On the transformation of series *

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§1 Since it is propounded to us to show the use of differential calculus, both in the whole field of analysis and in the doctrine of series, several auxiliary tools from common algebra which are usually are not discussed will have to be covered here. Although we have already covered a huge part in the *Introductio*, some things were nevertheless left aside there, either on purpose, because it is convenient to explain them just then when they are actually needed, or because all the things which will be necessary could not have been foreseen at that point. This concerns the transformation of series we devote this chapter to and by means of which a given series is transformed into innumerable others such that, if the sum of the propounded series is known, the resulting ones can all be summed at the same time. Indeed, having discussed this subject in advance, we will be able to develop the doctrine of series even further by means of differential and integral calculus.

§2 But we will mainly consider series whose terms are multiplied by successive powers of a certain variable quantity, since these extend further and are of greater utility.

Therefore, let the following general series be propounded, whose sum, either known or not, we want to put $= S$, and let

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \text{etc.}$$

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Now, put $x = \frac{y}{1+y}$ and because by an infinite series

$$\begin{aligned} x &= y - y^2 + y^3 - y^4 + y^5 - y^6 + \text{etc.} \\ x^2 &= y^2 - 2y^3 + 3y^4 - 4y^5 + 5y^6 - 6y^7 + \text{etc.} \\ x^3 &= y^3 - 3y^4 + 6y^5 - 10y^6 + 15y^7 - 21y^8 + \text{etc.} \\ x^4 &= y^4 - 4y^4 + 10y^6 - 20y^7 + 35y^8 - 35y^8 + \text{etc.} \end{aligned}$$

etc.,

these values, having substituted them and having arranged the series according to powers of y, will give

$$\begin{array}{llllll} S = ay & - ay^2 & + ay^3 & - ay^4 & + ay^5 & \text{etc.} \\ & + b & - 2b & + 3b & - 4b & \\ & & + c & - 3c & + 6c & \\ & & & + d & - 3d & \\ & & & & + e & \end{array}$$

§3 Since we put $x = \frac{y}{1+y}$, it will be $y = \frac{x}{1-x}$; having substituted this value for y, the propounded series

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \text{etc.}$$

will be transformed into this one:

$$S = a\frac{x}{1-x} + (b-a)\frac{x^2}{(1-x)^2} + (c-2b+a)\frac{x^3}{(1-x)^3} + \text{etc.},$$

in which the coefficient of the second term $b-a$ is the first difference of a from the series a, b, c, d, e etc., which difference we denoted by Δa above; the coefficient of the third term $c - 2b + a$ is the second difference $\Delta^2 a$; the coefficient of the fourth term is the third difference of $\Delta^3 a$ etc. Therefore, using the iterated differences of a which are formed from the series a, b, c, d, e etc., the transformed series will go over into this one

$$S = \frac{x}{1-x}a + \frac{x^2}{(1-x)^2}\Delta a + \frac{x^3}{(1-x)^3}\Delta^2 a + \frac{x^4}{(1-x)^4}\Delta^3 a + \text{etc.},$$

the sum of which series is therefore known, if the sum of the propounded series was known.

§4 Therefore, if the series a, b, c, d etc. was of such a nature that it finally leads to constant differences, what happens, if its general term was a polynomial, the series $\frac{x}{1-x}a + \frac{x^2}{(1-x)^2}\Delta a$+etc. will have terms vanishing eventually and hence its sum can be exhibited by a finite expression. Therefore, if the first differences of the series a, b, c, d etc. were already constant, the sum of this series $ax + bx^2 + cx^3 + dx^4 +$ etc. will be

$$= \frac{x}{1-x}a + \frac{x^2}{(1-x)^2}\Delta a.$$

But if just the second differences of the coefficients of that series become constant, the sum of the propounded series will be

$$= \frac{x}{1-x}a + \frac{x^2}{(1-x)^2}\Delta a + \frac{x^3}{(1-x)^3}\Delta\Delta a.$$

Therefore, the sums of series of this kind are easily found from the differences of the coefficients.

I. *Let the sum of this series be in question*

$$1x + 3x^2 + 5x^3 + 7x^4 + 9x^5 + \text{etc.},$$

$$\text{Diff. I} \quad 2, \quad 2, \quad 2, \quad 2 \quad \text{etc.}$$

Therefore, since the first differences are constant, because of $a = 1$ and $\Delta a = 2$, the sum of the propounded series will be

$$= \frac{x}{1-x} + \frac{2xx}{(1-x)^2} = \frac{x+xx}{(1-x)^2}.$$

II. *Let the sum of this series be in question*

$$1x + 4xx + 9x^3 + 16x^4 + 25x^5 + \text{etc.}$$

Diff. I	3,	5,	7,	9,	etc.
Diff. II	2	2	2	etc.	

