Consideration of certain series enjoying singular properties*

Leonhard Euler

§1 Many times the consideration of series we discover accidentally provides us with remarkable artifices, which afterwards can be used in the whole doctrine of series and lead to outstanding results. Therefore, since the doctrine of series is of greatest importance in Analysis, speculations of this kind are to be considered as completely worth one's while that they are developed with all eagerness. For this purpose, I decided to consider the following series, which both because of the singular properties it is detected to have and on the other hand because of the extraordinary applications it can have for us, seems to be worth one's complete attention. But the series I want to study is this

$$\frac{1-x}{1-a}+\frac{(1-x)(a-x)}{a-a^3}+\frac{(1-x)(a-x)(a^2-x)}{a^3-a^6}+\frac{(1-x)(a-x)(a^2-x)(a^3-x)}{a^6-a^{10}}+\text{etc.}$$

The structure of the numerators is manifest from the inspection alone; for; they are formed by multiplication of the terms of this series

$$1-x,\quad a-x,\quad a^2-x,\quad a^3-x,\quad a^4-x,\quad a^5-x,\quad a^6-x\quad \text{etc.}$$

The denominators on the other hand all consist of two terms, which are powers of a, whose exponents are the triangular numbers. Hence the term of order n of the propounded series will be

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$$\frac{(1-x)(a-x)(a^2-x)(a^3-x)\cdot(a^{n-1}-x)}{a^{n(n-1):2}-a^{n(n+1):2}}.$$

§2 At first it is certainly plain, if the quantity x is taken equal to a certain power of a, that then the series will terminate at a certain point, so that all following terms go over into zero. Therefore, let us in general put s for the sum of the propounded series that it is

$$s=\frac{1-x}{1-a}+\frac{(1-x)(a-x)}{a-a^3}+\frac{(1-x)(a-x)(a^2-x)}{a^3-a^6}+\frac{(1-x)(a-x)(a^2-x)(a^3-x)}{a^6-a^{10}}+\text{etc.},$$

and first set $x=1$ or $x=a^0$ and because of all the vanishing terms it will be $s=0$. Further, let $x=a$ that only the first term remains, and it will be $s=1$. Let $x=a^2$ and it will be

$$s=\frac{1-a^2}{1-a}+\frac{(1-a^2)(a-a^2)}{a-a^3}$$

or $s=2$. Put $x=a^3$ and this equation will result

$$s=\frac{1-a^3}{1-a}+\frac{(1-a^3)(a-a^3)}{a-a^3}+\frac{(1-a^3)(a-a^3)(a^2-a^3)}{a^3-a^6}.$$

The first of these terms gives $1+a+aa$, the second gives $1-a^3$ and the third $1-a-aa+a^3$; hence having collected all of them it will be $s=3$.

§3 In like manner, if one puts $x=a^4$, after the calculation one will find $s=4$ and having put $x=a^5$ $s=5$ will result. Hence it seems that it can be concluded by induction that, if x is set equal to a certain power of a, whose exponent is $=n$, that this exponent n itself will yield the value of s. For, if this would be true for each fractional number, then s would be equal to the logarithm of x having taken a for the number, whose logarithm is $=1$. So, if this was true, having put $a=10$ the sum of the series s would always have to express the common logarithm of x and it would be

$$\begin{aligned}s=-\frac{(1-x)}{9}-\frac{(1-x)(10-x)}{990}-\frac{(1-x)(10-x)(100-x)}{999000}\\-\frac{(1-x)(10-x)(100-x)(1000-x)}{9999000000}-\text{etc.}=\log x.\end{aligned}$$

But from the following it will become perspicuous that this equation can only hold, if x is a power of a having a positive integer exponent.

§4 But that, having put $x = a^n$, it can only be $s = n$, if n is a positive integer number, is easily concluded from the case $x = 0$. For, in this case, if the induction mentioned above would extend to completely all powers, it would have to be $s = -\infty$, since $-\infty$ always is the logarithm of zero. But having put $x = 0$ it will be

$$s = \frac{1}{1-a} + \frac{1}{1-aa} + \frac{1}{1-a^3} + \frac{1}{1-a^4} + \frac{1}{1-a^5} + \text{etc.};$$

even though this series cannot be summed, it will nevertheless clear immediately that its sum must be finite and therefore cannot express the logarithm of $x = 0$. In like manner, if having put $a = 10$ x is not put equal to a power of 10, the value will be found to differ rather notably from $\log x$ by actual summation of some terms. For, let $x = 9$ having put $a = 10$ and it will be

$$s = \frac{8}{9} + \frac{8 \cdot 1}{990} + \frac{8 \cdot 1 \cdot 91}{999000} + \frac{8 \cdot 1 \cdot 91 \cdot 991}{9999000000} + \frac{8 \cdot 1 \cdot 91 \cdot 991 \cdot 9991}{999990000000000} + \text{etc.};$$

if the terms are expressed in decimal numbers, it will be

$$\begin{aligned} s = {} & 0.888888888889 \\ & 0.008080808081 \\ & 0.000728728729 \\ & 0.000072152015 \\ & 0.000007208059 \\ & 0.000000720735 \\ & 0.000000072073 \\ & 0.000000007207 \\ & 0.000000000721 \\ & 0.000000000007 \\ \hline s = {} & 0.897778586588 \end{aligned}$$

which value certainly is greater than the logarithm of nine.

§5 Therefore, our series is of such a nature that, if integer powers of a are substituted for x, that the sum of the series becomes equal to the exponent of that power; of course, if it is

$$x = a^0, \quad a^1, \quad a^2, \quad a^3, \quad a^4, \quad a^5, \quad a^6, \quad a^7, \quad a^8 \quad \text{etc.},$$

it will be

$$s = \; 0, \quad 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad 8 \quad \text{etc.};$$

even though this is the property of logarithms, it does nevertheless only hold for this series, if the exponents of a are integer numbers. Therefore, if one considers a curved line, whose abscissas are $= s$ and whose ordinates are $= x$, the curve will intersect the logarithmic curve in innumerable points; of course, as often as the abscissa s is expressed by means of an integer number, so often the ordinate will pass through a point of intersection. Hence it is plain that the logarithmic curve is not even determined by infinitely many points, what is also true for every other curved line. Therefore, hence it is understood that any arbitrary series, even though its terms corresponding to integer coefficients are given, can be interpolated in infinitely many different ways, which subject I will treat comprehensively on another occasion[1].

