \frac{1}{25-p} + \frac{1}{49-p} + \frac{1}{81-p} + \text{etc.} = \frac{\pi\sqrt{p}}{4p\cot\frac{\pi\sqrt{p}}{2}},$$

$$\frac{1}{4-p} + \frac{1}{16-p} + \frac{1}{36-p} + \frac{1}{64-p} + \frac{1}{100-p} + \text{etc.} = \frac{1}{2p} - \frac{\pi\sqrt{p}}{4p\tan\frac{\pi\sqrt{p}}{2}},$$

$$\frac{1}{1+q} - \frac{1}{4+q} + \frac{1}{9+q} - \frac{1}{16+q} + \frac{1}{25+q} - \text{etc.} = \frac{1}{2q} - \frac{e^{\pi\sqrt{q}}\pi\sqrt{q}}{(e^{2\pi\sqrt{q}}-1)q},$$

$$\frac{1}{1+q}+\frac{1}{4+q}+\frac{1}{9+q}+\frac{1}{16+q}+\frac{1}{25+q}+\text{etc.}=\frac{(e^{2\pi\sqrt{q}}+1)\pi\sqrt{q}}{2(e^{2\pi\sqrt{q}}-1)q}-\frac{1}{2q},$$

$$\frac{1}{1+q}+\frac{1}{9+q}+\frac{1}{25+q}+\frac{1}{49+q}+\frac{1}{81+q}+\text{etc.}=\frac{(e^{\pi\sqrt{q}}-1)\pi\sqrt{q}}{4(e^{\pi\sqrt{q}}+1)q},$$

$$\frac{1}{4+q}+\frac{1}{16+q}+\frac{1}{36+q}+\frac{1}{64+q}+\frac{1}{100+q}+\text{etc.}=\frac{(e^{\pi\sqrt{q}}+1)\pi\sqrt{q}}{4(e^{\pi\sqrt{q}}-1)q}-\frac{1}{2q}.$$

§18 After I exhibited the rule above, how the sums of the powers of all terms of this series

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\text{etc.}$$

proceed, I will now investigate the rule connecting only the odd powers, that these sums, even not knowing the sums of the even ones, can be continued arbitrarily far; therefore, let it be

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\text{etc.}=A\pi,$$

$$1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\text{etc.}=B\pi^3,$$

$$1-\frac{1}{3^5}+\frac{1}{5^5}-\frac{1}{7^5}+\frac{1}{9^5}-\text{etc.}=C\pi^5,$$

$$1-\frac{1}{3^7}+\frac{1}{5^7}-\frac{1}{7^7}+\frac{1}{9^7}-\text{etc.}=D\pi^7,$$

$$1-\frac{1}{3^9}+\frac{1}{5^9}-\frac{1}{7^9}+\frac{1}{9^9}-\text{etc.}=E\pi^9$$

etc.

and it is to be investigated, how the coefficients A, B, C, D etc. proceed. For this aim, I will consider this series

$$A\pi z+B\pi^3z^3+C\pi^5z^5+D\pi^7z^7+\text{etc.},$$

whose sum I set to be $= s$; therefore, having multiplied these series by the corresponding powers of z respectively, it will be

$$s = \frac{z}{1-zz} - \frac{3z}{9-zz} + \frac{5z}{25-zz} - \frac{7z}{49-zz} + \text{etc.}$$

and

$$\frac{2s}{z} = \frac{1}{1-z} + \frac{1}{1+z} - \frac{1}{3-z} - \frac{1}{3+z} + \frac{1}{5-z} + \frac{1}{5+z} - \text{etc.}$$

But because from § 9 it is

$$\frac{\pi\cos\frac{(b-a)\pi}{2n}}{n\sin\frac{(b+a)\pi}{2n} - n\sin\frac{(b-a)\pi}{2n}}$$
$$= \frac{1}{a} + \frac{1}{n-b} - \frac{1}{n+b} - \frac{1}{2n-a} + \frac{1}{2n+a} + \frac{1}{3n-b} - \frac{1}{3n+b} - \text{etc.},$$

let

$$a = 1-z, \quad n = 2 \quad \text{and} \quad b = 1-z,$$

and this series will become that one; hence this equation follows

$$\frac{2s}{z} = \frac{\pi}{2\sin\frac{(1-z)\pi}{2}} \quad \text{and} \quad s = \frac{\pi z}{4\sin\frac{(1-z)\pi}{2}}$$

or

$$s = \frac{\pi z}{4\cos\frac{\pi z}{2}} = \frac{\frac{\pi z}{4}}{1 - \frac{\pi^2 z^2}{1\cdot 2\cdot 2^2} + \frac{\pi^4 z^4}{1\cdot 2\cdot 3\cdot 4\cdot 2^4} - \frac{\pi^6 z^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 2^2} + \text{etc.}};$$

because this fraction, if it the numerator is actually divided by the denominator, has to reproduce the assumed series

$$A\pi z + B\pi^3 z^3 + C\pi^5 z^5 + \text{etc.},$$

it will be

$$A = \frac{1}{4},$$
$$B = \frac{A}{2 \cdot 4},$$
$$C = \frac{B}{2 \cdot 4} - \frac{A}{2 \cdot 4 \cdot 6 \cdot 8},$$
$$D = \frac{C}{2 \cdot 4} - \frac{B}{2 \cdot 4 \cdot 6 \cdot 8} + \frac{A}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12},$$
$$E = \frac{D}{2 \cdot 4} - \frac{C}{2 \cdot 4 \cdot 6 \cdot 8} + \frac{B}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12} - \frac{A}{2 \cdot 4 \cdot 6 \cdots 16}$$

etc.

§19 Or if one puts

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{etc.} = \frac{A\pi}{2^2},$$
$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \text{etc.} = \frac{B\pi^3}{2^4},$$
$$1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \text{etc.} = \frac{C\pi^5}{2^6},$$
$$1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \text{etc.} = \frac{D\pi^7}{2^8}$$

etc.,

the coefficients A, B, C will be related to each other as follows:

$$
\begin{aligned}
A &= 1, \\
B &= \frac{A}{1 \cdot 2}, \\
C &= \frac{B}{1 \cdot 2} - \frac{A}{1 \cdot 2 \cdot 3 \cdot 4}, \\
D &= \frac{C}{1 \cdot 2} - \frac{B}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{A}{1 \cdot 2 \cdots 6}, \\
E &= \frac{D}{1 \cdot 2} - \frac{C}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{B}{1 \cdot 2 \cdots 6} - \frac{A}{1 \cdot 2 \cdots 8}
\end{aligned}
$$

etc.

But therefore, if those series are continued backwards that one reaches positive powers, the sums of all those sums will be $= 0$, so that, even if we proceed further in these formulas, no other values would result. Of course, it is

$$
\begin{aligned}
1 - 3 \ + 5 \ - 7 \ + 9 \ - \text{etc.} &= 0, \\
1 - 3^3 + 5^3 - 7^3 + 9^3 - \text{etc.} &= 0, \\
1 - 3^5 + 5^5 - 7^5 + 9^5 - \text{etc.} &= 0, \\
1 - 3^7 + 5^7 - 7^7 + 9^7 - \text{etc.} &= 0
\end{aligned}
$$

etc.

§20 But as the sums of the odd powers follow a certain rule, so also the even powers will enjoy a similar property, that they can all be defined not taking into account the odd powers. In order to find this rule, we will use a similar operation. Hence let be

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\text{etc.}=A\pi^2,$$

$$1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}+\text{etc.}=B\pi^4,$$

$$1+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{9^6}+\text{etc.}=C\pi^6,$$

$$1+\frac{1}{3^8}+\frac{1}{5^8}+\frac{1}{7^8}+\frac{1}{9^8}+\text{etc.}=D\pi^8$$

etc.

and investigate the sum of this series

$$A\pi^2z^2+B\pi^4z^4+C\pi^6z^6+D\pi^8z^8+\text{etc.}=s;$$

it will be

$$s=\frac{z^2}{1-z^2}+\frac{z^2}{9-z^2}+\frac{z^2}{25-z^2}+\frac{z^2}{49-z^2}+\text{etc.},$$

whence it will be from § 17

$$s=\frac{\pi z}{4\cot\dfrac{\pi z}{2}}$$

or by a series

$$s=\frac{\dfrac{\pi^2z^2}{1\cdot 2^3}-\dfrac{\pi^4z^4}{1\cdot 2\cdot 3\cdot 2^5}+\dfrac{\pi^6z^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 2^7}-\text{etc.}}{1-\dfrac{\pi^2z^2}{1\cdot 2\cdot 2^2}+\dfrac{\pi^4z^4}{1\cdot 2\cdot 3\cdot 4\cdot 2^4}-\dfrac{\pi^6z^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 2^6}+\text{etc.}};$$

because the assumed series itself has to result from this division, it will be

$$A=\frac{1}{8},$$

$$B=\frac{A}{2\cdot 4}-\frac{1}{2\cdot 4\cdot 6\cdot 4},$$

$$C = \frac{B}{2 \cdot 4} - \frac{A}{2 \cdot 4 \cdot 6 \cdot 8} + \frac{1}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 4},$$

$$D = \frac{C}{2 \cdot 4} - \frac{B}{2 \cdot 4 \cdot 6 \cdot 8} + \frac{1}{2 \cdot 4 \cdot 6 \cdots 12} - \frac{1}{2 \cdot 4 \cdots 14 \cdot 4}$$

etc.

§21 But this rule is seen easier, if one puts

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \text{etc.} = A\frac{\pi^2}{2^3},$$

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \text{etc.} = B\frac{\pi^4}{2^5},$$

$$1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \text{etc.} = C\frac{\pi^6}{2^7},$$

$$1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} + \text{etc.} = D\frac{\pi^8}{2^9}$$

etc.

For, here the coefficients A, B, C etc. will lead to the following progression:

$$A = 1,$$

$$B = \frac{A}{1 \cdot 2},$$

$$C = \frac{B}{1 \cdot 2} - \frac{A}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5},$$

$$D = \frac{C}{1 \cdot 2} - \frac{B}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{A}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \frac{1}{1 \cdot 2 \cdots 7}$$

etc.