Therefore, since $a = 1$, $\Delta a = 3$, $\Delta^2 a = 2$, the sum of the propounded series will be

$$= \frac{x}{1-x} + \frac{3xx}{(1-x)^2} + \frac{2x^3}{(1-x)^3} = \frac{x+xx}{(1-x)^3}.$$

III. *Let the sum of this series be in question*

$$S = 4x + 15x^2 + 40x^3 + 85x^4 + 156x^5 + 259x^6 + \text{etc.}$$

Diff. I	11,		25,		45,		71,		103		etc.
Diff. II		14,		20,		26,		32,		etc.	
Diff. III			6,		6,		6,		etc.		

Because $a = 4$, $\Delta a = 11$, $\Delta^2 a = 14$, $\Delta^3 a = 6$, the sum will be

$$S = \frac{4x}{1-x} + \frac{11xx}{(1-x)^2} + \frac{14x^3}{(1-x)^3} + \frac{6x^4}{(1-x)^4}$$

or

$$S = \frac{4x - xx + 4x^3 - x^4}{(1-x)^4} = \frac{x(1+xx)(4-x)}{(1-x)^4}.$$

§5 Although this way the sums of these infinite series are found, nevertheless using the same principles the finite counterparts of these series, i.e. series consisting of a finite number of terms, can be summed. For, let this series be propounded

$$S = ax + bx^2 + cx^3 + dx^4 + \cdots\cdots + ox^n,$$

and first let its sum be in question, if the series actually an infinite series; then the sum will be

$$= \frac{x}{1-x}a + \frac{x^2}{(1-x)^2}\Delta a + \frac{x^3}{(1-x)^3}\Delta^2 a + \text{etc.}$$

Now consider the terms of the same series following after the last ox^n which we want to put

$$px^{n+1} + qx^{n+2} + rx^{n+3} + sx^{n+4} + \text{etc.};$$

the sum of this series, if divided by x^n, can be found as before; this sum, multiplied by x^n again, will be

$$\frac{x^{n+1}}{1-x}p + \frac{x^{n+2}}{(1-x)^2}\Delta p + \frac{x^{n+3}}{(1-x)^3}\Delta^2 p + \text{etc.};$$

if the sum of this series is subtracted from the sum of the infinite series, the sum of the propounded portion in question will remain, i.e.

$$S = \frac{x}{1-x}(a - x^p) + \frac{x^2}{(1-x)^2}(\Delta a - x^n \Delta p) + \frac{x^3}{(1-x)^3}(\Delta^2 a - x^n \Delta^2 p) + \text{etc.}$$

I. *Let the sum of this finite series be in question*

$$S = 1x + 2x^2 + 3x^3 + 4x^4 + \cdots + nx^n.$$

Find the differences so of these coefficients as of the ones following the last term

$$\begin{array}{ccccccc|ccccccc}
1, & & 2, & & 3, & & 4, & \text{etc.} & n+1, & & n+2, & & n+3, & \text{etc.} \\
& 1, & & 1, & & 1, & & \text{etc.} & & 1, & & 1, & & \text{etc.}
\end{array}$$

and it will be $a = 1$, $\Delta a = 1$, $p = n+1$, $\Delta p = 1$, whence the sum in question is

$$s = \frac{x}{1-x}(1 - (n+1)x^n) + \frac{x^2}{(1-x)^2}(1 - x^n)$$

or

$$S = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}.$$

II. *Let the sum of this finite series be in question*

$$S = 1 + x + 4x + 9x^3 + 16x^4 + \cdots\cdots + n^2x^n.$$

At first, investigate the differences this way

$$\begin{array}{ccccccc|ccccc}
1, & & 4, & & 9, & & 16, \quad \text{etc.} & (n+1)^2, & & (n+2)^2, & & (n+3)^2, \quad \text{etc.} \\
& 3, & & 5, & & 7, & \text{etc.} & & 2n+3, & & 2n+5, & \text{etc.} \\
& & 2, & & 2, & & \text{etc.} & & & 2, & & \text{etc.}
\end{array}$$

having found which, the sum in question will be

$$S = \frac{x}{1-x}(1-(n+1)^2x^n) + \frac{x^2}{(1-x)^2}(3-(2n+3)x^n) + \frac{x^3}{(1-x)^3}(2-2x^n)$$