§6 But to get closer to a cognition of our series, it is possible to transform it into this form

$$s = \frac{1}{1-a}(1-x) + \frac{1}{1-a^2}(1-x)\left(1-\frac{x}{a}\right) + \frac{1}{1-a^3}(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right)$$
$$+\frac{1}{1-a^4}(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right)\left(1-\frac{x}{a^3}\right) + \text{etc.},$$

which is simpler than the preceding one, since here the triangular numbers do not occur. Now let us put ax instead of x and let t denote the sum of the series resulting from this; it will be

$$t = \frac{1}{1-a}(1-ax) + \frac{1}{1-a^2}(1-ax)(1-x) + \frac{1}{1-a^3}(1-ax)(1-x)\left(1-\frac{x}{a}\right)$$
$$+\frac{1}{1-a^4}(1-ax)(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right) + \text{etc.}$$

subtract the first series from the second and one will find

[1] Euler does so in his paper "Consideratio quarumdam serierum, quae singularibus proprietatibus sunt praeditae". This is E189 in the Eneström-Index.

$$t-s=x+\frac{x}{a}(1-x)+\frac{x}{aa}(1-x)\left(1-\frac{x}{a}\right)+\frac{x}{a^3}(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right)+\text{etc.};$$

subtract this series from 1, and since the residue is divisible by $1-x$, it will be

$$1+s-t=(1-x)\left(1-\frac{x}{a}-\frac{x}{aa}\left(1-\frac{x}{a}\right)-\frac{x}{a^3}\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right)-\text{etc.}\right).$$

This last factor is further divisible by $1-\frac{x}{a}$, whence it is

$$1+s-t=(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}-\frac{x}{a^3}\left(1-\frac{x}{aa}\right)-\text{etc.}\right).$$

Here, again the factor $1-\frac{x}{aa}$ is detected and having expressed this separately the factor $1-\frac{x}{a^3}$ will appear and so forth, whence it is finally found that it will be

$$1+s-t=(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right)\left(1-\frac{x}{a^3}\right)\left(1-\frac{x}{a^4}\right)\left(1-\frac{x}{a^5}\right)\quad\text{etc.}$$

§7 Therefore, hence it is plain, as often as x is taken equal to a certain power of a, that because of that one vanishing factor of this expression it will be

$$1+s-t=0 \quad \text{and} \quad t=1+s.$$

Hence, if having put $x=a^n$ while n denotes a positive integer number the sum of the propounded series was $s=n$, having put $x=a^{n+1}$ the sum of the series will be

$$t=s+1=n+1.$$

Therefore, having taken $n=0$ or $x=1$ the sum of the series is $s=0$, having put $x=a$ the sum of the series will be $s=1$; and hence it further follows, if one puts $x=a^2$, that it will be $s=2$, and if $x=a^3$, that it will be $s=3$. And now it is plain in general, what we found based on induction only before, if it was $x=a^n$, while n denotes a positive integer number, that it will always be $s=n$. But if n is not a positive integer and s denotes the sum of the series propounded initially having put $x=a^n$, then having put $x=a^{n+1}$ the sum of the series, which we want to put $=t$, will not be $=s+1$; for, it will be

$$t = 1 + s - (1 - a^n)(1 - a^{n-1})(1 - a^{n-2})(1 - a^{n-3})(1 - a^{n-4}) \quad \text{etc.}$$

Therefore, in these cases the value of the series manifestly deviates from the nature of logarithms.

§8 As here by multiplying the values of x by a we found the value of t from the value of s, so vice versa by dividing the values of x by a we will obtain the value of s from the value of t; and hence we will be able to descend to negative values of the exponent n. Of course, in the series propounded at the beginning or reduced to this form

$$s = \frac{1}{1-a}(1-x) + \frac{1}{1-a^2}(1-x)\left(1 - \frac{x}{a}\right) + \frac{1}{1-a^3}(1-x)\left(1 - \frac{x}{a}\right)\left(1 - \frac{x}{a^2}\right) + \text{etc.}$$

for the following cases let us indicate the sum of the series this way:

$$\begin{array}{llll} \text{if} & x = 1, & \text{it is} & s = A = 0, \\ & x = \frac{1}{a}, & & s = B, \\ & x = \frac{1}{a^2}, & & s = C, \\ & x = \frac{1}{a^3}, & & s = D, \\ & x = \frac{1}{a^4}, & & s = E, \\ & & \text{etc.} & \end{array}$$

If one now puts $x = \frac{1}{a}$, it will be $s = B$ and $t = A = 0$, since t results from s, if one writes ax instead of x; from the preceding it is

$$1 + B = \left(1 - \frac{1}{a}\right)\left(1 - \frac{1}{a^2}\right)\left(1 - \frac{1}{a^3}\right)\left(1 - \frac{1}{a^4}\right)\left(1 - \frac{1}{a^5}\right) \quad \text{etc.}$$

or

$$B = -1 + \left(1 - \frac{1}{a}\right)\left(1 - \frac{1}{a^2}\right)\left(1 - \frac{1}{a^3}\right)\left(1 - \frac{1}{a^4}\right)\left(1 - \frac{1}{a^5}\right) \quad \text{etc.;}$$

so, if it is $a = 10$, it will be

$$B = -0.109989900001001.$$

§9 Let $x = \frac{1}{a^2}$ and it will be $s = C$ and $t = B$, whence one will have

$$1 + C - B = \left(1 - \frac{1}{a^2}\right)\left(1 - \frac{1}{a^3}\right)\left(1 - \frac{1}{a^4}\right)\left(1 - \frac{1}{a^5}\right) \text{ etc.;}$$

add the first $1 + B$ to this and it will be

$$2 + C = \left(2 - \frac{1}{a}\right)\left(1 - \frac{1}{a^2}\right)\left(1 - \frac{1}{a^3}\right)\left(1 - \frac{1}{a^4}\right)\left(1 - \frac{1}{a^5}\right) \text{ etc.}$$

and

$$C = -2 + \left(2 - \frac{1}{a}\right)\left(1 - \frac{1}{a^2}\right)\left(1 - \frac{1}{a^3}\right)\left(1 - \frac{1}{a^4}\right)\left(1 - \frac{1}{a^5}\right) \text{ etc.}$$

Or having eliminated the series it will be

$$1 + B = \left(1 - \frac{1}{a}\right)(1 + C + -B);$$

or

$$C - 2B = \frac{1}{a}(1 + C - B).$$

In like manner, if one puts $x = \frac{1}{a^3}$, it will be $s = D$ and $t = C$, whence it will be

$$1 + D - C = \left(1 - \frac{1}{a^3}\right)\left(1 - \frac{1}{a^4}\right)\left(1 - \frac{1}{a^5}\right)\left(1 - \frac{1}{a^6}\right) \text{ etc.;}$$

the first series added to this one will yield

$$3 + D = \left(3 - \frac{1}{a} - \frac{2}{a^2} + \frac{1}{a^3}\right)\left(1 - \frac{1}{a^3}\right)\left(1 - \frac{1}{a^4}\right)\left(1 - \frac{1}{a^5}\right)\left(1 - \frac{1}{a^6}\right) \text{ etc.}$$

And since having put $x = \frac{1}{a^4}$ it is

$$1+E-D=\left(1-\frac{1}{a^4}\right)\left(1-\frac{1}{a^5}\right)\left(1-\frac{1}{a^6}\right)\left(1-\frac{1}{a^7}\right) \text{ etc.,}$$

it will be

$$4+E=\left(4-\frac{1}{a}-\frac{2}{a^2}-\frac{2}{a^3}+\frac{1}{a^4}+\frac{2}{a^5}-\frac{1}{a^6}\right)\left(1-\frac{1}{a^4}\right)\left(1-\frac{1}{a^5}\right)\left(1-\frac{1}{a^6}\right) \text{ etc.;}$$

and so it is possible to proceed arbitrarily far.