Therefore, if one assumes this power series for s

$$s = Az + Bz^3 + Cz^5 + Dz^7 + Ez^9 + \text{etc.},$$

it will be

$$s = \frac{z - \frac{z^3}{1 \cdot 2 \cdot 3} + \frac{z^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{z^7}{1 \cdot 2 \cdots 7} + \text{etc.}}{1 - \frac{z^2}{1 \cdot 2} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{z^6}{1 \cdot 2 \cdot 3 \cdots 6} + \text{etc.}}$$

and hence

$$s = \tan z \quad \text{oder} \quad z = \arctan s.$$

We will therefore have

$$dz = \frac{ds}{1 + ss} \quad \text{and} \quad dz + ssdz = ds;$$

since this equations have to be solved by s

$$s = Az + Bz^3 + Cz^5 + Dz^7 + Ez^9 + \text{etc.},$$

substitute the values for ds and ss and it will be

$$\begin{aligned}
\frac{ds}{dz} &= A + 3Bz^2 + 5C \quad z^4 + 7D \quad z^6 + 9E \quad z^8 + \text{etc.}, \\
ss \quad &= \quad + A^2z^2 + 2ABz^4 + 2ACz^6 + 2ADz^8 + \text{etc.}, \\
&\qquad\qquad\qquad\qquad + B^2 \quad z^6 + 2BC\, z^8 \\
1 \quad &= 1.
\end{aligned}$$

Therefore, having set up the equations the following other determinations of the letters A, B, C, D etc. will result:

$$\begin{aligned}
A &= 1, \\
B &= \frac{A^2}{3}, \\
C &= \frac{2AB}{5},
\end{aligned}$$

$$D = \frac{2AC + B^2}{7},$$

$$E = \frac{2AD + 2BC}{9},$$

$$F = \frac{2AE + 2BD + C^2}{11}$$

etc.

§22 But the sums of the series contained in this general form

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \text{etc.},$$

while n denotes an even number, depend on the series of the even powers. For, if it was

$$1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \text{etc.} = N\pi^n,$$

it will be

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \text{etc.} = \frac{2^n N\pi^n}{2^n - 1},$$

whence the sums of all these series, as long n is an even number, can be found by the quadrature of the circle; and additionally one will find the sums of the series of the odd powers from the sums of the same powers already found. But in order to find these sums directly, let us investigate how these sums proceed. Therefore, let

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.} = A\pi^2,$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{etc.} = B\pi^4,$$

$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \text{etc.} = C\pi^6,$$

$$1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \text{etc.} = D\pi^8$$

etc.

and I will consider this series

$$s = A\pi^2z^2 + B\pi^4z^4 + C\pi^6z^6 + D\pi^8z^8 + E\pi^{10}z^{10} + \text{etc.},$$

which having substituted the series for $A\pi^2$, $B\pi^4$, $C\pi^6$ etc. which these letters denoted and having added the homologous terms this equation will result

$$s = \frac{zz}{1-zz} + \frac{zz}{4-zz} + \frac{zz}{9-zz} + \frac{zz}{16-zz} + \frac{zz}{25-zz} + \text{etc.},$$

which series summed by § 17 gives

$$s = \frac{1}{2} - \frac{\pi z}{2\tan \pi z}$$

or, if the tangent of the arc πz is expressed by a series,

$$\begin{aligned} s &= \frac{1}{2} - \frac{1}{2}\cdot\frac{1 - \frac{\pi^2z^2}{1\cdot 2} + \frac{\pi^4z^4}{1\cdot 2\cdot 3\cdot 4} - \frac{\pi^6z^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \text{etc.}}{1 - \frac{\pi^2z^2}{1\cdot 2\cdot 3} + \frac{\pi^4z^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{\pi^6z^6}{1\cdot 2\cdots 7} + \text{etc.}} \\ &= \frac{\frac{\pi^2z^2}{1\cdot 2\cdot 3} - \frac{2\pi^4z^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{3\pi^6z^6}{1\cdot 2\cdots 7} - \frac{4\pi^8z^8}{1\cdot 2\cdots 9}}{1 - \frac{\pi^2z^2}{1\cdot 2\cdot 3} + \frac{\pi^4z^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{\pi^6z^6}{1\cdot 2\cdots 7} + \frac{\pi^8z^8}{1\cdot 2\cdots 9} - \text{etc.}}; \end{aligned}$$

because having expanded this expression it has give the assumed series itself

$$A\pi^2z^2 + B\pi^4z^4 + C\pi^6z^6 + D\pi^8z^8 + \text{etc.},$$

these equations for the coefficients will follow:

$$\begin{aligned} A &= \frac{1}{6}, \\ B &= \frac{A}{1\cdot 2\cdot 3} - \frac{2}{1\cdot 2\cdot 3\cdot 4\cdot 5}, \\ C &= \frac{B}{1\cdot 2\cdot 3} - \frac{A}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{3}{1\cdot 2\cdots 7}, \\ D &= \frac{C}{1\cdot 2\cdot 3} - \frac{B}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{A}{1\cdot 2\cdots 7} - \frac{4}{1\cdot 2\cdots 9} \end{aligned}$$

etc.

§23 But it is possible to exhibit another rule expressing how these same coefficients proceed and by means of this rule it will also be possible to find those coefficients a lot easier. Because it is

$$s = \frac{1}{2} - \frac{\pi z}{2\tan \pi z},$$

it will be

$$\tan \pi z = \frac{\pi z}{1-2s} \quad \text{and} \quad \pi z = \arctan \frac{\pi z}{1-2s}.$$

Now put $\pi z = u$; it will be

$$u = \arctan \frac{u}{1-2s}$$

and by differentiating

$$du = \frac{du - 2sdu + 2uds}{1 - 4s + 4ss + uu}$$

or

$$uudu + 4ssdu = 2sdu + 2uds,$$

which equation is solved by this value

$$s = Au^2 + Bu^4 + Cu^6 + Du^8 + Eu^{10} + \text{etc.},$$

having substituted this value it will be

$$\begin{aligned}
uu &= uu,\\
4ss &= \quad + 4A^2u^4 + 8ABu^6 + 8ACu^8 + 8ADu^{10} + 8AEu^{12} + \text{etc.},\\
&\qquad\qquad\qquad + 4B^2\, u^8 + 8BC\, u^{10} + 8BDu^{12}\\
&\qquad\qquad\qquad\qquad\qquad + 4C^2\, u^{12}\\
2s &= 2Au^2 + 2B\, u^4 + 2C\, u^6 + 2D\, u^8 + 2E\, u^{10} + 2F\, u^{12} + \text{etc.},\\
\frac{2uds}{du} &= 4Au^2 + 8B\, u^4 + 12C\, u^6 + 16D\, u^8 + 20E\, u^{10} + 24F\, u^{12} + \text{etc.},
\end{aligned}$$

whence the following equations result:

$$
\begin{aligned}
A &= \frac{1}{6},\\
B &= \frac{2A^2}{5},\\
C &= \frac{4AB}{7},\\
D &= \frac{4AC+2B^2}{9},\\
E &= \frac{4AD+4BC}{11},\\
F &= \frac{4AE+4BD+2C^2}{13},\\
G &= \frac{4AF+4BE+4CD}{15},\\
A &= \frac{4AG+4BF+4CE+2D^2}{17}
\end{aligned}
$$

etc.

§24 But the sums of the series of this kind, as far I gave them, are the following:

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\text{etc.}=\frac{2}{1\cdot 2\cdot 3}\cdot\frac{1}{2}\pi^2,$$

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\text{etc.}=\frac{2^3}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot\frac{1}{6}\pi^4,$$

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\text{etc.}=\frac{2^5}{1\cdot 2\cdots 7}\cdot\frac{1}{6}\pi^6,$$

$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\text{etc.}=\frac{2^7}{1\cdot 2\cdots 9}\cdot\frac{3}{10}\pi^8,$$

$$1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\frac{1}{5^{10}}+\text{etc.}=\frac{2^9}{1\cdot 2\cdots 11}\cdot\frac{5}{6}\pi^{10},$$

$$1+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{4^{12}}+\frac{1}{5^{12}}+\text{etc.}=\frac{2^{11}}{1\cdot 2\cdots 13}\cdot\frac{691}{210}\pi^{12},$$

$$1+\frac{1}{2^{14}}+\frac{1}{3^{14}}+\frac{1}{4^{14}}+\frac{1}{5^{14}}+\text{etc.}=\frac{2^{13}}{1\cdot 2\cdots 15}\cdot\frac{35}{2}\pi^{14},$$

$$1+\frac{1}{2^{16}}+\frac{1}{3^{16}}+\frac{1}{4^{16}}+\frac{1}{5^{16}}+\text{etc.}=\frac{2^{15}}{1\cdot 2\cdots 17}\cdot\frac{3617}{30}\pi^{16},$$

$$1+\frac{1}{2^{18}}+\frac{1}{3^{18}}+\frac{1}{4^{18}}+\frac{1}{5^{18}}+\text{etc.}=\frac{2^{17}}{1\cdot 2\cdots 19}\cdot\frac{43867}{42}\pi^{18},$$

$$1+\frac{1}{2^{20}}+\frac{1}{3^{20}}+\frac{1}{4^{20}}+\frac{1}{5^{20}}+\text{etc.}=\frac{2^{19}}{1\cdot 2\cdots 21}\cdot\frac{1222277}{110}\pi^{20},$$

$$1+\frac{1}{2^{22}}+\frac{1}{3^{22}}+\frac{1}{4^{22}}+\frac{1}{5^{22}}+\text{etc.}=\frac{2^{21}}{1\cdot 2\cdots 23}\cdot\frac{854513}{6}\pi^{22},$$

$$1+\frac{1}{2^{24}}+\frac{1}{3^{24}}+\frac{1}{4^{24}}+\frac{1}{5^{24}}+\text{etc.}=\frac{2^{23}}{1\cdot 2\cdots 25}\cdot\frac{1181820455}{546}\pi^{24}.$$

In these expressions only the rule how to form the factions in the middle is not manifest, how to form the remaining parts on the other hand is clear. But after I had considered these fractions in the middle

$$\frac{1}{2},\quad \frac{1}{6},\quad \frac{1}{6},\quad \frac{3}{10},\quad \frac{5}{6}\quad \text{etc.}$$

with more attention, I discovered that the same fractions occur in the general formula expressing the sum of any series from its general term, so that by means of the one expression the other can be constructed.

§25 Therefore, it will be worth the effort to investigate this agreement of these rather different-looking expressions with more attention. The one expression I gave for the summation of series can be explained follows: If the general term of any series or that term corresponding to indefinite numerical index x was $= X$ and the sum of the series from the first term to this term X is put $= S$, it will be

$$\begin{aligned} S = \int X dx + \frac{X}{1\cdot 2} &+ \frac{dX}{1\cdot 2\cdot 3\cdot 2dx} - \frac{d^3X}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6dx^3} \\ &+ \frac{d^5X}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 6dx^5} - \frac{d^7X}{1\cdot 2\cdot 3\cdots 9\cdot 6dx^7} \\ &+ \frac{5d^9X}{1\cdot 2\cdots 11dx^9} - \frac{691d^{11}X}{1\cdot 2\cdots 13\cdot 210dx^{11}} \\ &+ \frac{35d^{13}X}{1\cdot 2\cdots 15\cdot 2dx^{13}} - \frac{3617d^{15}X}{1\cdot 2\cdots 17\cdot 30dx^{15}} \\ &+ \frac{43867d^{17}X}{1\cdot 2\cdots 19\cdot 42dx^{17}} - \frac{1222277d^{19}X}{1\cdot 2\cdots 21\cdot 110dx^{19}} \\ &+ \frac{854513d^{21}X}{1\cdot 2\cdots 23\cdot 6dx^{21}} - \frac{1181820455d^{23}X}{1\cdot 2\cdots 25\cdot 546dx^{23}} \\ &\text{etc.,} \end{aligned}$$

in which expression it is apparently the same irregular fractions occur

$$\frac{1}{2},\quad \frac{1}{6},\quad \frac{1}{6},\quad \frac{3}{10},\quad \frac{5}{6} \quad \text{etc.,}$$

which occurred in the expression of the sums before, only with this difference that their signs alternate here, whereas there all had the same sign $+$. And this agreement allowed me to continue this general expression of the sum S up to the point I gave them here, whereas by the rule I had found at that time for the progression of these terms, I could do this only by a huge amount of work.