or

$$S = \frac{x + xx - (n+1)^2x^{n+1} + (2nn+2n-1)x^{n+2} - nnx^{n+3}}{(1-x)^3}.$$

§6 But if the propounded series does not have coefficients which are finally reduced to constant differences, the transformation exhibited here is not of any use to determine its sum. Furthermore, the sum can not even be approximated in a more convenient way applying said transformation than it is possible by actual addition of the terms of the propounded series itself. For, if in the series $ax + bx^2 + cx^3 + dx^4 + \text{etc.}$ it was $x < 1$, in which case only the summation, in the sense explained above, is actually possible, it will be $\frac{x}{1-x} > x$ and hence the new series converges less than the initial one. But if in the propounded series it was $x = 1$, all terms of the new series even become infinite, in which case this transformation will therefore be completely useless.

§7 Let us consider the series in which the signs $+$ and $-$ alternate and which will be deduced from the preceding by assuming x to be negative. Therefore, if it was

$$S = ax - bx^2 + cx^3 - dx^4 + ex^5 - \text{etc.},$$

the negative of which series results, if in the preceding series one takes a negative x, as before, let us take the differences Δa, $\Delta^2 a$, $\Delta^3 a$ etc. of the series of coefficients a, b, c, d, e etc., having attributed the signs to the powers of x, and the propounded series will be transformed into this one

$$S = \frac{x}{1+x}a - \frac{x^2}{(1+x)^2}\Delta a + \frac{x^3}{(1+x)^3}\Delta^2 a - \frac{x^4}{(1+x)^4}\Delta^3 a + \text{etc.},$$

whence it is seen that the propounded series can be summed in the same cases as the preceding one, of course, if the series a, b, c, d etc. has finally constant iterated differences.

§8 But in this case, this transformation yields a convenient approximation of the value of the propounded series $ax - bx^2 + cx^3 - dx^4 + ex^5 - fx^6 + \text{etc.}$; for, no matter how large the number x is, the fraction $\frac{x}{1+x}$, in powers of which the other series is expanded, becomes smaller than 1; and if $x = 1$, it will be $\frac{x}{1+x} = \frac{1}{2}$. But if $x < 1$, say $x = \frac{1}{n}$, it will be $\frac{x}{1+x} = \frac{1}{n+1}$ and hence the series found by means of the transformation will always converge more rapidly than the initial one. Let us especially consider the case, in which $x = 1$, which is especially useful for the summation of series, and let

$$S = a - b + c - d + e - f + \text{etc.},$$

and denote the first, second and following differences of a, which the progression a, b, c, d, e etc. yields, by Δa, $\Delta^2 a$, $\Delta^3 a$ etc.; having found these, it will be

$$S = \frac{1}{2}a - \frac{1}{4}\Delta a + \frac{1}{8}\Delta^2 a - \frac{1}{16}\Delta^3 a + \text{etc.},$$

which series, if it does not actually terminate, exhibits the approximate sum conveniently.

§9 Therefore, let us show the use of this last transformation, in which we took $x = 1$, in some examples and at first certainly in examples in which the true sum can be expressed finitely. Such series are divergent series, in which the numbers a, b, c, d etc. finally lead to constant differences; since the sums of these series can not be exhibited in the usual sense of the word sum, we understand the word sum here in this sense we gave it above [§ 111 of the first part], such that the word sum means the value of the finite expression, from whose the expansion the propounded series results.

I. *Therefore, let this series due to Leibniz be propounded*

$$S = 1 - 1 + 1 - 1 + 1 - 1 + \text{etc.};$$

because in this series all terms are equal, all differences will become $= 0$ and hence, because of $a = 1$, it will be $S = \frac{1}{2}$.

II. *Let this series be propounded, i.e.*

$$S = 1 - 2 + 3 - 4 + 5 - 6 + \text{etc.}$$

$$\text{Diff. I} \quad 1, \quad 1, \quad 1, \quad 1, \quad 1 \quad \text{etc.}$$

Therefore, because $a = 1$, $\Delta a = 1$, it will be $S = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

III. *Let this series be propounded*

$$S = 1 - 3 + 5 - 7 + 9 - \text{etc.}$$

$$\text{Diff. II} \quad 2, \quad 2, \quad 2, \quad 2, \quad \text{etc.}$$

Because of $a = 1$ and $\Delta a = 2$, we have $S = \frac{1}{2} - \frac{2}{4} = 0$.