§10 But one can exhibit a relation among three values of the sum of the series s for three successive values of x by means of a finite expression. For, while for the value x the sum is still $= s$, if one puts ax instead of x, let the sum of the series be $= t$, and if one puts aax instead of x, let the sum of the series be $= u$. Therefore, since we found this relation among s and t

$$1+s-t=(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right)\left(1-\frac{x}{a^3}\right)\left(1-\frac{x}{a^4}\right) \text{ etc.,}$$

if here we write ax for x, a similar relation among u and t will result

$$1+t-u=(1-ax)(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right)\left(1-\frac{x}{a^3}\right) \text{ etc.}$$

Therefore, hence it will be

$$1+t-u=(1-ax)(1+s-t)$$

or

$$u=2t-s+ax(1+s-t)$$

or

$$s=\frac{2t-u+ax(1-t)}{1-ax}.$$

And hence the following relations will result for the values A, B, C, D etc. assumed above.

If it is $x=\frac{1}{a^2}$, it will be

$$A=2B-C+\frac{1}{a}(1+C-B)$$

or

$$C = \frac{1 + (2a-1)B - aA}{a-1} = B + \frac{1 + a(B-A)}{a-1};$$

if it is $x = \frac{1}{a^3}$, it will be

$$D = C + \frac{1 + a^2(C-B)}{a^2-1};$$

if it is $x = \frac{1}{a^4}$, it will be

$$E = D + \frac{1 + a^3(D-C)}{a^3-1};$$

if it is $x = \frac{1}{a^5}$, it will be

$$F = E + \frac{1 + a^4(E-D)}{a^4-1}$$

etc.

But these relations can be expressed more conveniently in the following way:

$$\begin{aligned} C &= 2B - A + \frac{1+B-A}{a-1}, \\ D &= 2C - B + \frac{1+C-B}{a^2-1}, \\ E &= 2D - C + \frac{1+D-C}{a^3-1}, \\ F &= 2E - D + \frac{1+E-D}{a^4-1} \end{aligned}$$

etc.

Therefore, since it is $A = 0$, if only the value of the letter B was found

$$B = -1 + \left(1 - \frac{1}{a}\right)\left(1 - \frac{1}{a^2}\right)\left(1 - \frac{1}{a^3}\right)\left(1 - \frac{1}{a^4}\right) \text{ etc.},$$

hence the values of all following letters C, D, E, F etc. can be assigned exactly.

§11 But since, while n is a positive integer number, if one puts $x = a^n$, it is $s = n$, from our assumed series we will obtain this summable one

$$n = \frac{1-a^n}{1-a} + \frac{(1-a^n)(1-a^{n-1}}{1-a^2} + \frac{(1-a^n)(1-a^{n-1})(1-a^{n-2}}{1-a^3} + \text{etc.}$$

But then in the case, since it is $t = n+1$, it will be

$$1 = a^n + a^{n-1}(1-a^n) + a^{n-2}(1-a^n)(1-a^{n-1}) + a^{n-3}(1-a^n)(1-a^{n-1})(1-a^{n-2}) + \text{etc.},$$

which is seen to be manifestly correct having brought all terms to the same side; for, it will be

$$(1-a^n)(1-a^{n-1})(1-a^{n-2})(1-a^{n-3})(1-a^{n-4}) \;\; \text{etc.} = 0.$$

Hence this provides us with an opportunity to consider forms of this kind more generally. For, let

$$A, \quad B, \quad C, \quad D, \quad E, \quad F \quad \text{etc.}$$

be a series of certain quantities and let

$$(1-A)(1-B)(1-C)(1-D)(1-E)\text{etc.} = S.$$

And hence it will be obtained

$$1 - A - B(1-A) - C(1-A)(1-B) - D(1-A)(1-B)(1-C) - \text{etc.} = S;$$

for, this formula will most easily be reduced to that one. Therefore, we will have

$$A + B(1-A) + C(1-A)(1-B) + B(1-A)(1-B)(1-C) + \text{etc.} = 1 - S.$$

§12 Therefore, if a certain one of these quantities A, B, C etc. becomes equal to 1, it will be $S = 0$ and a series will result, whose sum is $= 1$. For the sake of an example take this series

$$A, \quad B, \quad C, \quad D, \quad E, \quad F \quad \text{etc.}$$

$$\frac{1}{2}, \quad \frac{2}{3}, \quad \frac{3}{4}, \quad \frac{4}{5}, \quad \frac{5}{6}, \quad \frac{6}{7} \quad \text{etc.;}$$

since the infinitesimal of these fractions is $= 1$, it will be $S = 0$ and the following series will result

$$1 = \frac{1}{2} + \frac{2}{2 \cdot 3} + \frac{3}{2 \cdot 3 \cdot 4} + \frac{4}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{5}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \text{etc.},$$

whose truth is certainly easily seen; for, it results this way: let

$$z = 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \text{etc.;}$$

it will be

$$z - 1 = \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \text{etc.}$$

and hence by subtraction it results

$$1 = \frac{1}{2} + \frac{2}{2 \cdot 3} + \frac{3}{2 \cdot 3 \cdot 4} + \frac{4}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{5}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \text{etc..}$$

§13 Let

$$A = \frac{1}{9}, \quad B = \frac{1}{25}, \quad C = \frac{1}{49}, \quad D = \frac{1}{81} \quad \text{etc.;}$$

it will be

$$S = \frac{8}{9} \cdot \frac{24}{25} \cdot \frac{48}{49} \cdot \frac{80}{81} \cdot \frac{120}{121} \cdot \text{etc.} = \frac{\pi}{4}$$

while π denotes the circumference of the circle whose diameter is $= 1$. Therefore, hence this series for the quadrature of the circle will result:

$$-\frac{\pi}{4} + 1 = \frac{1}{9} + \frac{8}{9 \cdot 25} + \frac{8 \cdot 24}{9 \cdot 25 \cdot 49} + \frac{8 \cdot 24 \cdot 48}{9 \cdot 25 \cdot 49 \cdot 81} + \text{etc.}$$

or

$$-\frac{9}{4}\pi + 8 = \frac{2 \cdot 4}{5 \cdot 5} + \frac{2 \cdot 4 \cdot 4 \cdot 6}{5 \cdot 5 \cdot 7 \cdot 7} + \frac{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8}{5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9} + \text{etc.}$$

Therefore, because there are innumerable products of this kind, whose value S can be exhibited, from each one an infinite series, whose sum can be assigned, will be derived this way. Therefore, a very broad field to find arbitrarily many summable series is opened.