§26 But even though this mere observation of this extraordinary agreement could suffice to show the agreement in the following terms, which are not known, it will nevertheless be better to actually prove it from the a general principle; hence then it is understood not to happen by chance but necessarily. But I obtained this last expression in the following way. Because S denotes the sum of so many terms in any series, as unities are contained in the exponent x, and the last of these terms is $= X$, it is manifest, if in the expression for S one puts $x-1$ instead of x, that then the same sum S without that last term X, or $S-X$, has to result. But having put $x-1$ instead of x the quantity S will become

$$S - \frac{dS}{1dx} + \frac{ddS}{1\cdot 2dx^2} - \frac{d^3S}{1\cdot 2\cdot 3dx^3} + \frac{d^4S}{1\cdot 2\cdot 3\cdot 4dx^4} - \text{etc.},$$

which is therefore equal to $S-X$; hence one has this equation

$$X = \frac{dS}{1dx} - \frac{ddS}{1\cdot 2dx^2} + \frac{d^3S}{1\cdot 2\cdot 3dx^3} - \frac{d^4S}{1\cdot 2\cdot 3\cdot 4dx^4} + \text{etc.}$$

Now to express this representation of S in terms of X, I assume this equation

$$S = \int Xdx + \alpha X + \frac{\beta dX}{dx} + \frac{\gamma ddX}{dx^2} + \frac{\delta d^3X}{dx^3} + \text{etc.};$$

having substituted this equation in the one above one will find

$$\begin{aligned}
X = X &+ \frac{\alpha dX}{dx} + \frac{\beta ddX}{dx^2} + \frac{\gamma d^3X}{dx^3} + \frac{\delta d^4X}{dx^4} + \text{etc.}\\
&- \frac{dX}{1\cdot 2dx} - \frac{\alpha ddX}{1\cdot 2dx^2} - \frac{\beta d^3X}{1\cdot 2dx^3} - \frac{\gamma d^4X}{1\cdot 2dx^4}\\
&+ \frac{ddX}{1\cdot 2\cdot 3dx^2} + \frac{\alpha d^3X}{1\cdot 2\cdot 3dx^3} + \frac{\beta d^4X}{1\cdot 2\cdot 3dx^4}\\
&- \frac{d^4X}{1\cdot 2\cdot 3\cdot 4dx^3} - \frac{\alpha d^4X}{1\cdot 2\cdot 3\cdot 4dx^4}\\
&+ \frac{d^4X}{1\cdot 2\cdot 3\cdot 4\cdot 5dx^4}
\end{aligned}$$

§27 From this equality the following determinations of the coefficients α, β, γ, δ etc. are derived

$$\begin{aligned}
\alpha &= \frac{1}{1\cdot 2},\\
\beta &= \frac{\alpha}{1\cdot 2} - \frac{1}{1\cdot 2\cdot 3},\\
\gamma &= \frac{\beta}{1\cdot 2} - \frac{\alpha}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4},\\
\delta &= \frac{\gamma}{1\cdot 2} - \frac{\beta}{1\cdot 2\cdot 3} + \frac{\alpha}{1\cdot 2\cdot 3\cdot 4} - \frac{1}{1\cdot 2\cdots 5},\\
\varepsilon &= \frac{\delta}{1\cdot 2} - \frac{\gamma}{1\cdot 2\cdot 3} + \frac{\beta}{1\cdot 2\cdot 3\cdot 4} - \frac{\alpha}{1\cdot 2\cdots 5} + \frac{1}{1\cdot 2\cdots 6}\\
&\text{etc.}
\end{aligned}$$

and from these formulas I then calculated the values of these letters, investing a lot of work. And only by observation alone, it happened in this paper, I recognized that all the second values γ, ε, η vanish. But from principles formulated this can now be proved rigorously, if another rule is found for this progression. To do so, I will consider this series

$$s = 1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 + \text{etc.}$$

and the preceding rule for the coefficients gives

$$s = \frac{1}{1 - \frac{z}{1\cdot 2} + \frac{zz}{1\cdot 2\cdot 3} - \frac{z^3}{1\cdot 2\cdot 3\cdot 4} + \frac{z^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \text{etc.}},$$

which equation becomes

$$s = \frac{z}{1-e^{-z}} \quad \text{or} \quad s = \frac{e^z z}{e^z - 1}.$$

Hence it arises

$$e^z s - a = e^z z \quad \text{and} \quad e^z = \frac{s}{s-z} \quad \text{and also} z = \log s - \log(s-z).$$

But by differentiating the last equation one will find

$$dz = \frac{ds}{s} - \frac{ds - dz}{s - z}$$

or

$$ssdz - szdz = sdz - zds,$$

which equation has to be solved by the assumed value

$$s = 1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 + \text{etc.};$$

therefore, substitute this value in this equation

$$\frac{zds}{dz} - s - sz + ss = 0$$

and one will obtain

$$\begin{aligned}
\frac{zds}{dz} &= \quad\; + \alpha z + 2\beta z^2 + 3\gamma z^3 + 4\delta z^4 + 5\varepsilon z^5 + \text{etc.},\\
-s &= -1 + \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \varepsilon z^5 - \text{etc.},\\
-sz &= \quad\; - z - \alpha z^2 - \beta z^3 - \gamma z^4 - \delta z^5 - \text{etc.},\\
+s^2 &= 1 + 2\alpha z + 2\beta z^2 + 2\gamma z^3 + 2\delta z^4 + 2\varepsilon z^5 + \text{etc.}\\
&\qquad\qquad + \alpha^2 \quad + 2\alpha\beta \quad + 2\alpha\gamma \quad + 2\alpha\delta\\
&\qquad\qquad\qquad\qquad\qquad + \beta^2 \quad + 2\beta\gamma
\end{aligned}$$

Therefore, one can conclude that it will be

$$\begin{aligned}
\alpha &= \frac{1}{2},\\
\beta &= \frac{\alpha - \alpha^2}{3},\\
\gamma &= \frac{\beta - 2\alpha\beta}{4},\\
\delta &= \frac{\gamma - 2\alpha\gamma - \beta\beta}{5},
\end{aligned}$$

$$\varepsilon = \frac{\delta - 2\alpha\delta - 2\beta\gamma}{6},$$

$$\zeta = \frac{\varepsilon - 2\alpha\varepsilon - 2\beta\delta - \gamma\gamma}{7},$$

$$\eta = \frac{\zeta - 2\alpha\zeta - 2\beta\varepsilon - 2\gamma\delta}{8}$$

etc.

§28 Because it is $\alpha = \frac{1}{2}$, it will be $1 - 2\alpha = 0$; because this value occurs in all following terms, it will be

$$\alpha = \frac{1}{2},$$

$$\beta = \frac{1}{12},$$

$$\gamma = 0,$$

$$\delta = -\frac{\beta\beta}{5},$$

$$\varepsilon = -\frac{2\beta\gamma}{6},$$

$$\zeta = \frac{-2\beta\delta - \gamma\gamma}{7},$$

$$\eta = \frac{-2\beta\varepsilon - 2\gamma\delta}{8},$$

$$\theta = \frac{-2\beta\zeta - 2\gamma\varepsilon - \delta\delta}{9},$$

$$\iota = \frac{-2\beta\eta - 2\gamma\zeta - 2\delta\varepsilon}{10}$$

etc.

Because now it is $\gamma = 0$, it is evident that it also is $\varepsilon = 0$ and hence further $\eta = 0$, $\iota = 0$ etc. so that all remaining second terms, beginning from γ are $= 0$, what evident clear from the preceding rule only by observation, but is now proved to happen necessarily. Hence, while it still is $\alpha = \frac{1}{2}$, we will find the following equations:

$$\begin{aligned}
\beta &= \frac{1}{12},\\
\delta &= -\frac{\beta^2}{5},\\
\zeta &= -\frac{2\beta\delta}{7},\\
\theta &= \frac{2\beta\zeta - 2\delta\delta}{9},\\
\varkappa &= \frac{-2\beta\theta - 2\delta\zeta}{11}\\
&\text{etc.}
\end{aligned}$$

Therefore, if one puts

$$\beta = \frac{A}{2}, \quad \delta = -\frac{B}{2^3}, \quad \zeta = \frac{C}{2^5}, \quad -\theta = -\frac{D}{2^7}, \quad \varkappa = \frac{E}{2^9} \quad \text{etc.,}$$

so that

$$S = \int Xdx = \frac{X}{2} + \frac{AdX}{2dx} - \frac{Bd^3X}{2^3dx^3} + \frac{Cd^5X}{2^5dx^5} - \frac{Dd^7X}{2^7dx^7} + \frac{Ed^9X}{2^9dx^9} - \frac{Fd^{11}X}{2^{11}dx^{11}} + \text{etc.},$$

the coefficients A, B, C, D will give this rule

$$\begin{aligned}
A &= \frac{1}{6},\\
B &= \frac{2A^2}{5},\\
C &= \frac{4AB}{7},\\
D &= \frac{4AC + 2B^2}{9},
\end{aligned}$$

$$E = \frac{4AD + 4BC}{11},$$

$$F = \frac{4AE + 4BD + 2C^2}{13},$$

$$G = \frac{4AF + 4BE + 4CD}{15}$$

etc.

So the letters A, B, C, D etc. obtain the values which we attributed to them above in § 22 and 23. And hence we now proved the agreement of the coefficients completely rigorously and hence the agreement cannot be ascribed to chance anymore.

§29 Although we can assign the sum of this series

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \text{etc.}$$

conveniently enough, if n is an even number, we are nevertheless not able to conclude anything from these same principles to find the sums, if n is an odd number. It is natural to conjecture that these series also depend on the quadrature of the circle in such a way and that their sum is $= N\pi^n$ also in the cases, in which n is an odd number; but if we actually calculate these sums by approximations, we will see that the coefficient N does not become a rational number, if n is not an even number, what will be seen more clearly from this table:

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \text{etc.} = 1,644934067 = \frac{\pi^2}{6},$$

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \text{etc.} = 1,202056903 = \frac{\pi^3}{26,79435},$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \text{etc.} = 1,082323234 = \frac{\pi^4}{90},$$

$$1 + \frac{1}{2^5} + \frac{1}{3^5} + \text{etc.} = 1,036927755 = \frac{\pi^5}{295,1215},$$

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\text{etc.}=1,017343062=\frac{\pi^6}{945},$$
$$1+\frac{1}{2^7}+\frac{1}{3^7}+\text{etc.}=1,008349277=\frac{\pi^7}{2995,285},$$
$$1+\frac{1}{2^8}+\frac{1}{3^8}+\text{etc.}=1,004077356=\frac{\pi^8}{9450},$$
$$1+\frac{1}{2^9}+\frac{1}{3^9}+\text{etc.}=1,002008393=\frac{\pi^9}{29749,35},$$
$$1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\text{etc.}=1,000994575=\frac{\pi^{10}}{93555},$$
$$1+\frac{1}{2^{11}}+\frac{1}{3^{11}}+\text{etc.}=1,000494189=\frac{\pi^{11}}{294058,7},$$
$$1+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\text{etc.}=1,000246087=\frac{\pi^{12}}{924041\frac{544}{691}}.$$

And furthermore no relation among the sums of the odd powers similar to that seen in the case of the even powers is detected.

