On the investigation of summable series *

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§19 If the sum of a series, whose terms contain the variable quantity x, was known, and which series will therefore be a function of x, then, whatever value is attributed to x, one will always be able to assign the sum of the series. Therefore, if one writes $x + dx$ instead of x, the sum of the resulting series will be equal to the sum of the first series and the differential: Hence it follows that the differential of the sum will be = the differential of the series. Because this way so the sum as each term will be multiplied by dx, if one divides by dx everywhere, one will have a new series, whose sum will be known. In like manner, if this series is differentiated again and is divided by dx, a new series will result together with its sum and this way new likewise summable series will be found from one summable series involving the variable quantity x, if that series is differentiated several times.

§20 In order to understand all this more clearly, let the variable geometric progression be propounded, whose sum is known; for,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \text{etc.}$$

If this equation is now differentiated with respect to x, it will be

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$$\frac{dx}{(1-x)^2} = dx + 2xdx + 3x^2dx + 4x^3dx + 5x^4dx + \text{etc.},$$

and having divided by dx, one will have

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \text{etc.}$$

If one differentiates the last equation once more and divides by dx, this equation will result

$$\frac{2}{(1-x)^3} = 2 + 2 \cdot 3x + 3 \cdot 4x^2 + 4 \cdot 5x^3 + 5 \cdot 6x^4 + \text{etc.}$$

or

$$\frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + \text{etc.}$$

where the coefficients are the triangular numbers. If one differentiates once again and divides by $3dx$, one will obtain

$$\frac{1}{(1-x)^4} = 1 + 4x + 10x^2 + 20x^3 + 35x^4 + \text{etc.},$$

whose coefficients are the first pyramidal numbers. And, proceeding further this way, the same series result, which are known to originate from the expansion of the fraction $\frac{1}{(1-x)^n}$.

§21 This investigation will extend even further, if, before the differentiation is done, the series and the sum are multiplied by a certain power of x or even a function of x. Therefore, because

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \text{etc.},$$

multiply by x^m on both sides and it will be

$$\frac{x^m}{1-x} = x^m + x^{m+1} + x^{m+2} + x^{m+3} + x^{m+4} + \text{etc.}$$

Now differentiate this series and, having divided the result by dx, it will be

$$\frac{mx^{m-1} - (m-1)x^m}{(1-x)^2} = mx^{m-1} + (m+1)x^m + (m+2)x^{m+1} + (m+3)x^{m+2} + \text{etc.}$$

Now divide by x^{m-1}; one will have

$$\frac{m-(m-1)x}{(1-x)^2} = \frac{m}{1-x} + \frac{x}{(1-x)^2} = m + (m+1)x + (m+2)x^2 + \text{etc.}$$

Before differentiating again multiply this equation by x^n, so that

$$\frac{mx^n}{1-x} + \frac{x^{n+1}}{(1-x)^2} = mx^n + (m+1)x^{n+1} + (m+2)x^{n+2} + \text{etc.}$$

Now, let us differentiate, and having divided by dx, it will be

$$\frac{mnx^{n-1}}{1-x} + \frac{(m+n+1)x^n}{(1-x)^2} + \frac{2x^{n+1}}{(1-x)^3}$$
$$= mnx^{n-1} + (m+1)(n+1)x^n + (m+2)(n+2)x^{n+1} + \text{etc.}$$

But, having divided by x^{n-1}, it will be

$$\frac{mn}{1-x} + \frac{(m+n+1)x}{(1-x)^2} + \frac{2xx}{(1-x)^3}$$
$$= mn + (m+1)(n+1)x + (m+2)(n+2)x^2 + \text{etc.}$$

and it will be possible to proceed further this way; indeed one will always find the same series which result from the expansions of the fractions constituting the sum.

§22 Since the sum of the geometric progression assumed at first can be assigned up to any given term, this way also series consisting of a finite number of terms will be summed. Because

$$\frac{1-x^{n+1}}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n,$$

after the differentiation and having divided the result by dx it will be

$$\frac{1}{(1-x)^2} - \frac{(n+1)x^n - nx^{n+1}}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1}.$$

Therefore, the sum of the powers of natural numbers up to a certain term can be found. For, multiply this series by x, so that

$$\frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}=x+2x^2+3x^3+\cdots+nx^n,$$

which, having differentiated it again and divided by dx, will give

$$\frac{1+x-(n+1)x^n+(2nn+2n-1)x^{n+1}-nnx^{n+2}}{(1-x)^3}=1+4x+9x^2+\cdots+n^2x^{n-1};$$

this equation multiplied by x will give

$$\frac{x+x^2-(n+1)^2x^{n+1}+(2nn+2n-1)x^{n+2}-nnx^{n+3}}{(1-x)^3}=x+4x^2+9x^3+\cdots+n^2x^n,$$

which equality, if differentiated, divided by dx and multiplied by x, will lead to this series

$$x+8x^2+27x^3+\cdots+n^2x^n,$$

whose sum can therefore be assigned. And from this in like manner it is possible to find the indefinite sum of the fourth powers and higher powers.