§14 But I return to the series mentioned initially

$$s = \frac{1}{1-a}(1-x) + \frac{1}{1-a^2}(1-x)\left(1-\frac{x}{a}\right) + \frac{1}{1-a^3}(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right) + \text{etc.},$$

which I want to transform into another form, in which the terms proceed according to the powers of x. This first could certainly be done by actual expansion of the single terms, but since this way the single coefficients would arise expressed in terms of infinite series, the formula found above will be used more conveniently for this purpose

$$u = 2t - s + ax(1-t+s) \quad \text{or} \quad u - 2t + s = ax + ax(s-t),$$

where t results from s, if one puts ax instead of x, and in like manner u results from t, if it is put ax instead of x again. Hence, if for the series in question we assume

$$s = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.},$$

it will be

$$t = A + Bax + Ca^2x^2 + Da^3x^3 + Ea^4x^4 + Fa^5x^5 + \text{etc.}$$

and

$$u = A + Ba^2x + Ca^4x^2 + Da^6x^3 + Ea^8x^4 + Fa^{10}x^5 + \text{etc.}$$

From these one will therefore conclude

$$u - 2t + s = B(1-a)^2x + C(1-aa)^2x^2 + D(1-a^3)^2x^3 + E(1-a^4)^2x^4 + \text{etc.},$$
$$ax(1+s-t) = ax + Ba(1-a)x^2 + Ca(1-aa)x^3 + Da(1-a^3)x^4 + \text{etc.}$$

From the equality of these series it is concluded that it will be:

$$B = \frac{a}{(1-a)^2}, \quad C = \frac{Ba(1-a)}{(1-aa)^2}, \quad D = \frac{Ca(1-aa)}{(1-a^3)^2}, \quad E = \frac{Da(1-a^3)}{(1-a^4)^2} \quad \text{etc.}$$

§15 Therefore, hence the following values of the assumed coefficients will be obtained:

$$\begin{aligned}
B &= \frac{a}{(1-a)^2}, \\
C &= \frac{a^2}{(1-a)(1-aa)^2}, \\
D &= \frac{a^3}{(1-a)(1-aa)(1-a^3)^2}, \\
E &= \frac{a^4}{(1-a)(1-aa)(1-a^3)(1-a^4)^2}, \\
F &= \frac{a^5}{(1-a)(1-aa)(1-a^3)(1-a^4)(1-a^5)^2} \\
&\text{etc.}
\end{aligned}$$

But the first term A is hence not defined. And since A yields the value of s, if one puts $x = 0$, it is perspicuous that it will be

$$A = \frac{1}{1-a} + \frac{1}{1-a^2} + \frac{1}{1-a^3} + \frac{1}{1-a^4} + \frac{1}{1-a^5} + \text{etc.}$$

Therefore, having defined these values the series propounded initially

$$s = \frac{1}{1-a}(1-x) + \frac{1}{1-a^2}(1-x)\left(1-\frac{x}{a}\right) + \frac{1}{1-a^3}(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right) + \text{etc.}$$

will be transformed into this form

$$\begin{gathered}
s = \frac{1}{1-a} + \frac{1}{1-a^2} + \frac{1}{1-a^3} + \frac{1}{1-a^4} + \frac{1}{1-a^5} + \text{etc.} \\
+\frac{ax}{(1-a)^2} + \frac{a^2x^2}{(1-a)(1-aa)^2} + \frac{a^3x^3}{(1-a)(1-aa)(1-a^3)^2} + \frac{a^4x^4}{(1-a)(1-aa)(1-a^3)(1-a^4)^2} + \text{etc.}
\end{gathered}$$

§16 Therefore, since having put $x = a^n$ while n denotes a positive integer number it is $s = n$, one will have this summation

$$n + \frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \frac{1}{a^4-1} + \frac{1}{a^5-1} + \text{etc.}$$

$$= \frac{a^{n+1}}{(a-1)^2} - \frac{a^{2n+2}}{(a-1)(aa-1)^2} + \frac{a^{3n+3}}{(a-1)(aa-1)(a^3-1)^2} - \frac{a^{4n+4}}{(a-1)(aa-1)(a^3-1)(a^4-1)^2} + \text{etc.}$$

Therefore, if it was $n = 0$, it will be

$$\frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \text{etc.}$$
$$= \frac{a}{(a-1)^2} - \frac{a^2}{(a-1)(aa-1)^2} + \frac{a^3}{(a-1)(a^2-1)(a^3-1)^2} - \text{etc.},$$

and if one puts $n = 1$, it will be

$$\frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \text{etc.}$$
$$= \frac{a^2}{(a-1)^2} - \frac{a^4}{(a-1)(aa-1)^2} + \frac{a^6}{(a-1)(a^2-1)(a^3-1)^2} - \text{etc.} - 1.$$

Therefore, in general it will be

$$\frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \frac{1}{a^4-1} + \text{etc.}$$
$$= \frac{a^{n+1}}{(a-1)^2} - \frac{a^{2n+2}}{(a-1)(a^2-1)^2} + \frac{a^{3n+3}}{(a-1)(a^2-1)(a^3-1)^2} - \text{etc.} - n$$

while n denotes an arbitrary positive integer.

§17 If one puts $n-1$ instead of n, one will have

$$\frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \frac{1}{a^4-1} + \text{etc.}$$
$$= \frac{a^{n}}{(a-1)^2} - \frac{a^{2n}}{(a-1)(a^2-1)^2} + \frac{a^{3n}}{(a-1)(a^2-1)(a^3-1)^2} - \text{etc.} - n + 1;$$

if from this series the upper one is subtracted, this expression will result

$$1 = \frac{a^n}{a-1} - \frac{a^{2n}}{(a-1)(a^2-1)} + \frac{a^{3n}}{(a-1)(a^2-1)(a^3-1)} - \frac{a^{4n}}{(a-1)(a^2-1)(a^3-1)(a^4-1)} + \text{etc.}$$

Therefore, the sum of this series is equal to the 1, whatever value is attributed to a and whatever positive integer number is substituted for n. But in the case, in which it is $n = 1$, this summation is easily seen. For, since it is

$$1 = \frac{a}{a-1} - \frac{a^2}{(a-1)(a^2-1)} + \frac{a^3}{(a-1)(a^2-1)(a^3-1)} - \text{etc.},$$

it clearly follows from the consideration of this series

$$z = 1 - \frac{1}{a-1} + \frac{1}{(a-1)(a^2-1)} - \frac{1}{(a-1)(a^2-1)(a^3-1)} + \text{etc.},$$

whence it is

$$1 - z = \frac{1}{a-1} - \frac{1}{(a-1)(a^2-1)} + \frac{1}{(a-1)(a^2-1)(a^3-1)} - \frac{1}{(a-1)(a^2-1)(a^3-1)(a^4-1)} + \text{etc.},$$

which added to each other will give

$$1 = \frac{a}{a-1} - \frac{aa}{(a-1)(a^2-1)} + \frac{a^3}{(a-1)(a^2-1)(a^3-1)} - \frac{a^4}{(a-1)(a^2-1)(a^3-1)(a^4-1)} + \text{etc.}$$