§30 But it seems that it is possible to conclude something about the sums of the odd powers if the signs alternate. Because the first sum of the odd powers,

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\text{etc.},$$

has a known sum, of course $\log 2$, it seems to be very probable that also the sums of the following odd powers depend on the logarithm of two and maybe furthermore on the quadrature of the circle. But before we conclude anything here, let us investigate the sums of the even powers and let it be

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\text{etc.}=A\pi^2,$$
$$1-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\text{etc.}=B\pi^4,$$

$$1-\frac{1}{2^6}+\frac{1}{3^6}-\frac{1}{4^6}+\text{etc.}=C\pi^6,$$

$$1-\frac{1}{2^8}+\frac{1}{3^8}-\frac{1}{4^8}+\text{etc.}=D\pi^8,$$

$$1-\frac{1}{2^{10}}+\frac{1}{3^{10}}-\frac{1}{4^{10}}+\text{etc.}=E\pi^{10}$$

etc.,

where the values of the letters A, B, C, D etc. can be deduced easily from the known values for the same series, in which the sings of the terms do not alternate; but it will be more convenient to find an own rule for this case. Therefore, I will consider the following series

$$s=A\pi^2z^2+B\pi^4z^4+C\pi^6z^6+D\pi^8z^8+\text{etc.},$$

which having substituted the series will become this one

$$s=\frac{zz}{1-zz}-\frac{zz}{4-zz}+\frac{zz}{9-zz}-\frac{zz}{16-zz}+\text{etc.},$$

which series summed by § 17 will give

$$s=\frac{\pi z}{2\sin\pi z}-\frac{1}{2}$$

or having expressed the sine by its power series

$$s=-\frac{1}{2}+\frac{\frac{1}{2}}{1-\frac{\pi^2z^2}{1\cdot2\cdot3}+\frac{\pi^4z^4}{1\cdot2\cdot3\cdot4\cdot5}-\frac{\pi^6z^6}{1\cdot2\cdots7}+\text{etc.}}.$$

If now in the series of letters A, B, C, D, E etc. the preceding term or the one before the first A is put $=\Delta$, it will be

$$\Delta=\frac{1}{2},$$

$$A=\frac{\Delta}{1\cdot2\cdot3}=\frac{1}{12},$$

$$B=\frac{A}{1\cdot2\cdot3}-\frac{\Delta}{1\cdot2\cdot3\cdot4\cdot5},$$

$$C = \frac{B}{1 \cdot 2 \cdot 3} - \frac{A}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{\Delta}{1 \cdot 2 \cdots 7},$$

$$D = \frac{C}{1 \cdot 2 \cdot 3} - \frac{B}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{A}{1 \cdot 2 \cdots 7} - \frac{\Delta}{1 \cdot 2 \cdots 9}$$

etc.

But the value of Δ was not merely assumed to be $\frac{1}{2}$, but indeed expresses the sum of the preceding series, which is

$$1 - 1 + 1 - 1 + 1 - 1 + \text{etc.} = \Delta \pi^0 = \frac{1}{2},$$

the sums of all remaining series on the other hand, which precede this one, are $= 0$, of course

$$1 - 2^2 + 3^2 - 4^2 + \text{etc.} = 0,$$

$$1 - 2^4 + 3^4 - 4^4 + \text{etc.} = 0,$$

$$1 - 2^6 + 3^6 - 4^6 + \text{etc.} = 0$$

etc.

§31 Therefore, it follows that the sum of any series can be deduced from the preceding ones in this way: If it was

$$1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \text{etc.} = \alpha \pi^n,$$

$$1 - \frac{1}{2^{n-2}} + \frac{1}{3^{n-2}} - \frac{1}{4^{n-2}} + \text{etc.} = \beta \pi^{n-2},$$

$$1 - \frac{1}{2^{n-4}} + \frac{1}{3^{n-4}} - \frac{1}{4^{n-4}} + \text{etc.} = \gamma \pi^{n-4},$$

$$1 - \frac{1}{2^{n-6}} + \frac{1}{3^{n-6}} - \frac{1}{4^{n-6}} + \text{etc.} = \delta \pi^{n-6}$$

etc.,

it will be

$$\alpha = \frac{\beta}{1\cdot 2\cdot 3} - \frac{\gamma}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{\delta}{1\cdot 2\cdots 7} - \frac{\varepsilon}{1\cdot 2\cdots 9} + \frac{\zeta}{1\cdot 2\cdots 9} - \text{etc.}$$

So to find the sum of this series

$$1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} - \frac{1}{6^3} + \text{etc.},$$

one will have all series, which precede it according to this rule; these series are

$$\begin{aligned}
&1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \text{etc.} = B\pi^3,\\
&1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \text{etc.} = A\pi,\\
&1 - 2 + 3 - 4 + \text{etc.} = \frac{\alpha}{\pi},\\
&1 - 2^3 + 3^3 - 4^3 + \text{etc.} = \frac{\beta}{\pi^3},\\
&1 - 2^5 + 3^5 - 4^5 + \text{etc.} = \frac{\gamma}{\pi^5}
\end{aligned}$$

etc.,

and it will be

$$B = \frac{A}{1\cdot 2\cdot 3} - \frac{\alpha}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{\beta}{1\cdot 2\cdots 7} - \frac{\gamma}{1\cdot 2\cdots 9} + \text{etc.}$$

But the sums of all these series can be exhibited; for, it is

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\text{etc.}=\log 2,$$

$$1-2+3-4+\text{etc.}=\frac{1}{4}=\frac{2\cdot 1}{\pi^2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\text{etc.}\right),$$

$$1-2^3+3^3-4^3+\text{etc.}=\frac{-1}{8}=\frac{-2\cdot 1\cdot 2\cdot 3}{\pi^4}\left(1+\frac{1}{3^4}+\frac{1}{5^4}+\text{etc.}\right),$$

$$1-2^5+3^5-4^5+\text{etc.}=\frac{1}{4}=\frac{2\cdot 1\cdot 2\cdot 3\cdot 4\cdot 5}{\pi^6}\left(1+\frac{1}{3^6}+\frac{1}{5^6}+\text{etc.}\right),$$

$$1-2^7+3^7-4^7+\text{etc.}=\frac{-17}{16}=\frac{2\cdot 1\cdot 2\cdots 7}{\pi^8}\left(1+\frac{1}{3^8}+\frac{1}{5^8}+\text{etc.}\right)$$

etc.

And therefore it will be

$$A=\frac{\log 2}{\pi},$$

$$\alpha=\frac{2\cdot 1}{\pi}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\text{etc.}\right),$$

$$\beta=\frac{-2\cdot 1\cdot 2\cdot 3}{\pi}\left(1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\text{etc.}\right),$$

$$\gamma=\frac{2\cdot 1\cdot 2\cdot 3\cdot 4\cdot 5}{\pi}\left(1+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\text{etc.}\right),$$

$$\delta=\frac{-2\cdot 1\cdot 2\cdots 7}{\pi}\left(1+\frac{1}{3^8}+\frac{1}{5^8}+\frac{1}{7^8}+\text{etc.}\right),$$

$$\varepsilon=\frac{2\cdot 1\cdot 2\cdots 9}{\pi}\left(1+\frac{1}{3^{10}}+\frac{1}{5^{10}}+\frac{1}{7^{10}}+\text{etc.}\right)$$

etc.

§32 But we exhibited the sums of even powers of the fractions, whose denominators are odd numbers, above. Let

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\text{etc.}=P\pi^2,$$

$$1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\text{etc.}=Q\pi^4,$$

$$1+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\text{etc.}=R\pi^6,$$

$$1+\frac{1}{3^8}+\frac{1}{5^8}+\frac{1}{7^8}+\text{etc.}=S\pi^8$$

etc.:

by § 21 it will be

$$P\pi^2+Q\pi^4+R\pi^6+S\pi^8+\text{etc.}=\frac{\pi}{4}\tan\frac{\pi}{2}.$$

But the letters α, β, γ, δ will obtain the following values

$$\alpha = \frac{2\cdot 1}{\pi} P\pi^2,$$

$$\beta = \frac{-2\cdot 1\cdot 2\cdot 3}{\pi} Q\pi^4,$$

$$\gamma = \frac{2\cdot 1\cdot 2\cdot 3\cdot 4\cdot 5}{\pi} R\pi^6,$$

$$\delta = \frac{-2\cdot 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}{\pi} S\pi^8$$

etc.

Therefore, using the rule for their progression, if we put

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \text{etc.} = A\pi = \log 2,$$

$$1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} - \frac{1}{6^3} + \text{etc.} = B\pi^3,$$

$$1 - \frac{1}{2^5} + \frac{1}{3^5} - \frac{1}{4^5} + \frac{1}{5^5} - \frac{1}{6^5} + \text{etc.} = C\pi^5,$$

$$1 - \frac{1}{2^7} + \frac{1}{3^7} - \frac{1}{4^7} + \frac{1}{5^7} - \frac{1}{6^7} + \text{etc.} = D\pi^7$$

etc.,

we will be able to determine these coefficients A, B, C, D etc. in such a way that it is

$$A = \frac{2}{\pi}\left(\frac{P\pi^2}{2 \cdot 3} + \frac{Q\pi^4}{4 \cdot 5} + \frac{R\pi^6}{6 \cdot 7} + \frac{S\pi^8}{8 \cdot 9} + \text{etc.}\right)$$

$$B = \frac{A}{1 \cdot 2 \cdot 3} - \frac{2}{\pi}\left(\frac{P\pi^2}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{Q}{4 \cdot 5 \cdot 6 \cdot 7} + \frac{R\pi^6}{6 \cdot 7 \cdot 8 \cdot 9} + \text{etc.}\right),$$

$$C = \frac{B}{1 \cdot 2 \cdot 3} - \frac{A}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{2}{\pi}\left(\frac{P\pi^2}{2 \cdot 3 \cdots 7} + \frac{Q}{4 \cdot 5 \cdots 9} + \frac{R\pi^6}{6 \cdot 7 \cdots 11} + \text{etc.}\right),$$

$$D = \frac{C}{1 \cdot 2 \cdot 3} - \frac{B}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{A}{1 \cdot 2 \cdots 7} - \frac{2}{\pi}\left(\frac{P\pi^2}{2 \cdot 3 \cdots 9} + \frac{Q}{4 \cdot 5 \cdots 11} + \text{etc.}\right)$$

etc.