IV. *Let this series of the triangular numbers be propounded, i.e.*

$$1 - 3 + 6 - 10 + 15 - 21 + \text{etc.}$$

$$\begin{array}{llllllll} \text{Diff. I} & 2, & & 3, & & 4, & & 5, \quad 6, \quad \text{etc.} \\ \text{Diff. II} & & 1 & & 1, & & 1, & \quad 1 \quad \text{etc.} \end{array}$$

Here, because of $a = 1$, $\Delta a = 2$ and $\Delta\Delta a = 1$, it will be $S = \frac{1}{2} - \frac{2}{4} + \frac{1}{8} = \frac{1}{8}$.

V. *Let the series of the squares be propounded, i.e.*

$$S = 1 - 4 + 9 - 16 + 25 - 36 + \text{etc.}$$

$$\begin{array}{llllllll} \text{Diff. I} & 3, & & 5, & & 7, & & 9, \quad 11, \quad \text{etc.} \\ \text{Diff. II} & & 2 & & 2, & & 2, & \quad 2 \quad \text{etc.} \end{array}$$

Because of $a = 1$, $\Delta a = 3$, $\Delta\Delta a = 2$ it will be $S = \frac{1}{2} - \frac{3}{4} + \frac{2}{8} = 0$.

VI. *Let this series of the fourth powers be propounded, i.e.*

$$S = 1 - 16 + 81 - 256 + 625 - 1296 + \text{etc.}$$

Diff. I	15,	65,	175,	175,	369,	671,	etc.
Diff. II	50,	110,	194,	302,	etc.		
Diff. III	60,	84,	108,	etc.			
Diff. IV	24,	24,	etc.				

Therefore, it will be $S = \frac{1}{2} - \frac{15}{4} + \frac{50}{8} - \frac{60}{16} + \frac{24}{32} = 0$.

§10 If the series diverges more rapidly, as the geometric series and other similar ones, applying the transformation it is immediately transformed into a series converging more, which, if it did not already converge quickly enough, in like manner will be converted into another converging more rapidly.

I. *Let this geometric series be propounded*

$$s = 1 - 2 + 4 - 8 + 16 - 32 + \text{etc.}$$

Diff. I	1,	2,	4,	8,	16,	etc.
Diff. II	1,	2,	4,	8,	etc.	
Diff. III	1,	2,	4,	etc.		

Therefore, because in all these differences the first term is $= 1$, the sum of the series will be expressed this way

$$S = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64} + \text{etc.},$$

the sum of which series is $= \frac{1}{3}$; for, it results from the expansion of the fraction $\frac{1}{2+1}$, whereas the propounded series results from $\frac{1}{1+2}$.

II. *Let this recurring series be propounded*

$$S = 1 - 2 + 5 - 12 + 29 - 70 + 169 - \text{etc.}$$

Diff. I	1,		3,		7,		17,		41,		99,		etc.
Diff. II		2,		4,		10,		24,		58,		etc.	
Diff. III			2,		6,		14,		34,		etc.		
Diff. IV				4,		8,		20,		etc.			
Diff. V					4,		12,		etc.				
Diff. VI						8,		etc.					
							etc.						

Therefore, the first terms of the continued differences constitute this double geometric series 1, 1, 2, 2, 4, 4, 8, 8, 16, 16 etc., whence it will be

$$S = \frac{1}{2} - \frac{1}{4} + \frac{2}{8} - \frac{2}{16} + \frac{4}{32} - \frac{4}{64} + \frac{8}{128} - \text{etc.};$$

therefore, because, except for the first, each two terms cancel each other, it will be $S = \frac{1}{2}$. Indeed, the propounded series results from the expansion of the fraction $\frac{1}{1+2-1} = \frac{1}{2}$, as we showed in the discussion of the nature of recurring series.

III. *Let the hypergeometric series be propounded, i.e.*

$$S = 1 - 2 + 6 - 24 + 120 - 720 + 5040 - \text{etc.},$$

whose continued differences we will investigate more conveniently this way:

	Diff. I	Diff. II	Diff. III	
1	1	3	11	
2	4	14	64	
6	18	78	426	
24	96	504	3216	
120	600	3720	27240	etc.
720	4320	30960	256320	
5040	35280	287280	2656080	
40320	322560	2943360		
362880	3265920			
3628800				

Having continued these differences further, it will be

$$S = \frac{1}{2} - \frac{1}{4} + \frac{3}{8} - \frac{11}{16} + \frac{53}{32} - \frac{309}{64} + \frac{2119}{128} - \frac{16687}{256} + \frac{148329}{512} - \frac{1468457}{1024} + \frac{16019531}{2048} - \frac{190899411}{4096} + \text{etc.}$$