§18 But further the truth of this series can be shown for the remaining values of n in the following way. If it was

$$1 = \frac{a^n}{a-1} - \frac{a^{2n}}{(a-1)(a^2-1)} + \frac{a^{3n}}{(a-1)(a^2-1)(a^3-1)} - \text{etc.},$$

I say that it will also be

$$1 = \frac{a^{n+1}}{a-1} - \frac{a^{2n+2}}{(a-1)(a^2-1)} + \frac{a^{3n+3}}{(a-1)(a^2-1)(a^3-1)} - \text{etc.},$$

Since now by assumption it is

$$1 = \frac{a^n}{a-1} - \frac{a^{2n}}{(a-1)(a^2-1)} + \frac{a^{3n}}{(a-1)(a^2-1)(a^3-1)} - \text{etc.},$$

it will also be

$$0 = a^n - \frac{a^{2n}}{a-1} + \frac{a^{3n}}{(a-1)(a^2-1)} - \text{etc.},$$

which series added to each other will give

$$1 = \frac{a^{n+1}}{a-1} - \frac{a^{2n+2}}{(a-1)(a^2-1)} + \frac{a^{3n+3}}{(a-1)(a^2-1)(a^3-1)} - \text{etc..}$$

Since this equation

$$1 = \frac{a^{n}}{a-1} - \frac{a^{2n}}{(a-1)(a^2-1)} + \frac{a^{3n}}{(a-1)(a^2-1)(a^3-1)} - \text{etc.}$$

was shown to be true in the case $n = 1$, it will also be true in the case $n = 2$ and hence further in the cases $n = 3$, $n = 4$ etc., so that, whatever positive integer number is substituted for n, the sum will always be $= 1$.

§19 Since I ordered the series propounded initially $s = \frac{1}{1-a}(1-x) + \text{etc.}$ according to powers of x using the property demonstrated above

$$u - 2t + s = ax + ax(s-t),$$

it will not be out of place to derive the same transformation directly from the series s itself; for, so we will get to the summation of innumerable new series. Therefore, it will be necessary that the terms of the series s are actually expanded by multiplication; that this can be done more easily, I will consider an arbitrary term

$$\frac{1}{(1-a^m)}(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right)\left(1-\frac{x}{a^3}\right)\cdots\left(1-\frac{x}{a^{m-1}}\right)$$

Therefore, I will put

$$P = (1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right)\left(1-\frac{x}{a^3}\right)\cdots\left(1-\frac{x}{a^{m-1}}\right)$$

and it will be

$$\log P = \log(1-x) + \log\left(1-\frac{x}{a}\right) + \log\left(1-\frac{x}{a^2}\right) + \cdots + \left(1-\frac{x}{a^{m-1}}\right)$$

and by differentiation it will be

$$\frac{dP}{P} = \frac{-dx}{1-x} - \frac{dx}{a-x} - \frac{dx}{aa-x} - \cdots - \frac{dx}{a^{m-1}-x}$$

or

$$\frac{dP}{P} = -dx \left\{ \begin{array}{cccccccccccccc} & 1 & + & x & + & x^2 & + & x^3 & + & x^4 & + & x^5 & + \text{etc.} \\ + & \frac{1}{a} & + & \frac{x}{a^2} & + & \frac{x^2}{a^3} & + & \frac{x^3}{a^4} & + & \frac{x^4}{a^5} & + & \frac{x^5}{a^6} & + \text{etc.} \\ + & \frac{1}{a^2} & + & \frac{x}{a^4} & + & \frac{x^2}{a^6} & + & \frac{x^3}{a^8} & + & \frac{x^4}{a^{10}} & + & \frac{x^5}{a^{12}} & + \text{etc.} \\ & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\ + & \frac{1}{a^{m-1}} & + & \frac{x}{a^{2m-2}} & + & \frac{x^2}{a^{3m-3}} & + & \frac{x^3}{a^{4m-4}} & + & \frac{x^4}{a^{5m-5}} & + & \frac{x^5}{a^{6m-6}} & + \text{etc.} \end{array} \right\}.$$

Now by summing the single vertical series it will result

$$dP = -Pdx \left(\frac{a^m - 1}{a^m - a^{m-1}} + \frac{a^{2m} - 1}{a^{2m} - a^{2m-2}} x + \frac{a^{3m} - 1}{a^{3m} - a^{3m-3}} x^2 + \frac{a^{4m} - 1}{a^{4m} - a^{4m-4}} x^3 + \text{etc.} \right).$$

§20 Now assume this series for P

$$P = \alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{etc.}$$

and it will be

$$\frac{dP}{dx} = \beta + 2\gamma x + 3\delta x^2 + 4\varepsilon x^3 + 5\zeta x^4 + \text{etc.}$$

Now after the substitution it will be

$$\begin{aligned} \beta + \frac{a^m - 1}{a^m - a^{m-1}} \alpha &= 0, \\ 2\gamma + \frac{a^m - 1}{a^m - a^{m-1}} \beta + \frac{a^{2m} - 1}{a^{2m} - a^{2m-2}} \alpha &= 0, \\ 3\delta + \frac{a^m - 1}{a^m - a^{m-1}} \gamma + \frac{a^{2m} - 1}{a^{2m} - a^{2m-2}} \beta + \frac{a^{3m} - 1}{a^{3m} - a^{3m-3}} \alpha &= 0 \text{etc.}, \end{aligned}$$

and since having put $x = 0$ it is $P = 1$, it is plain that it is $\alpha = 1$. Therefore, it will be

$$\beta = \frac{-a^m + 1}{a^m - a^{m-1}}$$

and

$$2\gamma - \frac{(a^m-1)^2}{(a^m-a^{m-1})^2} + \frac{a^{2m}-1}{a^{2m}-a^{2m-2}} = 0$$

or

$$2\gamma = \frac{a^m-1}{a^m-a^{m-1}}\left(\frac{a^m-1}{a^m-a^{m-1}} - \frac{a^m+1}{a^m+a^{m-1}}\right) = \frac{2a^m(a^{m-1}-1)(a^m-1)}{(a^m-a^{m-1})(a^{2m}-a^{2m-2})}$$

and hence

$$\gamma = \frac{(a^m-1)(a^{m-1}-1)}{(a^m-a^{m-1})(a^m-a^{m-2})}.$$

In like manner the remaining coefficients, although not without a lot of work, will be found and will finally be detected to be expressed conveniently.