§33 But before we conclude anything from this, let us show that the rule found here is indeed correct and gives the true values in an example. So let us take the first formula, and because it is $A = \frac{\log 2}{\pi}$, one will have this equation.

$$\frac{\log 2}{2} = \frac{P\pi^2}{2 \cdot 3} + \frac{Q\pi^4}{4 \cdot 5} + \frac{R\pi^6}{6 \cdot 7} + \frac{S\pi^8}{8 \cdot 9} + \text{etc.}$$

By approximating the true values it is

$$\begin{aligned}
\log 2 &= 0,693147181,\\
P\pi^2 &= 1,233700550,\\
Q\pi^4 &= 1,014678032,\\
R\pi^6 &= 1,001447077,\\
S\pi^8 &= 1,000155179,\\
T\pi^{10} &= 1,000017041,\\
V\pi^{12} &= 1,000001886,\\
W\pi^{14} &= 1,000000209,\\
X\pi^{16} &= 1,000000023,\\
Y\pi^{18} &= 1,000000003
\end{aligned}$$

etc.

At first let us find the integral part of $P\pi^2$, $Q\pi^4$ etc.; we will have

$$\frac{1}{2\cdot 3}+\frac{1}{4\cdot 5}+\frac{1}{6\cdot 7}+\frac{1}{8\cdot 9}+\text{etc.},$$

the sum of which series is known, of course it is

$$= 1-\log 2 \quad \text{or} \quad 0,306852819;$$

now let us take the remainders of the same terms, which divided by the respective denominators will give

0,038950092
0,000733902
34454
2155
155
12
1
0,039720771;

add $1 - \log 2$

$$0{,}306852819$$

$$\overline{\phantom{0{,}346573590.}}$$

$$0{,}346573590.$$

But on the other hand

$$\frac{\log 2}{2} = 0{,}346573590.$$

whence the equality is clearly seen.

§34 Therefore, because now the truth of the proposition asserted in § 32 is demonstrated, we found a rule, how the sums of the series

$$1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \text{etc.},$$

while n denotes any odd number, proceed. But since we only know by observation that it is

$$\frac{\log 2}{2} = \frac{P\pi^2}{2 \cdot 3} + \frac{Q\pi^4}{4 \cdot 5} + \frac{R\pi^6}{6 \cdot 7} + \frac{S\pi^8}{8 \cdot 9} + \text{etc.}$$

or

$$\log 2 = \left\{ \begin{array}{llllllll} + & \frac{1}{3}\Big(1 & +\frac{1}{3^2} & +\frac{1}{5^2} & +\frac{1}{7^2} & +\frac{1}{9^2} & +\text{etc.}\Big) \\ + & \frac{1}{10}\Big(1 & +\frac{1}{3^4} & +\frac{1}{5^4} & +\frac{1}{7^4} & +\frac{1}{9^4} & +\text{etc.}\Big) \\ + & \frac{1}{21}\Big(1 & +\frac{1}{3^6} & +\frac{1}{5^6} & +\frac{1}{7^6} & +\frac{1}{9^6} & +\text{etc.}\Big) \\ + & \frac{1}{36}\Big(1 & +\frac{1}{3^8} & +\frac{1}{5^8} & +\frac{1}{7^8} & +\frac{1}{9^8} & +\text{etc.}\Big) \\ + & \frac{1}{55}\Big(1 & +\frac{1}{3^{10}} & +\frac{1}{5^{10}} & +\frac{1}{7^{10}} & +\frac{1}{9^{10}} & +\text{etc.}\Big) \end{array} \right\}$$

etc.

it will be worth one's while, to find a prove of this statement. Therefore, let us put

$$s = \frac{P\pi^2}{2\cdot 3} + \frac{Q\pi^4}{4\cdot 5} + \frac{R\pi^6}{6\cdot 7} + \frac{S\pi^8}{8\cdot 9} + \text{etc.}$$

and consider the following transformations

$$\frac{d.\pi s}{d\pi} = \frac{P\pi^2}{2} + \frac{Q\pi^4}{4} + \frac{R\pi^6}{6} + \text{etc.},$$

$$\frac{dd.\pi s}{d\pi^2} = P\pi^2 + Q\pi^4 + R\pi^6 + \text{etc.}$$

Because this last series, if it is multiplied by π, has the sum $\frac{\pi}{4}\tan\frac{\pi}{2}$, which expression is true, even if π is a variable quantity, as we assumed here, it will hence be

$$dd.\pi s = \frac{d\pi^2}{4}\tan\frac{\pi}{2}$$

and therefore

$$d.\pi s = \frac{d\pi}{4}\int d\pi \tan\frac{\pi}{2}$$

and finally

$$s = \frac{1}{4\pi}\int d\pi \int d\pi \tan\frac{\pi}{2};$$

and the root of this equation is easily found; for, it is

$$s = \frac{\log 2}{2}.$$

§35 Now first let us consider this formula

$$\int d\pi \tan\frac{\pi}{2},$$

which becomes this one

$$\int \frac{d\pi \sin\frac{\pi}{2}}{\cos\frac{\pi}{2}} = -2\log\cos\frac{\pi}{2};$$

but having substituted this integral we will have

$$s = \frac{-1}{\pi} \int \frac{d\pi}{2} \log \cos \frac{\pi}{2}.$$

To integrate this formula I put

$$\tan \frac{\pi}{2} = t;$$

it will be

$$\cos \frac{\pi}{2} = \frac{1}{\sqrt{1+tt}}$$

and

$$-\log \cos \frac{\pi}{2} = \log \sqrt{1+tt} = \frac{1}{2} \log(1+tt)$$

and

$$\frac{d\pi}{2} = \frac{dt}{1+tt};$$

hence it will be

$$s = \frac{1}{2\pi} \int \frac{dt}{1+tt} \log(1+tt),$$

and therefore the question was reduced the solution of the integral $\int \frac{dt \log(1+tt)}{1+tt}$ using such a constant that the integral vanishes for $t = 0$; having done so one has to substitute $t = \tan \frac{\pi}{2}$ again and because $\frac{\pi}{2}$ = to an arc of 90°, it will be $t = \infty$. But this formula, because it is

$$\log(1+tt) = \frac{tt}{1+tt} + \frac{t^4}{2(1+tt)^4} + \frac{t^6}{3(1+tt)^3} + \frac{t^8}{4(1+tt)^4} + \text{etc.},$$

is reduced to this one

$$\int \frac{dt}{1+tt} \log(1+tt)$$
$$= \int \frac{ttdt}{(1+tt)^2} + \frac{1}{2} \int \frac{t^4dt}{(1+tt)^3} + \frac{1}{3} \int \frac{t^6dt}{(1+tt)^4} + \frac{1}{4} \int \frac{t^6dt}{(1+tt)^5} + \text{etc.}$$

But by means of the reduction of integral formulas[4] it is in general

$$\int \frac{t^{2m}}{(1+tt)^{m+1}} = \frac{-t^{2m-1}}{2m(1+tt)^m} + \frac{2m-1}{2m} \int \frac{t^{2m-2}}{(1+tt)^m}.$$

Therefore, because it is

$$\int \frac{dt}{1+tt} = \frac{\pi}{2},$$

it will be

$$\int \frac{ttdt}{(1+tt)^2} = \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1}{2} \cdot \frac{t}{1+tt},$$

$$\int \frac{t^4dt}{(1+tt)^3} = \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{\pi}{2} - \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{t}{1+tt} - \frac{1}{4} \cdot \frac{t^3}{(1+tt)^2},$$

$$\int \frac{t^6dt}{(1+tt)^4} = \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{\pi}{2} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{t}{1+tt} - \frac{1 \cdot 5}{4 \cdot 6} \cdot \frac{t^3}{(1+tt)^2} - \frac{1}{6} \cdot \frac{t^5}{(1+tt)^3},$$

$$\int \frac{t^8dt}{(1+tt)^5} = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{\pi}{2} - \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{t}{1+tt} - \frac{1 \cdot 5 \cdot 7}{4 \cdot 6 \cdot 8} \cdot \frac{t^3}{(1+tt)^2}$$

$$- \frac{1 \cdot 7}{6 \cdot 8} \cdot \frac{t^5}{(1+tt)^3} - \frac{1}{8} \cdot \frac{t^7}{(1+tt)^4},$$

$$\int \frac{t^{10}dt}{(1+tt)^6} = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} \cdot \frac{\pi}{2} - \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} \cdot \frac{t}{1+tt} - \frac{1 \cdot 5 \cdot 7 \cdot 9}{4 \cdot 6 \cdot 8 \cdot 10} \cdot \frac{t^3}{(1+tt)^2}$$

$$- \frac{1 \cdot 7 \cdot 9}{6 \cdot 8 \cdot 10} \cdot \frac{t^5}{(1+tt)^3} - \frac{1 \cdot 9}{8 \cdot 10} \cdot \frac{t^7}{(1+tt)^4} - \frac{1}{10} \cdot \frac{t^9}{(1+tt)^5}$$

etc.

Form these substitutions it will result

$$\int \frac{dt}{1+tt} \log(1+tt)$$

$$= \frac{\pi}{2}\left(\frac{1}{2 \cdot 1} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 2} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 3} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 4} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 5} + \text{etc.}\right)$$

[4]By this Euler means integration by parts.

$$-\quad\frac{t}{1+tt}\left(\frac{1}{2\cdot 1}+\frac{1\cdot 3}{2\cdot 4\cdot 2}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 3}+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 4}+\text{etc.}\right)$$

$$-\frac{t^3}{4(1+tt)^2}\left(\frac{1}{2}+\frac{5}{6\cdot 3}+\frac{5\cdot 7}{6\cdot 8\cdot 4}+\frac{5\cdot 7\cdot 9}{6\cdot 8\cdot 10\cdot 5}+\text{etc.}\right)$$

$$-\frac{t^5}{6(1+tt)^3}\left(\frac{1}{3}+\frac{7}{8\cdot 4}+\frac{7\cdot 9}{8\cdot 10\cdot 5}+\frac{7\cdot 9\cdot 11}{8\cdot 10\cdot 12\cdot 6}+\text{etc.}\right)$$

$$-\frac{t^7}{8(1+tt)^4}\left(\frac{1}{4}+\frac{9}{10\cdot 5}+\frac{9\cdot 11}{10\cdot 12\cdot 6}+\frac{9\cdot 11\cdot 13}{10\cdot 12\cdot 14\cdot 7}+\text{etc.}\right)$$

etc.