Collect the two initial terms and it will be $S = \frac{1}{4} + A$, where

$$A = \frac{3}{8} - \frac{11}{16} + \frac{53}{32} - \frac{309}{64} + \frac{2119}{128} - \text{etc.}$$

If now in the same way the differences are taken, it will be

$$A = \frac{3}{2^4} - \frac{5}{2^6} + \frac{21}{2^8} - \frac{99}{2^{10}} + \frac{615}{2^{12}} - \frac{4401}{2^{14}} + \frac{36585}{2^{16}} - \frac{342207}{2^{18}} + \frac{3565323}{2^{20}} - \frac{40866525}{2^{22}} + \text{etc.}$$

Collect the two initial terms, because they converge, and it will be $A = \frac{7}{2^6} + B$ while $B = \frac{21}{2^8} - \frac{99}{2^{10}} + \text{etc.}$; taking the differences of this series again, it will be

$$B = \frac{21}{2^9} - \frac{15}{2^{12}} + \frac{159}{2^{15}} - \frac{429}{2^{18}} + \frac{5241}{2^{21}} - \frac{26283}{2^{24}} + \frac{338835}{2^{27}} - \frac{2771097}{2^{30}} + \text{etc.}$$

Collect the four initial terms into one and put $B = \frac{153}{2^{12}} + \frac{843}{2^{20}} + C$ while

$$C = \frac{5241}{2^{21}} - \frac{26283}{2^{24}} + \text{etc.}$$

and actually summing some terms it will approximately be $C = \frac{15645}{2^{24}} - \frac{60417}{2^{30}}$. Therefore, from these the sum of the series will finally be concluded to be $S = 0.40082055$, which can nevertheless only be considered to be accurate hardly further than to three or four digits because of the divergence of the series; it is nevertheless certainly smaller than the correct value. For, in another paper I found this sum to be $= 0.4036524077$, where not even the last digit deviates from the true value.

§11 But this transformation is especially useful for the transformation of an already but slowly converging series into others which converge a lot more rapidly. Since indeed the following terms are smaller than the preceding ones, the first differences become negative; therefore, in the following the nature of the sign is carefully to be taken into account.

I. *Let this series be propounded*

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \text{etc.}$$

$$\begin{array}{llllll}
\text{Diff. I} & -\frac{1}{2}, & -\frac{1}{2\cdot 3}, & -\frac{1}{3\cdot 4}, & -\frac{1}{4\cdot 5}, & \frac{1}{5\cdot 6} \quad \text{etc.} \\[2ex]
\text{Diff. II} & +\frac{1}{3}, & \frac{2}{2\cdot 3\cdot 4}, & \frac{2}{3\cdot 4\cdot 5}, & \frac{2}{4\cdot 5\cdot 6} \quad \text{etc.} & \\[2ex]
\text{Diff. III} & -\frac{1}{4}, & -\frac{2\cdot 3}{2\cdot 3\cdot 4\cdot 5}, & -\frac{2\cdot 3}{3\cdot 4\cdot 5\cdot 6} \quad \text{etc.} & & \\[2ex]
\text{Diff. IV} & +\frac{1}{5} \quad \text{etc.} & & & & \\
 & \text{etc.} & & & &
\end{array}$$

Therefore, it will be

$$S = \frac{1}{2} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 8} + \frac{1}{4\cdot 16} + \frac{1}{5\cdot 32} + \text{etc.};$$

but we already showed in the *Introductio* that both series exhibit the hyperbolic logarithm of two.

II. *Let this series for the circle be propounded, i.e.*

$$S = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.}$$

$$\text{Diff. I} \quad -\frac{2}{1\cdot 3}, \quad -\frac{2}{3\cdot 5}, \quad -\frac{2}{5\cdot 7}, \quad -\frac{2}{7\cdot 9}, \quad -\frac{2}{9\cdot 11} \quad \text{etc.}$$

$$\text{Diff. II} \quad +\frac{2\cdot 4}{1\cdot 3\cdot 5}, \quad \frac{2\cdot 4}{3\cdot 5\cdot 7}, \quad \frac{2\cdot 4}{5\cdot 7\cdot 9}, \quad \frac{2\cdot 4}{7\cdot 9\cdot 11} \quad \text{etc.}$$

$$\text{Diff. III} \quad -\frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5\cdot 7}, \quad \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7\cdot 9} \quad \text{etc.}$$