§21 Therefore, that this determination of the coefficients can be done more easily, I will apply the method used in this paper already several times. Of course, in the series

$$P = \alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{etc.}$$

instead of x I set $\frac{x}{a}$ and the sum of the resulting series I assume to be $= Q$, namely

$$Q = \alpha + \frac{\beta x}{a} + \frac{\gamma x^2}{a^2} + \frac{\delta x^3}{a^3} + \frac{\varepsilon x^4}{a^4} + \text{etc.}$$

But since it is

$$P = (1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right)\cdots\left(1-\frac{x}{a^{m-1}}\right),$$

it will be

$$Q = \left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right)\left(1-\frac{x}{a^3}\right)\cdots\left(1-\frac{x}{a^m}\right)$$

and hence

$$P\left(1-\frac{x}{a^m}\right) = Q(1-x) \quad \text{or} \quad a^mP - Px - a^mQ + a^mQx = 0;$$

here, substitute the series assumed for P and Q and it will be

$$\left.\begin{array}{rrrrr} \alpha a^m + & \beta a^m x & + \ \gamma a^m x^2 & + \ \delta a^m x^3 & + \text{ etc.} \\ - & \alpha x & - \ \beta x^2 & - \ \gamma x^3 & - \text{ etc.} \\ -\alpha a^m - & \beta a^{m-1} x & - \ \gamma a^{m-2} x^2 & + \ \delta a^{m-3} x^3 & + \text{ etc.} \\ + & \alpha a^m x & + \ \beta a^{m-1} x^2 & + \ \gamma a^{m-2} x^3 & + \text{ etc.} \end{array}\right\} = 0.$$

From the comparison of the homogeneous terms[2] it is found

$$\begin{aligned} \beta &= \frac{-\alpha(a^m-1)}{a^{m-1}(a-1)}, \\ \gamma &= \frac{-\beta(a^{m-1}-1)}{a^{m-2}(aa-1)}, \\ \delta &= \frac{-\gamma(a^{m-1}-1)}{a^{m-3}(a^3-1)}, \\ \varepsilon &= \frac{-\delta(a^{m-3}-1)}{a^{m-4}(a^4-1)}, \\ &\text{etc.} \end{aligned}$$

§22 Therefore, since it is $\alpha = 1$, the coefficients will behave this way:

$$\begin{aligned} \alpha &= 1, \\ \beta &= \frac{-(a^m-1)}{a^{m-1}(a-1)}, \\ \gamma &= \frac{+(a^m-1)(a^{m-1}-1)}{a^{2m-3}(a-1)(aa-1)}, \\ \delta &= \frac{-(a^m-1)(a^{m-1}-1)(a^{m-2}-1)}{a^{3m-6}(a-1)(aa-1)(a^3-1)}, \\ \varepsilon &= \frac{+(a^m-1)(a^{m-1}-1)(a^{m-2}-1)(a^{m-3}-1)}{a^{4m-10}(a-1)(a^2-1)(a^3-1)(a^4-1)} \\ &\text{etc.} \end{aligned}$$

Therefore, the general term of the series s,

[2]By this Euler means the terms containing the same respective power of x.

$$\frac{1}{1-a^m}(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right)\cdots\left(1-\frac{x}{a^{m-1}}\right),$$

in expanded form will give this progression

$$\frac{1}{1-a^m}-\frac{x}{a^{m-1}(1-a)}+\frac{(1-a^{m-1})x^2}{a^{2m-3}(1-a)(1-a^2)}-\frac{(1-a^{m-1})(1-a^{m-2})x^3}{a^{3m-6}(a-1)(1-a^2)(1-a^3)}+\text{etc.}$$

Therefore, if the numbers 1, 2, 3, 4 etc. are successively substituted for m, the following formulas or terms of the series s will result:

$$\begin{aligned}
&\text{First Term:} && =\frac{1}{1-a}-\frac{x}{1-a},\\
&\text{Second Term:} && =\frac{1}{1-a^2}-\frac{x}{a(1-a)}+\frac{(1-a)x^2}{a(1-a)(1-a^2)},\\
&\text{Third Term:} && =\frac{1}{1-a^3}-\frac{x}{a^2(1-a)}+\frac{(1-a^2)x^2}{a^3(1-a)(1-a^2)}-\frac{(1-a)(1-a^2)x^3}{a^3(1-a)(1-a^2)(1-a^3)},\\
&\text{Fourth Term:} && =\frac{1}{1-a^4}-\frac{x}{a^3(1-a)}+\frac{(1-a^3)xx}{a^5(1-a)(1-a^2)}-\frac{(1-a^2)(1-a^3)x^3}{a^6(1-a)(1-a^2)(1-a^3)}\\
& && \quad +\frac{(1-a)(1-a^2)(1-a^3)x^4}{a^6(1-a)(1-a^2)(1-a^3)(1-a^4)}\\
& && \quad \text{etc.}
\end{aligned}$$

§23 Therefore, if all these terms are collected into one sum, a large amount of infinite series will result, which taken at the same time will be equal to the series propounded initially. Of course, because it is

$$s=\frac{1}{1-a}(1-x)+\frac{1}{1-a^2}(1-x)\left(1-\frac{x}{a}\right)+\frac{1}{1-a^3}\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right)+\text{etc.},$$

it will be

$$\begin{aligned}
s=&\frac{1}{1-a}+\frac{1}{1-a^2}+\frac{1}{1-a^3}+\frac{1}{1-a^4}+\frac{1}{1-a^5}+\text{etc.}\\
&-\frac{x}{1-a}\left(1+\frac{1}{a}+\frac{1}{a^2}+\frac{1}{a^3}+\frac{1}{a^4}+\text{etc.}\right)
\end{aligned}$$

$$+\frac{x^2}{a(1-a)(1-a^2)}\left(\frac{1-a}{1}+\frac{1-a^2}{a^2}+\frac{1-a^3}{a^4}+\frac{1-a^4}{a^6}+\text{etc.}\right)$$
$$-\frac{x^3}{a^3(1-a)(1-a^2)(1-a^3)}\left(\frac{(1-a)(1-a^2)}{1}+\frac{(1-a^2)(1-a^3)}{a^3}+\frac{(1-a^3)(1-a^4)}{a^6}+\text{etc.}\right)$$
$$+\frac{x^4}{a^6(1-a)(1-a^2)(1-a^3)(1-a^4)}\left(\frac{(1-a)(1-a^2)(1-a^3)}{1}+\frac{(1-a^2)(1-a^3)(1-a^4)}{a^4}+\text{etc.}\right)$$
$$\text{etc.}$$

Therefore, since this series must be identical to the one found before, from the agreement of these single series the sums will be found to be

$$1+\frac{1}{a}+\frac{1}{a^2}+\frac{1}{a^3}+\frac{1}{a^4}+\text{etc.}=\frac{-a}{1-a},$$
$$\frac{1-a}{1}+\frac{1-a^2}{a^2}+\frac{1-a^3}{a^4}+\frac{1-a^4}{a^6}+\text{etc.}=\frac{+a^3}{1-aa},$$
$$\frac{(1-a)(1-a^2)}{1}+\frac{(1-a^2)(1-a^3)}{a^3}+\frac{(1-a^3)(1-a^4)}{a^6}+\text{etc.}=\frac{-a^6}{1-a^3},$$
$$\frac{(1-a)(1-a^2)(1-a^3)}{1}+\frac{(1-a^2)(1-a^3)(1-a^4)}{a^4}+\text{etc.}=\frac{+a^{10}}{1-a^4},$$
$$\frac{(1-a)(1-a^2)(1-a^3)(1-a^4)}{1}+\frac{(1-a^2)(1-a^3)(1-a^4)(1-a^5)}{a^5}+\text{etc.}=\frac{-a^{15}}{1-a^5}$$
$$\text{etc.}$$