§36 First let the sum of the following sum be in question

$$\frac{1}{2\cdot 1}+\frac{1\cdot 3}{2\cdot 4\cdot 2}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 3}+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 4}+\text{etc.}$$

and let us put

$$s=\frac{x}{2\cdot 1}+\frac{1\cdot 3x^2}{2\cdot 4\cdot 2}+\frac{1\cdot 3\cdot 5x^3}{2\cdot 4\cdot 6\cdot 3}+\frac{1\cdot 3\cdot 5\cdot 7x^4}{2\cdot 4\cdot 6\cdot 8\cdot 4}+\text{etc.};$$

it will be

$$s=\int\frac{dx}{x\sqrt{1-x}}-\log x,$$

as it will be seen by actual expansion. But it is

$$\int\frac{dx}{x\sqrt{1-x}}=c-\log(1+\sqrt{1-x})+\log(1-\sqrt{1-x})$$

and hence

$$s=c-\log(1+\sqrt{1-x})+\log(1-\sqrt{1-x})-\log x,$$

where the constant c has to be defined in such a way that for $x=0$ s becomes $=0$. So let x become infinitely small; it will be

$$\sqrt{1-x}=1-\frac{x}{2}$$

and

$$\log(1-\sqrt{1-x}) = \log\frac{x}{2} = \log x - \log 2$$

and

$$\log(1+\sqrt{1-x}) = \log 2,$$

whence $c = 2\log 2$. Now put $x = 1$; it will be $s = 2\log 2$ and

$$\frac{1}{2\cdot 1} + \frac{1\cdot 3}{2\cdot 4\cdot 2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 3} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 4} + \text{etc.} = 2\log 2.$$

But using this series the sum of the remaining series are determined in such a way that it is

$$\frac{1}{2} + \frac{5}{6\cdot 3} + \frac{5\cdot 7}{6\cdot 8\cdot 4} + \frac{5\cdot 7\cdot 9}{6\cdot 8\cdot 10\cdot 5} + \text{etc.} = \frac{2\cdot 4}{1\cdot 3}\cdot 2\log 2 - \frac{2\cdot 4}{1\cdot 3\cdot 2},$$

$$\frac{1}{3} + \frac{7}{8\cdot 4} + \frac{7\cdot 9}{8\cdot 10\cdot 5} + \frac{7\cdot 9\cdot 11}{8\cdot 10\cdot 12\cdot 6} + \text{etc.} = \frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5}\cdot 2\log 2 - \frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5\cdot 2} - \frac{6}{5\cdot 2},$$

$$\frac{1}{4} + \frac{9}{10\cdot 5} + \frac{9\cdot 11}{10\cdot 12\cdot 6} + \frac{9\cdot 11\cdot 13}{10\cdot 12\cdot 14\cdot 7} + \text{etc.} = \frac{2\cdot 4\cdot 6\cdot 8}{1\cdot 3\cdot 5\cdot 7}\cdot 2\log 2 - \frac{2\cdot 4\cdot 6\cdot 8}{1\cdot 3\cdot 5\cdot 7\cdot 2} - \frac{6\cdot 8}{5\cdot 7\cdot 2} - \frac{8}{7\cdot 3}$$

etc.

Having substituted these sums it will arise

$$\int \frac{dt}{1+tt}\log(1+tt)$$

$$= \frac{\pi}{2}\cdot 2\log 2 - \frac{t}{1+tt}\cdot 2\log 2$$

$$- \frac{t^3}{(1+tt)^2}\left(\frac{2}{3}\cdot 2\log 2 - \frac{1}{3\cdot 1}\right)$$

$$- \frac{t^5}{(1+tt)^3}\left(\frac{2\cdot 4}{3\cdot 5}\cdot 2\log 2 - \frac{4}{3\cdot 5\cdot 1} - \frac{1}{5\cdot 2}\right)$$

$$-\frac{t^7}{(1+tt)^4}\left(\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\cdot 2\log 2-\frac{4\cdot 6\cdot 8}{3\cdot 5\cdot 7\cdot 1}-\frac{6}{5\cdot 7\cdot 2}\right)$$

$$-\frac{t^9}{(1+tt)^5}\left(\frac{2\cdot 4\cdot 6\cdot 8}{3\cdot 5\cdot 7\cdot 9}\cdot 2\log 2-\frac{4\cdot 6\cdot 8}{3\cdot 5\cdot 7\cdot 9\cdot 1}-\frac{6\cdot 8}{5\cdot 7\cdot 9\cdot 2}-\frac{1}{9\cdot 4}\right)$$

etc.

§37 But because for our purpose after the integration one has to put $t=\infty$, it will be

$$\int \frac{dt}{1+tt}\log(1+tt)=\pi\log 2$$

and

$$s=\frac{1}{2}\pi\int\frac{dt}{1+tt}\log(1+tt)=\frac{\log 2}{2},$$

which is that value itself we saw to have to result (§ 34). For, the remaining terms in the expression we found for

$$\int\frac{dt}{1+tt}\log(1+tt),$$

if one puts $t=\infty$, all vanish, because in the single denominators of the single terms t has more dimensions than in the numerators and in addition to that the coefficients decrease. For, if this would not happen, we could conclude that the sum of all terms, which all vanish, is $=0$. For, if for example one takes only the first parts of the numerical coefficients, that this series results

$$\frac{t}{1+tt}+\frac{2t^3}{3(1+tt)^2}+\frac{2\cdot 4t^5}{3\cdot 5(1+tt)^5}+\frac{2\cdot 4\cdot 6t^7}{3\cdot 5\cdot 7(1+tt)^4}+\text{etc.},$$

then its sum becomes finite in the case $t=\infty$ and $=\frac{\pi}{2}$, even though the single terms vanish; but in the case of integer coefficients and the hence resulting rapidly converging series the whole series would also become $=0$.

§38 Now let us find the sum of this series

$$1-\frac{1}{2^3}+\frac{1}{3^3}-\frac{1}{4^3}+\text{etc.}=B\pi^3,$$

which sum by § 32 will be

$$B\pi^3 = \frac{\pi^2 \log 2}{1 \cdot 2 \cdot 3} - 2\pi^2 \left(\frac{P\pi^2}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{Q\pi^4}{4 \cdot 5 \cdot 6 \cdot 7} + \frac{R\pi^6}{6 \cdot 7 \cdot 8 \cdot 9} + \text{etc.} \right).$$

To find the value of this quantity let

$$s = \frac{P\pi^2}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{Q\pi^4}{4 \cdot 5 \cdot 6 \cdot 7} + \frac{R\pi^6}{6 \cdot 7 \cdot 8 \cdot 9} + \text{etc.};$$

it will be

$$\begin{aligned}
\frac{d.\pi^3 s}{d\pi} &= \frac{P\pi^4}{2 \cdot 3 \cdot 4} + \frac{Q\pi^6}{4 \cdot 5 \cdot 6} + \frac{R\pi^8}{6 \cdot 7 \cdot 8} + \text{etc.}, \\
\frac{dd.\pi^3 s}{d\pi^2} &= \frac{P\pi^3}{2 \cdot 3} + \frac{Q\pi^5}{4 \cdot 5} + \frac{R\pi^7}{6 \cdot 7} + \text{etc.}, \\
\frac{d^3.\pi^3 s}{d\pi^3} &= \frac{P\pi^2}{2} + \frac{Q\pi^4}{4} + \frac{R\pi^6}{6} + \text{etc.}, \\
\frac{d^4.\pi^3 s}{d\pi^4} &= P\pi + Q\pi^3 + R\pi^5 + \text{etc.} = \frac{1}{4} \tan \frac{\pi}{2}.
\end{aligned}$$

By going backwards it will therefore be

$$\begin{aligned}
\frac{d^3.\pi^3 s}{d\pi^3} &= \frac{1}{4} \int d\pi \tan \frac{\pi}{2}, \\
\frac{dd.\pi^3 s}{d\pi^2} &= \frac{1}{4} \int d\pi \int d\pi \tan \frac{\pi}{2}, \\
\frac{d.\pi^3 s}{d\pi} &= \frac{1}{4} \int d\pi \int d\pi \int d\pi \tan \frac{\pi}{2}, \\
\pi^3 s &= \frac{1}{4} \int d\pi \int d\pi \int d\pi \int d\pi \tan \frac{\pi}{2}.
\end{aligned}$$

And hence one will find the sum of the propounded series to be

$$B\pi^3 = \frac{\pi^2 \log 2}{6} - \frac{1}{2\pi} \int d\pi \int d\pi \int d\pi \int d\pi \tan \frac{\pi}{2},$$

and all integrals have to be taken in such a way that they vanish for $\pi = 0$.

§39 Put $\frac{\pi}{2} = q$ such that after the integrations q denotes the fourth part of the circumference of the circle whose diameter $= 1$, or the arc of 90 degrees. And further let it be

$$\sin q = y \quad \text{and} \quad \cos q = x = \sqrt{1-yy};$$

it will be

$$\tan\frac{\pi}{2} = \frac{y}{x}.$$

Hence because of $\pi = 2q$ the sum of our series will be

$$B\pi^3 = \frac{2qq\log 2}{3} - \frac{4}{q}\int dq\int dq\int dq\int\frac{ydq}{x}.$$

So let us put

$$\int dq\int dq\int dq\int\frac{ydq}{x} = u;$$

it will be

$$B\pi^3 = \frac{2qq\log 2}{3} - \frac{4u}{q},$$

where, in order to find the quantity u, all integrals have to be taken in such a way that the single integrals vanish for $q = 0$ and $y = 0$; but having calculated the integrals it will be $y = 1$ and $x = 0$. But it is

$$\int\frac{ydq}{x} = \int\frac{ydy}{1-yy} = -\log\sqrt{1-yy} = \log\frac{1}{x}$$

and

$$\log\frac{1}{x} = \frac{yy}{2} + \frac{y^4}{4} + \frac{y^6}{6} + \frac{y^8}{8} + \frac{y^{10}}{10} + \text{etc.}$$

Because now it is

$$u = \int dq\int dq\int dq\log\frac{1}{x},$$

by a reduction of the integrals it will be

$$u = q\int dq\int dq\log\frac{1}{x} - \int qdq\int dq\log\frac{1}{x}$$

and further

$$\int dq \int dq \log\frac{1}{x} = q\int dq \log\frac{1}{x} - \int q dq \log\frac{1}{x},$$
$$\int q dq \int dq \log\frac{1}{x} = \frac{qq}{2}\int dq \log\frac{1}{x} - \frac{1}{2}\int qq dq \log\frac{1}{x},$$

hence

$$u = \frac{1}{2}qq\int dq \log\frac{1}{x} - q\int q dq \log\frac{1}{x} + \frac{1}{2}\int qq dq \log\frac{1}{x},$$

that we now have three simple integral formulas we have to integrate.