Therefore, the sum of the series will also be

$$S = \frac{1}{2} + \frac{1}{3\cdot 2} + \frac{1\cdot 2}{3\cdot 5\cdot 2} + \frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7\cdot 2} + \text{etc.}$$

or

$$2S = 1 + \frac{1}{3} + \frac{1\cdot 2}{3\cdot 5} + \frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7} + \frac{1\cdot 2\cdot 3\cdot 4}{3\cdot 5\cdot 7\cdot 9} + \text{etc.}$$

III. *Let the value of this infinite series be in question*

$$S = \log 2 - \log 3 + \log 4 - \log 5 + \log 6 - \log 7 + \log 8 - \log 9 + \text{etc.}$$

Since the differences become too irregular at the beginning, let us actually sum the terms up to $\log 10$ using tables, the value of which sum will be found to be $= -0.3911005$, and it will be

$$S = -0.3911005 + \log 10 - \log 11 + \log 12 - \log 13 + \log 14 - \log 15 + \text{etc.}$$

Take those logarithms from tables and look for their differences this way:

	Diff. I	Diff. II	Diff. III	Diff. IV	Diff. V
$\log 10 = 1.0000000$	+	−	+	−	+
$\log 11 = 1.0413927$	413927				
		36042			
$\log 12 = 1.0791812$	377885		5779		
		30263		1292	
$\log 13 = 1.1139434$	347622		4487		368
		25776		924	
$\log 14 = 1.1461280$	321846		3563		
		22213			
$\log 15 = 1.1760913$	299633				

From these one finds

$$\log 10 - \log 11 + \log 12 - \log 13 + \text{etc.}$$
$$= \frac{1.0000000}{2} - \frac{413927}{4} - \frac{36042}{8} - \frac{5779}{16} - \frac{1292}{32} - \frac{368}{64} = 0.4891606.$$

Therefore, the value of the propounded series will be

$$S = \log 2 - \log 3 + \log 4 - \log 5 + \text{etc.} = 0.0980601,$$

to which logarithm the number 1.253315 corresponds.

§12 As we obtained these transformations by substituting the fraction $\frac{y}{1\pm y}$ for x in the series, so innumerable other transformations will result, if other functions of y are substituted for x. Let again this series be propounded

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + fx^6 + \text{etc.}$$

and put $x = y(1-y)$, having done which the following series will result

$$\begin{aligned} S = ay &- ayy \\ &+ byy - 2by^3 + by^4 \\ &\qquad + cy^3 - 3cy^4 + 3cy^5 - cy^6 \\ &\qquad\qquad + dy^4 - 4dy^5 + 6dy^6 \\ &\qquad\qquad\qquad + ey^5 - 5ey^6 \\ &\qquad\qquad\qquad\qquad + fy^6 \quad \text{etc.} \end{aligned}$$

Therefore, if the one of these series was summable, at the same time the sum of the other will be known. Hence, if one puts

$$S = x + x^2 + x^3 + x^4 + x^5 + \text{etc.} = \frac{x}{1-x},$$

it will be

$$S = y - y^3 - y^4 + y^6 + y^7 - y^9 - y^{10} + \text{etc.}$$

The sum of this series will therefore also be $= \frac{y-yy}{1-y+yy}$.

§13 If the one series terminates anywhere, the sum of the other can be exhibited explicitly. Let us put $a = 1$ and that in the found series all terms after the first vanish, so that $S = y$; hence, because of $x = y - yy$, the sum of the first will be $= \frac{1}{2} - \sqrt{\frac{1}{4} - x}$. But, because of $a = 1$, it will be

$$\begin{aligned}
b &= \quad 1 = \frac{1}{4}\cdot 2^2 \\
c &= \quad 2 = \frac{1\cdot 3}{4\cdot 6}\cdot 2^4 \\
d &= \quad 5 = \frac{1\cdot 3\cdot 5}{4\cdot 6\cdot 8}\cdot 2^6 \\
e &= \ 14 = \frac{1\cdot 3\cdot 5\cdot 7}{4\cdot 6\cdot 8\cdot 10}\cdot 2^8 \\
f &= \ 42 = \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{4\cdot 6\cdot 8\cdot 10\cdot 12}\cdot 2^{10} \\
g &= 132 = \frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11}{4\cdot 6\cdot 8\cdot 10\cdot 12\cdot 14}\cdot 2^{12}
\end{aligned}$$

etc.,

whence the first series will go over into this one

$$S = \frac{1}{2} - \sqrt{\frac{1}{4} - x} = x + x^2 + 2x^3 + 5x^4 + 14x^5 + 42x^6 + 132x^7 + \text{etc.},$$

which same series is found, if the surdic quantity $\sqrt{\frac{1}{4} - x}$ is expanded into a series and is subtracted from $\frac{1}{2}$.