§24 These series can be cast into the following forms, from which the structure of the progression will be seen more clearly:

$$\frac{a}{a-1}=1+\frac{1}{a}+\frac{1}{a^2}+\frac{1}{a^3}+\frac{1}{a^4}+\text{etc.}$$
$$\frac{a^2}{a^2-1}=\left(1-\frac{1}{a}\right)+\frac{1}{a}\left(1-\frac{1}{a^2}\right)+\frac{1}{a^2}\left(1-\frac{1}{a^3}\right)+\frac{1}{a^3}\left(1-\frac{1}{a^4}\right)+\frac{1}{a^4}\left(1-\frac{1}{a^5}\right)+\text{etc.}$$
$$\frac{a^3}{a^3-1}=\left(1-\frac{1}{a}\right)\left(1-\frac{1}{a^2}\right)+\frac{1}{a}\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{a^3}\right)+\frac{1}{a^2}\left(1-\frac{1}{a^3}\right)\left(1-\frac{1}{a^4}\right)+\text{etc.},$$
$$\frac{a^4}{a^4-1}=\left(1-\frac{1}{a}\right)\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{a^3}\right)+\frac{1}{a}\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{a^3}\right)\left(1-\frac{1}{a^4}\right)+\text{etc.},$$
$$\frac{a^5}{a^5-1}=\left(1-\frac{1}{a}\right)\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{a^3}\right)\left(1-\frac{1}{a^4}\right)+\frac{1}{a}\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{a^3}\right)\left(1-\frac{1}{a^4}\right)\left(1-\frac{1}{a^5}\right)+\text{etc.}$$
$$\text{etc.}$$

Hence it is concluded that it will be in general

$$\frac{a^{m+1}}{a^{m+1}-1} = \frac{1}{1-\frac{1}{a^{m+1}}} = \left(1-\frac{1}{a}\right)\left(1-\frac{1^2}{a}\right)\cdots\left(1-\frac{1}{a^m}\right)$$
$$+\frac{1}{a}\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{a^3}\right)\cdots\left(1-\frac{1}{a^{m+1}}\right)+\frac{1}{a^2}\left(1-\frac{1}{a^3}\right)\left(1-\frac{1}{a^4}\right)\cdots\left(1-\frac{1}{a^{m+2}}\right)$$
$$+\frac{1}{a^3}\left(1-\frac{1}{a^4}\right)\left(1-\frac{1}{a^5}\right)\cdots\left(1-\frac{1}{a^{m+3}}\right)+\text{etc.}$$

§25 The sum of this series can also be investigated in this way. For the sake of brevity let $\frac{1}{a}=b$ and put the sum in question

$$z = (1-b)(1-b^2)\cdots(1-b^m)+b(1-b^2)(1-b^3)\cdots(1-b^{m+1})$$
$$+b^2(1-b^3)(1-b^4)\cdots(1-b^{m+2})+b^3(1-b^4)(1-b^5)\cdots(1-b^{m+3})+\text{etc.}$$

Multiply by $1-b^{m+1}$ on both sides and it will result

$$(1-b^{m+1})z = (1-b)(1-b^2)\cdots(1-b^m)(1-b^{m+1})+(1-b^2)(1-b^3)\cdots(1-b^{m+1})(b-b^{m+2})$$
$$+(1-b^3)(1-b^4)\cdots(1-b^{m+2})(b^2-b^{m+3})+\text{etc.}$$

But it is

$$\begin{aligned} b\ \ -b^{m+2} &= 1-b^{m+2}-(1-b), \\ b^2-b^{m+3} &= 1-b^{m+3}-(1-bb), \\ b^3-b^{m+4} &= 1-b^{m+4}-(1-b^3) \\ &\text{etc.}; \end{aligned}$$

these values substituted for the last products will give

$$\begin{aligned} (1-b^{m+1})z = {} & (1-b)\,(1-b^2)\cdots(1-b^{m+1})+(1-b^2)(1-b^3)\cdots(1-b^{m+2}) \\ & -(1-b)\,(1-b^2)\cdots(1-b^{m+1})-(1-b^2)(1-b^3)\cdots(1-b^{m+2}) \\ & +(1-b^3)(1-b^4)\cdots(1-b^{m+3})-(1-b^4)(1-b^5)\cdots(1-b^{m+4}) \\ & -(1-b^3)(1-b^4)\cdots(1-b^{m+3})-(1-b^4)(1-b^5)\cdots(1-b^{m+4}) \\ & \text{etc.} \end{aligned}$$

Therefore, since all terms cancel each other, only the last one will remain

$$(1-b^{m+1})z = (1-b^{\infty})(1-b^{\infty+1})\cdots(1-b^{m+\infty}),$$

whence it is plain, if it was $b<1$, this means $a>1$, as we assumed, that it will be $(1-b^{m+1})z = 1$ and hence

$$z = \frac{1}{1-b^{m+1}} = \frac{a^{m+1}}{a^{m+1}-1},$$

as we had found before.

§26 From the things, which were given in § 21, one easily finds a power series in x, which is equal to this infinite product

$$P = (1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right)\left(1-\frac{x}{a^3}\right)\left(1-\frac{x}{a^4}\right) \quad \text{etc.}$$

For, having put

$$P = 1 - \alpha x + \beta x^2 - \gamma x^3 + \delta x^4 - \varepsilon x^5 + \text{etc.}$$

write ax instead of x and let the resulting value be $= Q$; it will be

$$Q = (1-ax)(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right)\left(1-\frac{x}{a^3}\right) \quad \text{etc.} = P - axP$$

and

$$Q = 1 - \alpha a x + \beta a^2 x^2 - \gamma a^3 x^3 + \delta a^4 x^4 - \varepsilon a^5 x^5 + \text{etc.};$$

but it is

$$axP = ax - \alpha a x^2 + \beta a x^3 - \gamma a x^4 + \delta a x^5 - \text{etc.},$$
$$-P = -1 + \alpha x - \beta x^2 + \gamma x^2 + \gamma x^3 - \delta x^4 + \varepsilon x^5 - \text{etc.},$$

whence it is

$$\alpha = \frac{a}{a-1}, \quad \beta = \frac{\alpha a}{a^2-1}, \quad \gamma = \frac{\beta a}{a^3-1}, \quad \delta = \frac{\gamma a}{a^4-1} \quad \text{etc.}$$

Therefore, the infinite product

$$P = (1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right) \quad \text{etc.}$$

is resolved into this infinite series:

$$P = 1 - \frac{ax}{a-1} + \frac{a^2x^2}{(a-1)(a^2-1)} - \frac{a^3x^3}{(a-1)(a^2-1)(a^3-1)} + \frac{a^4x^4}{(a-1)(a^2-1)(a^3-1)(a^4-1)} - \text{etc.}$$

§27 Therefore, if this product P is put equal to zero, this infinite equation

$$0 = 1 - \frac{ax}{a-1} + \frac{a^2x^2}{(a-1)(a^2-1)} - \frac{a^3x^3}{(a-1)(a^2-1)(a^3-1)} + \text{etc.}$$

will have only real roots in x and the values of x will be equal to the terms of this geometric progression

$$1, \quad a, \quad a^2, \quad a^3, \quad a^4, \quad a^5, \quad a^6, \quad a^7 \quad \text{etc.;}$$

hence, if one puts $x = a^n$ while n denotes an arbitrary positive integer number, it will be

$$0 = 1 - \frac{a^{n+1}}{a-1} + \frac{a^{2n+2}}{(a-1)(a^2-1)} - \frac{a^{3n+3}}{(a-1)(a^2-1)(a^3-1)} + \text{etc.},$$

whose truth was already demonstrated above.