§40 So let us consider these three single formulas separately and start with this one $\int dq \log\frac{1}{x}$; even if we already integrated it above [§ 35], let us nevertheless integrate it again without considering sines and cosines; this will simplify the integration of the remaining ones. Therefore, it is

$$\int dq \log\frac{1}{x} = \int dq\left(\frac{yy}{2} + \frac{y^4}{4} + \frac{y^6}{6} + \frac{y^8}{8} + \frac{y^{10}}{10} + \text{etc.}\right)$$

To find this integral just consider the general term - $\int y^{n+2}dq$ - and because it is

$$dq = \frac{dy}{x} = \frac{-dx}{y} \quad \text{and} \quad xx + yy = 1,$$

it will be

$$\int y^{n+2}dq = -\int y^{n+1}dx = -y^{n+1}x + (n+1)\int y^n x dy;$$

but it is

$$\int y^n x dy = \int y^n x^2 dq = \int y^n dq - \int y^{n+2}dq$$

- because of $xx = 1 - yy$; therefore, it is

$$\int y^{n+2}dq = -y^{n+1}x + (n+1)\int y^n dq - (n+1)\int y^{n+2}dq$$

and

$$\int y^{n+2}dq = \frac{-y^{n+2}}{n+2} + \frac{n+1}{n+2}\int y^n dq.$$

Hence the integral of one term is reduced to the integral of the preceding, and because having done the integration x becomes $= 0$, it will be for this case

$$\int y^{n+2}dq = \frac{n+1}{n+2}\int y^n dq.$$

Therefore, one will find all parts of the integral from this formula as follows

$$\int y^2 dq = \frac{1}{2}q,$$

$$\int y^4 dq = \frac{1\cdot 3}{2\cdot 4}q,$$

$$\int y^6 dq = \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}q,$$

$$\int y^8 dq = \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}q$$

etc.

Therefore, one will have

$$\int dq \log\frac{1}{x} = \frac{1}{q}\left(\frac{1}{2\cdot 1} + \frac{1\cdot 3}{2\cdot 4\cdot 2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 3} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 4} + \text{etc.}\right);$$

because the sum of this series was already found above (§ 36) to be $= 2\log 2$, it will be

$$\int dq \log\frac{1}{x} = q\log 2.$$

§41 Now let us proceed to the second integral formula $\int qdq \log\frac{1}{x}$, which becomes

$$\int qdq \log\frac{1}{x} = \int qdq\left(\frac{yy}{2} + \frac{y^4}{4} + \frac{y^6}{6} + \frac{y^8}{8} + \text{etc.}\right);$$

and let us again consider the general term

$$\int y^{n+2} q dq = -\int y^{n+1} q dx = -y^{n+1} qx + \int y^{n+1} x dq + (n+1) \int y^n qx dy$$

$$= -y^{n+1} qx + \frac{y^{n+2}}{n+2} + (n+1) \int y^n q dq - (n+1) \int y^{n+2} q dq.$$

Therefore, having put $y = 1$ and $x = 0$ it will be

$$\int y^{n+2} q dq = \frac{1}{(n+2)^2} + \frac{n+1}{n+2} \int y^n q dq.$$

The integrals of the single terms hence will be:

$$\int y^2 q dq = \frac{1}{2^2} + \frac{1}{2} \cdot \frac{q^2}{2},$$

$$\int y^4 q dq = \frac{1}{4^2} + \frac{3}{4 \cdot 2^2} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{q^2}{2},$$

$$\int y^6 q dq = \frac{1}{6^2} + \frac{5}{6 \cdot 4^2} + \frac{3 \cdot 5}{4 \cdot 6 \cdot 2^2} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{q^2}{2},$$

$$\int y^8 q dq = \frac{1}{8^2} + \frac{7}{8 \cdot 6^2} + \frac{5 \cdot 7}{6 \cdot 8 \cdot 4^2} + \frac{3 \cdot 5 \cdot 7}{4 \cdot 6 \cdot 8 \cdot 2^2} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{q^2}{2}$$

etc.

Hence one will obtain the integral

$$\int q dq \log \frac{1}{x} = + \frac{qq}{4} \left(\frac{1}{2 \cdot 1} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 2} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 3} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 4} + \text{etc.} \right)$$

$$+ \frac{1}{2 \cdot 2^2} \left(\frac{1}{1} + \frac{3}{4 \cdot 2} + \frac{3 \cdot 5}{4 \cdot 6 \cdot 3} + \frac{3 \cdot 5 \cdot 7}{4 \cdot 6 \cdot 8 \cdot 4} + \text{etc.} \right)$$

$$+ \frac{1}{2 \cdot 4^2} \left(\frac{1}{2} + \frac{5}{6 \cdot 3} + \frac{5 \cdot 7}{6 \cdot 8 \cdot 4} + \frac{5 \cdot 7 \cdot 9}{6 \cdot 8 \cdot 10 \cdot 5} + \text{etc.} \right)$$

$$+ \frac{1}{2 \cdot 6^2} \left(\frac{1}{3} + \frac{7}{8 \cdot 4} + \frac{7 \cdot 9}{8 \cdot 10 \cdot 5} + \frac{7 \cdot 9 \cdot 11}{8 \cdot 10 \cdot 12 \cdot 6} + \text{etc.} \right)$$

etc.

or also in this form

$$\int qdq \log\frac{1}{x} = \frac{qq}{4}\Big(\frac{1}{2\cdot 1} + \frac{1\cdot 3}{2\cdot 4\cdot 2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 3} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 4} + \text{etc.}\Big)$$

$$+ \frac{1}{2^2\cdot 2} + \frac{1}{4^2\cdot 4} + \frac{1}{6^2\cdot 6} + \frac{1}{8^2\cdot 8} + \text{etc.}$$

$$+ \frac{3}{2^2\cdot 4^2} + \frac{5}{4^2\cdot 6^2} + \frac{7}{6^2\cdot 8^2} + \frac{9}{8^2\cdot 10^2} + \text{etc.}$$

$$+ \frac{3\cdot 5}{2^2\cdot 4\cdot 6^2} + \frac{5\cdot 7}{4^2\cdot 6\cdot 8^2} + \frac{7\cdot 9}{6^2\cdot 8\cdot 10^2} + \frac{9\cdot 11}{8^2\cdot 10\cdot 12^2} + \text{etc.}$$

$$+ \frac{3\cdot 5\cdot 7}{2^2\cdot 4\cdot 6\cdot 8^2} + \frac{5\cdot 7\cdot 9}{4^2\cdot 6\cdot 8\cdot 10^2} + \frac{7\cdot 9\cdot 11}{6^2\cdot 8\cdot 10\cdot 12^2} + \frac{9\cdot 11\cdot 13}{8^2\cdot 10\cdot 12\cdot 14^2} + \text{etc.}$$

etc.

but these series actually contain the object in question, namely the summation of the cubes of the terms of the harmonic series.

§42 If we proceed as in the first form, all series become summable (§ 36) and one will have

$$\int qdq \log\frac{1}{x} = \frac{qq}{2}\log 2 + \frac{1}{2^2}\left(\frac{2}{1}\log 2\right)$$

$$+ \frac{1}{4^2}\left(\frac{2\cdot 4}{1\cdot 3}\log 2 - \frac{4}{3\cdot 2}\right)$$

$$+ \frac{1}{6^2}\left(\frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5}\log 2 - \frac{4\cdot 6}{3\cdot 5\cdot 2} - \frac{6}{5\cdot 4}\right)$$

$$+ \frac{1}{8^2}\left(\frac{2\cdot 4\cdot 6\cdot 8}{1\cdot 3\cdot 5\cdot 7}\log 2 - \frac{4\cdot 6\cdot 8}{3\cdot 5\cdot 7\cdot 2} - \frac{6\cdot 8}{5\cdot 7\cdot 4} - \frac{8}{7\cdot 6}\right)$$

etc.,

which, if the series are again summed column by column, give

$$\int qdq\log\frac{1}{x}=\frac{qq}{2}\log 2+\log 2\left(\frac{1}{2}+\frac{2}{3\cdot 4}+\frac{2\cdot 4}{3\cdot 5\cdot 6}+\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7\cdot 9}+\text{etc.}\right)$$
$$-\frac{1}{2\cdot 3}\left(\frac{1}{4}+\frac{4}{5\cdot 6}+\frac{4\cdot 6}{5\cdot 7\cdot 8}+\frac{4\cdot 6\cdot 8}{5\cdot 7\cdot 9\cdot 11}+\text{etc.}\right)$$
$$-\frac{1}{4\cdot 5}\left(\frac{1}{6}+\frac{6}{7\cdot 8}+\frac{6\cdot 8}{7\cdot 9\cdot 10}+\frac{6\cdot 8\cdot 10}{7\cdot 9\cdot 11\cdot 12}+\text{etc.}\right)$$
$$-\frac{1}{6\cdot 7}\left(\frac{1}{8}+\frac{8}{9\cdot 10}+\frac{8\cdot 10}{9\cdot 11\cdot 12}+\frac{8\cdot 10\cdot 12}{9\cdot 11\cdot 13\cdot 14}+\text{etc.}\right)$$
$$\text{etc.}$$

But it is

$$\frac{1}{2}+\frac{2}{3\cdot 4}+\frac{2\cdot 4}{3\cdot 5\cdot 6}+\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7\cdot 8}+\text{etc.}=\frac{qq}{2},$$

whence it will be

$$\frac{1}{4}+\frac{4}{5\cdot 6}+\frac{4\cdot 6}{5\cdot 7\cdot 8}+\text{etc.}=\frac{3}{2}\cdot\frac{qq}{2}-\frac{3}{2}\cdot\frac{1}{2},$$
$$\frac{1}{6}+\frac{6}{7\cdot 8}+\frac{6\cdot 8}{7\cdot 9\cdot 10}+\text{etc.}=\frac{3\cdot 5}{2\cdot 4}\cdot\frac{qq}{2}-\frac{3\cdot 5}{2\cdot 4}\cdot\frac{1}{2}-\frac{5}{4}\cdot\frac{1}{4},$$
$$\frac{1}{8}+\frac{8}{9\cdot 10}+\frac{8\cdot 10}{9\cdot 11\cdot 12}+\text{etc.}=\frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}\cdot\frac{qq}{2}-\frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}\cdot\frac{1}{2}-\frac{5\cdot 7}{4\cdot 6}\cdot\frac{1}{4}-\frac{7}{6}\cdot\frac{1}{6}$$
$$\text{etc.}$$

Therefore, one will have

$$\int qdq\log\frac{1}{x} = qq\log 2 - \frac{qq}{2}\left(\frac{1}{2^2} + \frac{3}{2\cdot 4^2} + \frac{3\cdot 5}{2\cdot 4\cdot 6^2} + \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8^2} + \text{etc.}\right)$$

$$+\frac{1}{2^2}\cdot\frac{1}{2}$$

$$+\frac{3}{2\cdot 4^2}\cdot\frac{1}{2} + \frac{1}{4^2}\cdot\frac{1}{4}$$

$$+\frac{3\cdot 5}{2\cdot 4\cdot 6^2}\cdot\frac{1}{2} + \frac{5}{4\cdot 6^2} + \frac{1}{6^2}\cdot\frac{1}{6}$$

$$+\frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8^2}\cdot\frac{1}{2} + \frac{5\cdot 7}{4\cdot 6\cdot 8^2}\cdot\frac{1}{4} + \frac{7}{6\cdot 8^2}\cdot\frac{1}{6} + \frac{1}{8^2}\cdot\frac{1}{8}$$

etc.