§14 For the transformation to extend further, let us put $x = y(1+ny)^\nu$ and the propounded series

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \text{etc.}$$

will be transformed into the following

$$\begin{aligned}
S = {} & ay + \frac{\nu}{1}nay^2 + \frac{\nu(\nu-1)}{1\cdot 2}n^2ay^3 + \frac{\nu(\nu-1)(\nu-2)}{1\cdot 2\cdot 3}n^3ay^4 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{1\cdot 2\cdot 3\cdot 4}n^4ay^5 \\
& + by^2 + \frac{2\nu}{1}nby^3 + \frac{2\nu(2\nu-1)}{1\cdot 2}n^2by^4 + \frac{2\nu(2\nu-1)(2\nu-2)}{1\cdot 2\cdot 3}n^3by^5 \\
& + cy^3 + \frac{3\nu}{1}ncy^4 + \frac{3\nu(3\nu-1)}{1\cdot 2}n^2cy^5 \\
& + dy^4 + \frac{4\nu}{1}ndy^5 \\
& \text{etc.}
\end{aligned}$$

Therefore, if the sum of this series was known, one will at the same time also have the sum of the first one and vice versa. Since n and ν can be taken arbitrarily, from one summable series innumerable other summable ones can be found.

§15 One can also do transformations of such a kind that the sum of the found series becomes irrational in the following way.

Let this series be propounded

$$S = ax + bx^3 + cx^5 + dx^7 + ex^9 + fx^{11} + \text{etc.};$$

it will be

$$Sx = ax^2 + bx^4 + cx^6 + dx^8 + ex^{10} + fx^{12} + \text{etc.}$$

Now, put

$$x = \frac{y}{\sqrt{1-nyy}};$$

it will be $xx = \frac{y^2}{1-nyy}$ and the propounded series will be transformed into this one

$$\frac{Sy}{\sqrt{1-nyy}} = ay^2 + nay^4 + n^2ay^6 + n^3ay^8 + n^4ay^{10} + \text{etc.}$$

$$+ by^4 + 2nby^6 + 3n^2by^8 + 4n^3by^{10} + \text{etc.}$$

$$+ cy^6 + 3ncy^8 + 6n^2cy^{10} + \text{etc.}$$

$$+ dy^8 + 4ndy^{10} + \text{etc.}$$

$$+ ey^{10} + \text{etc.}$$

etc.

Therefore, if the sum S was known from the first series, one will at the same time have the sum of the following series

$$\frac{S}{\sqrt{1-nyy}} = ay + (na+b)y^3 + (n^2a+2nb+c)y^5 + (n^3a+3n^2b+3nc+d)y^7 + \text{etc.}$$

§16 If one takes $n = -1$, the coefficients of this series will be the iterated differences of a from the series a, b, c, d etc.; but if in the propounded series the signs alternate, the coefficients will be these differences for $n = 1$. Therefore, let Δa, $\Delta^2 a$, $\Delta^3 a$, $\Delta^4 a$ etc. denote the first, second, third etc. differences of a of the series of the numbers a, b, c, d, e, f etc. And if it was

$$S = ax + bx^3 + cx^5 + dx^7 + ex^9 + \text{etc.},$$

having put $x = \frac{y}{\sqrt{1+yy}}$, it will be

$$\frac{S}{\sqrt{1+yy}} = ay + \Delta a \cdot y^3 + \Delta^2 a \cdot y^5 + \Delta^3 a \cdot y^7 + \text{etc.}$$

But if it was

$$S = ax - bx^3 + cx^5 - dx^7 + ex^9 - \text{etc.}$$

and one puts $x = \frac{y}{\sqrt{1-yy}}$, it will be

$$\frac{S}{\sqrt{1-yy}} = ay - \Delta a \cdot y^3 + \Delta^2 a \cdot y^5 - \Delta^3 a \cdot y^7 + \text{etc.}$$

Therefore, if the series a, b, c, d, e etc. finally leads to constant differences, both series can be summed explicitly; but this summation also follows from the preceding paragraphs.

§17 Let us put that the coefficients a, b, c, d etc. constitute this series

$$1, \quad \frac{1}{3}, \quad \frac{1}{5}, \quad \frac{1}{7}, \quad \frac{1}{9} \quad \text{etc.}$$

and, as we already saw above [§ 11, II.], it will be

$$a=1, \quad \Delta a=-\frac{2}{3}, \quad \Delta^2 a=\frac{2\cdot 4}{3\cdot 5}, \quad \Delta^3 a=-\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7} \quad \text{etc.},$$

whence we will sum the following two series.