§28 But especially that series innumerable other were found to be equal (§ 16) to is remarkable; this series is

$$\frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \frac{1}{a^4-1} + \frac{1}{a^5-1} + \text{etc.;}$$

its sum, if $a > 1$, even though it is finite and can easily be assigned by approximations, can nevertheless not be expressed by rational or irrational numbers. Therefore, it seems especially worth that the Geometers investigate the nature of that transcendental quantity by which its sum is expressed.

§29 But I will show, how the sum of series of this kind can be found approximately quickly, and I will certainly consider this series in a bit broader sense. Let

$$s = \frac{1}{a-z} + \frac{1}{a^2-z} + \frac{1}{a^3-z} + \frac{1}{a^4-z} + \frac{1}{a^5-z} + \text{etc.}$$

Convert the single terms into geometric series and it will be

$$\begin{aligned} s = {} & \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \frac{1}{a^4} + \frac{1}{a^5} + \text{etc.} \\ & + z\left(\frac{1}{a^2} + \frac{1}{a^4} + \frac{1}{a^6} + \frac{1}{a^8} + \frac{1}{a^{10}} + \text{etc.}\right) \\ & + z^2\left(\frac{1}{a^3} + \frac{1}{a^6} + \frac{1}{a^9} + \frac{1}{a^{12}} + \frac{1}{a^{15}} + \text{etc.}\right) \\ & \qquad \text{etc.,} \end{aligned}$$

which series summed again will give

$$s = \frac{1}{a-1} + \frac{z}{aa-1} + \frac{zz}{a^3-1} + \frac{z^3}{a^4-1} + \frac{z^4}{a^5-1} + \text{etc.}$$

Therefore, if it was $z = 1$, these two series reduce to the same and this transformation causes no difference.

§30 To sum this series let us put that n terms of the first form have actually already been summed, whose sum we want to set $= A$, so that it is

$$A = \frac{1}{a-z} + \frac{1}{a^2-z} + \frac{1}{a^4-z} + \cdots + \frac{1}{a^n-z}.$$

Therefore, the whole sum in question will be

$$s = A + \frac{1}{a^{n+1}-z} + \frac{1}{a^{n+2}-z} + \frac{1}{a^{n+3}-z} + \frac{1}{a^{n+4}-z} + \text{etc.}$$

Now expand these fractions into geometric series and it will be

$$\begin{aligned} s = A + & \quad \frac{1}{a^{n+1}} + \frac{1}{a^{n+2}} + \frac{1}{a^{n+3}} + \frac{1}{a^{n+4}} + \text{etc.} \\ & + z\left(\frac{1}{a^{2n+2}} + \frac{1}{a^{2n+4}} + \frac{1}{a^{2n+6}} + \frac{1}{a^{2n+8}} + \text{etc.}\right) \\ & + z^2\left(\frac{1}{a^{3n+3}} + \frac{1}{a^{3n+6}} + \frac{1}{a^{3n+9}} + \frac{1}{a^{3n+12}} + \text{etc.}\right) \\ & \qquad\qquad \text{etc.,} \end{aligned}$$

which series summed again will give

$$s = A + \frac{1}{a^n(a-1)} + \frac{z}{a^{2n}(aa-1)} + \frac{zz}{a^{3n}(a^3-1)} + \frac{z^3}{a^{4n}(a^4-1)} + \text{etc.},$$

which converges the more quickly than the first the greater the number n was.

§31 Let $a = 2$, that it is

$$s = \frac{1}{2-z} + \frac{1}{4-z} + \frac{1}{8-z} + \frac{1}{16-z} + \text{etc.}$$

Therefore, if it was

$$A = \frac{1}{2-z} + \frac{1}{4-z} + \frac{1}{8-z} + \cdots + \frac{1}{2^n - z},$$

it will be

$$s = A + \frac{1}{1 \cdot 2^n} + \frac{z}{3 \cdot 2^{2n}} + \frac{z^2}{7 \cdot 2^{3n}} + \frac{z^3}{15 \cdot 2^{4n}} + \frac{z^4}{31 \cdot 2^{5n}} + \text{etc.}$$

But let us put $z = 1$, so that the sum of this series is in question

$$s = 1 + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \frac{1}{31} + \frac{1}{63} + \text{etc.}$$

For the sake of an example let us actually add the four initial terms that it is $n = 4$; it will be

$$\begin{aligned}
1 &= 1.000000000000000 \\
\frac{1}{3} &= 0.333333333333333 \\
\frac{1}{7} &= 0.142857142857142 \\
\frac{1}{15} &= 0.066666666666666 \\
\hline
A &= 1.542857142857141
\end{aligned}$$

Hence it will be

$$s = A + \frac{1}{16 \cdot 1} + \frac{1}{16^2 \cdot 3} + \frac{1}{16^3 \cdot 7} + \frac{1}{16^4 \cdot 15} + \text{etc.}$$

and these terms in decimal fractions will give

$$\begin{aligned}
&0.063838009558149 \\
A &= 1.542857142857142 \\
\hline
\text{Therefore } s &= 1.606695152415291
\end{aligned}$$

§32 Furthermore, if the single terms of the series

$$s = \frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \text{etc.}$$

are resolved into geometric series and the equal powers of a are collected, one will find this form

$$s = \frac{1}{a} + \frac{2}{a^2} + \frac{2}{a^3} + \frac{3}{a^4} + \frac{2}{a^5} + \frac{4}{a^6} + \frac{2}{a^7} + \frac{4}{a^8} + \frac{3}{a^9} + \text{etc.},$$

which series has this property that the numerator of each fraction indicates, how many divisors the exponent of a in the denominator has. So the numerator of the fraction $\frac{4}{a^6}$ is $= 4$, since the exponent 6 has the four divisors 1, 2, 3, 6. Hence, if the exponent of a in the denominator is a prime number, the numerator will always be $= 2$; but for non-prime numbers it will be greater than two. Hence it easily becomes clear, if $a = 10$, that it will be

$$s = 0.1223242434262445262644428344628.$$