§43 But maybe the difficulty to find a closed expression will be reduced, if we combine those three integral formulas. Therefore, let us take the third formula

$$\int qqdq\log\frac{1}{x},$$

which goes over into this one

$$\int qqdq\left(\frac{yy}{2} + \frac{y^4}{4} + \frac{y^6}{6} + \frac{y^8}{8} + \text{etc.}\right).$$

Consider the formula

$$\int y^{n+2}qqdq,$$

which becomes

$$-\int y^{n+1}qqdx = -y^{n+1}qqx + 2\int y^{n+1}qxdq + (n+1)\int y^n qqxdy$$
$$= -y^{n+1}qqx + 2\int y^{n+1}qdy + (n+1)\int y^n qqdq - (n+1)\int y^{n+2}qqdq;$$

hence it will be

$$\int y^{n+2}qqdq = \frac{-y^{n+1}qqx}{n+2} + \frac{2}{n+2}\int y^{n+1}qdy + \frac{n+1}{n+2}\int y^n qqdq.$$

But it is

$$\int y^{n+1}qdy = \frac{y^{n+2}q}{n+2} - \frac{1}{n+2}\int y^{n+2}dq = \frac{q}{n+2} - \frac{1}{n+2}\cdot\frac{1\cdot3\cdot5\cdot7\cdots(n+1)}{2\cdot4\cdot6\cdot8\cdots(n+2)}q,$$

having put $y=1$ (§ 40). As a logical consequence it will be

$$\int y^{n+2}qqdq = \frac{2q}{(n+2)}\left(1-\frac{1\cdot3\cdot5\cdot7\cdots(n+1)}{2\cdot4\cdot6\cdot8\cdots(n+2)}\right) + \frac{n+1}{n+2}\int y^n qqdq$$

and the single terms of the integral in question will be

$$\int y^2 qqdq = \frac{1}{2^2}2q - \frac{1}{2^2\cdot 2}2q + \frac{1}{2}\cdot\frac{q^3}{3},$$

$$\int y^4 qqdq = \frac{1}{4^4}2q - \frac{1}{4^2}\cdot\frac{1\cdot3}{2\cdot4}2q + \frac{1}{2^2}\cdot\frac{3}{4}2q - \frac{1}{2^2}\cdot\frac{1\cdot3}{2\cdot4}2q + \frac{1\cdot3}{2\cdot4}\cdot\frac{q^3}{3},$$

$$\int y^6 qqdq = \frac{2q}{6}\left(1-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right) + \frac{5}{6}\cdot\frac{2q}{4^2}\left(1-\frac{1\cdot3}{2\cdot4}\right) + \frac{3\cdot5}{4\cdot6}\cdot\frac{2q}{2^2}\left(1-\frac{1}{2}\right) + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{q^3}{3}$$

etc.

Finally, having substituted and arranged the terms one will finally find

$$\begin{aligned}\int qqdq\log\frac{1}{x} = \quad & \frac{q^3}{6}\left(\frac{1}{2\cdot1} + \frac{1\cdot3}{2\cdot4\cdot2} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot3} + \text{etc.}\right)\\
+\frac{1}{2^2}q\left(1-\frac{1}{2}\right) & \left(1+\frac{3}{4\cdot2} + \frac{3\cdot5}{4\cdot6\cdot3} + \frac{3\cdot5\cdot7}{4\cdot6\cdot8\cdot4} + \text{etc.}\right)\\
+\frac{1}{4^2}q\left(1-\frac{1\cdot3}{2\cdot4}\right) & \left(\frac{1}{2}+\frac{5}{6\cdot3} + \frac{5\cdot7}{6\cdot8\cdot4} + \frac{5\cdot7\cdot9}{6\cdot8\cdot10\cdot5} + \text{etc.}\right)\\
+\frac{1}{6^2}q\left(1-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right) & \left(\frac{1}{3}+\frac{7}{8\cdot4} + \frac{7\cdot9}{8\cdot10\cdot5} + \frac{7\cdot9\cdot11}{8\cdot10\cdot12\cdot6} + \text{etc.}\right)\end{aligned}$$

etc.

But because it is

$$= \frac{1}{2}qq\int dq\log\frac{1}{x} - q\int qdq\log\frac{1}{x} + \frac{1}{2}\int qqdq\log\frac{1}{x},$$

having added the integrals, as they were found, it will be

$$\begin{aligned}
u = & \quad \frac{q^3}{12}\left(\frac{1}{2\cdot 1} + \frac{1\cdot 3}{2\cdot 4\cdot 2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 3} + \text{etc.}\right)\\
& - \frac{q}{2\cdot 2^2}\cdot\frac{1}{2}\left(\frac{1}{1} + \frac{3}{4\cdot 2} + \frac{3\cdot 5}{4\cdot 6\cdot 3} + \frac{3\cdot 5\cdot 7}{4\cdot 6\cdot 8\cdot 4} + \text{etc.}\right)\\
& - \frac{q}{2\cdot 4^2}\cdot\frac{1\cdot 3}{2\cdot 4}\left(\frac{1}{2} + \frac{5}{6\cdot 3} + \frac{5\cdot 7}{6\cdot 8\cdot 4} + \frac{5\cdot 7\cdot 9}{6\cdot 8\cdot 10\cdot 5} + \text{etc.}\right)\\
& - \frac{q}{2\cdot 6^2}\cdot\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{1}{3} + \frac{7}{8\cdot 4} + \frac{7\cdot 9}{8\cdot 10\cdot 5} + \frac{7\cdot 9\cdot 11}{8\cdot 10\cdot 12\cdot 6} + \text{etc.}\right)\\
& \qquad \text{etc.}
\end{aligned}$$

But having integrated the series as above [§ 36] it will be

$$\begin{aligned}
u = \frac{q^3}{6}\log 2 & - \frac{q}{2\cdot 2^2} \quad 2\log 2,\\
& - \frac{q}{2\cdot 2^4}\left(2\log 2 - \frac{1}{2\cdot 1}\right),\\
& - \frac{q}{2\cdot 2^6}\left(2\log 2 - \frac{1}{2\cdot 1} - \frac{1\cdot 3}{2\cdot 4\cdot 2}\right),\\
& - \frac{q}{2\cdot 2^8}\left(2\log 2 - \frac{1}{2\cdot 1} - \frac{1\cdot 3}{2\cdot 4\cdot 2} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 3}\right),\\
& - \frac{q}{2\cdot 2^{10}}\left(2\log 2 - \frac{1}{2\cdot 1} - \frac{1\cdot 3}{2\cdot 4\cdot 2} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 3} - \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 4}\right)\\
& \qquad \text{etc.}
\end{aligned}$$

But it is

$$\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \text{etc.} = \frac{qq}{6},$$

whence

$$\begin{aligned}
u = {} & \frac{q}{2\cdot 2} && \left(\frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \text{etc.}\right)\\
& + \frac{1\cdot 3q}{2\cdot 4\cdot 4} && \left(\frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \text{etc.}\right)\\
& + \frac{1\cdot 3\cdot 5q}{2\cdot 4\cdot 6\cdot 6} && \left(\frac{1}{8^2} + \frac{1}{10^2} + \frac{1}{12^2} + \text{etc.}\right)\\
& + \frac{1\cdot 3\cdot 5\cdot 7q}{2\cdot 4\cdot 6\cdot 8\cdot 8} && \left(\frac{1}{10^2} + \frac{1}{12^2} + \frac{1}{14^2} + \text{etc.}\right)\\
& && \text{etc.}
\end{aligned}$$

Therefore, it will be

$$\begin{aligned}
u = {} & \frac{q}{2\cdot 2} && \left(\frac{qq}{6} - \frac{1}{2^2}\right)\\
& + \frac{1\cdot 3q}{2\cdot 4\cdot 4} && \left(\frac{qq}{6} - \frac{1}{2^2} - \frac{1}{4^2}\right)\\
& + \frac{1\cdot 3\cdot 5q}{2\cdot 4\cdot 6\cdot 6} && \left(\frac{qq}{6} - \frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{6^2}\right)\\
& && \text{etc.}
\end{aligned}$$

or having actually summed the first vertical series

$$\begin{aligned}
u = \frac{q^3}{6}\log 2 & - \frac{q}{2\cdot 2}\cdot\frac{1}{2^2}\\
& - \frac{1\cdot 3q}{2\cdot 4\cdot 4} \left(\frac{1}{2^2} + \frac{1}{4^2}\right)\\
& - \frac{1\cdot 3\cdot 5q}{2\cdot 4\cdot 6\cdot 6} \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2}\right)\\
& \qquad \text{etc.}
\end{aligned}$$

§44 Because now the sum of our propounded series

$$1-\frac{1}{2^3}+\frac{1}{3^3}-\frac{1}{4^3}+\frac{1}{5^3}-\text{etc.}$$

is

$$=B\pi^3=\frac{2qq\log 2}{3}-\frac{4u}{q},$$

the same sum will become

$$\begin{aligned}
=&+\frac{1}{2\cdot 2}\cdot 1\\
&+\frac{1\cdot 3}{2\cdot 4\cdot 4}\left(1+\frac{1}{2^2}\right)\\
&+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6}\left(1+\frac{1}{2^2}+\frac{1}{3^2}\right)\\
&+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 8}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}\right)\\
&\text{etc.}
\end{aligned}$$

Or, because it is

$$\frac{1}{2\cdot 2}+\frac{1\cdot 3}{2\cdot 4\cdot 4}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6}+\text{etc.}=\log 2,$$

the sum of the propounded series will be

$$\begin{aligned}
B\pi^3=\log 2&+\frac{1}{2^2}\left(\log 2-\frac{1}{2\cdot 2}\right)+\frac{1}{3^2}\left(\log 2-\frac{1}{2\cdot 2}-\frac{1\cdot 3}{2\cdot 4\cdot 4}\right)\\
&+\frac{1}{4^2}\left(\log 2-\frac{1}{2\cdot 2}-\frac{1\cdot 3}{2\cdot 4\cdot 4}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6}\right)+\text{etc.},
\end{aligned}$$

or this same sum can be expressed as follows

$$B\pi^3 = \frac{\pi^2}{6}\log 2 - \frac{1}{2^2}\cdot\frac{1}{2\cdot 2}$$
$$-\frac{1}{3^2}\left(\frac{1}{2\cdot 2}+\frac{1\cdot 3}{2\cdot 4\cdot 4}\right)$$
$$-\frac{1}{4^2}\left(\frac{1}{2\cdot 2}+\frac{1\cdot 3}{2\cdot 4\cdot 4}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6}\right)$$
$$-\frac{1}{5^2}\left(\frac{1}{2\cdot 2}+\frac{1\cdot 3}{2\cdot 4\cdot 4}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6}+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 8}\right)$$
$$\text{etc.}$$

But because, no matter how we transform this series, we are not able to reduce it to a simple series, whose sum is known, we stop our attempts here, contented by these many expressions equivalent to the propounded series

$$1 - \frac{1}{2^3} - \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} - \text{etc.}$$