I. Let $S=x+\frac{1}{3}x^3+\frac{1}{5}x^5+\frac{1}{7}x^7+\text{etc.}$; it will be $S=\frac{1}{2}\log\frac{1+x}{1-x}$. Now, having put $x=\frac{y}{\sqrt{1+yy}}$, it will be

$$S=\frac{1}{2}\log\frac{\sqrt{1+yy}+y}{\sqrt{1+yy}-y}=\log(\sqrt{1+yy}+y),$$

whence it will be

$$\frac{\log(\sqrt{1+yy}+y)}{\sqrt{1+yy}}=y-\frac{2}{3}y^3+\frac{2\cdot 4}{3\cdot 5}y^5-\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}y^7+\text{etc.}$$

II. Let $S=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\text{etc.}$; it will be $S=\arctan x$. Now, having put $x=\frac{y}{\sqrt{1-yy}}$, it will be

$$S=\arctan\frac{y}{\sqrt{1-yy}}=\arcsin y=\arccos\sqrt{1-yy}.$$

Therefore, one will obtain this summation

$$\frac{\arcsin y}{\sqrt{1-yy}}=y+\frac{2}{3}y^3+\frac{2\cdot 4}{3\cdot 5}y^5+\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}y^7+\text{etc.}$$

§18 One can also substitute transcendental functions of y for x and can discover other summations more difficult to find this way; but nevertheless, for the new series to not become too complex, one has to pick functions whose powers can easily be exhibited, as it is the case for the exponential quantities e^y. Therefore, having propounded this series

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + fx^6 + \text{etc.},$$

put $x = e^{ny}y$, where e denotes the number whose hyperbolic logarithm is $= 1$; it will be $x^2 = e^{2ny}y^2$, $x^3 = e^{3ny}y^3$ etc. Indeed, in general, as it is known,

$$e^z = 1 + z + \frac{z^2}{1 \cdot 2} + \frac{z^3}{1 \cdot 2 \cdot 3} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.}$$

Therefore, the propounded series will be transformed into this one

$$\begin{aligned} S = \ & ay + 1nay^2 + \frac{1}{2}n^2ay^3 + \frac{1}{6}n^3ay^4 + \frac{1}{24}n^4ay^5 + \text{etc.} \\ & + by^2 + \frac{2}{1}\,nby^3 + \frac{4}{2}n^2by^4 + \frac{8}{6}n^3by^5 + \text{etc.} \\ & + cy^3 + \frac{3}{1}\,ncy^4 + \frac{9}{2}n^2cy^5 + \text{etc.} \\ & + dy^4 + \frac{4}{1}\,ndy^5 + \text{etc.} \\ & + ey^5 + \text{etc.} \end{aligned}$$

etc.

I. Let the geometric series be propounded, i.e. $S = x + x^2 + x^3 + x^4 + x^5 +$ etc.; it will be $S = \frac{x}{1-x}$. Now, put $n = -1$ so that $x = e^{-y}y$ and $S = \frac{e^{-y}y}{1-e^{-y}y} = \frac{y}{e^y - y}$; one will find this sum

$$\frac{y}{e^y - y} = y - \frac{1}{2}y^3 - \frac{1}{6}y^4 + \frac{5}{24}y^5 + \frac{19}{120}y^6 - \text{etc.},$$

the law of which series is not recognized[1].

[1]By this Euler means that it is not possible to see an explicit formula for the general coefficient of this power series.

II. In the other series, let all terms except for the first be $= 0$; it will be

$$b = -na, \quad c = \frac{3}{2}n^2a, \quad d = -\frac{8}{3}n^3a, \quad e = \frac{125}{24}n^4a, \quad f = -\frac{54}{5}n^5a \quad \text{etc.}$$

Because therefore the sum is $S = ay$ and $x = ye^{ny}$, it will be

$$y = x - nx^2 + \frac{3}{2}n^2x^3 - \frac{8}{3}n^3x^4 + \frac{125}{24}n^4x^5 - \frac{54}{5}n^5x^6 + \text{etc.}$$

Since in these series the structure of the progression is not obvious, the summations deduced from this substitution have hardly any use. But the transformations derived from the substitution $x = \frac{y}{1 \pm y}$, which not only yield extraordinary summations but also appropriate ways to approximate the sums of series, stand out especially. Therefore, having mentioned these things in advance without resorting to differential calculus, we want to proceed to show the use of this calculus in the doctrine of series.