§23 Therefore, this method can be applied to all series containing a variable quantity and whose sum is known, of course. Because, aside from the geometric series, all recurring series enjoy the same property that they can be summed not only up to infinity but also to any given term, one will be able to find innumerable other summable series from these by the same method. Because a lot of work would be necessary, if we wanted to study this in more detail, let us consider only one single example.

Let this series be propounded

$$\frac{x}{1-x-xx}=x+x^2+2x^3+3x^4+5x^5+8x^6+13x^7+\text{etc.},$$

which equation, if differentiated and divided by dx, will give

$$\frac{1+xx}{(1-x-xx)^2}=1+2x+6x^2+12x^3+25x^4+48x^5+91x^6+\text{etc.}$$

But obviously all series resulting this way will also be recurring, whose sums can therefore even be found more naturally.

§24 Therefore, in general, if the sum of a certain series of this form

$$ax + bx^2 + cx^3 + dx^4 + \text{etc.}$$

was known, which sum we want to put $= S$, one will be able to find the sum of the same series, if each term is multiplied by terms of an arithmetic progression. For, let

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \text{etc.};$$

multiply by x^m; it will be

$$Sx^m = ax^{m+1} + bx^{m+2} + cx^{m+3} + dx^{m+4} + \text{etc.};$$

differentiate this equation and divide by dx to find

$$mSx^{m-1} + x^m \frac{dS}{dx} = (m+1)ax^m + (m+2)bx^{m+1} + (m+3)cx^{m+2} + \text{etc.};$$

divide by x^{m-1} and it will be

$$mS + \frac{xdS}{dx} = (m+1)ax + (m+2)bx^2 + (m+3)cx^3 + \text{etc.}$$

Therefore, if one wants to find the sum of the following series

$$\alpha ax + (\alpha+\beta)bx^2 + (\alpha+2\beta)cx^3 + (\alpha+3\beta)dx^4 + \text{etc.},$$

multiply the above series by β and put $m\beta + \beta = \alpha$, so that $M = \frac{\alpha-\beta}{\beta}$, and the sum of this series will be

$$= (\alpha-\beta)S + \frac{\beta xdS}{dx}.$$

§25 One will also be able to find the sum of this propounded series, if each term is multiplied by a term of a progression of second order, whose second differences are just constant, of course. For, because we already found

$$mS + \frac{xdS}{dx} = (m+1)ax + (m+2)bx^2 + (m+3)cx^3 + \text{etc.},$$

multiply this equation by x^n that

$$mSx^n + \frac{x^{n+1}dS}{dx} = (m+1)ax^{n+1} + (m+2)bx^{n+2} + \text{etc.};$$

differentiate this equation, where dx is assumed to be constant, and divide by dx; this gives

$$mnSx^{n-1} + \frac{(m+n+1)x^nS}{dx} + \frac{x^{n+1}ddS}{dx^2}$$
$$= (m+1)(n+1)ax^n + (m+2)(n+2)bx^{n+1} + \text{etc.}$$

Divide by x^{n-1} and multiply by k, so that

$$mnkS + \frac{(m+n+1)kxdS}{dx} + \frac{kx^2ddD}{dx^2}$$
$$= (m+1)(n+1)kax + (m+2)(n+2)kbx^2 + (m+3)(n+3)kcx^3 + \text{etc.}$$

Now, compare this series to the initial one; it will be

	Diff. I	Diff. II
$kmn + 1km + 1kn + 1k = \alpha$		
	$km + kn + 3k = \beta$	
$knm + 2km + 2kn + 4k = \alpha + 1\beta$		$2k = \gamma$
	$km + kn + 5k = \beta + \gamma$	
$lnm + 3km + 3kn + 9k = \alpha + 2\beta + \gamma$		

Therefore, $k = \frac{1}{2}\gamma$ and $m + n = \frac{2\beta}{\gamma} - 3$ and

$$mn = \frac{\alpha}{k} - m - n - 1 = \frac{2\alpha}{\gamma} - \frac{2\beta}{\gamma} + 2 = \frac{2(\alpha - \beta + \gamma)}{\gamma}.$$

Therefore, the sum of the series in question will be

$$(\alpha - \beta + \gamma)S + \frac{(\beta - \gamma)xdS}{dx} + \frac{\gamma x^2 ddS}{2dx^2}.$$

§26 In like manner, one will be able to find the sum of this series

$$Aa + Bbx + Ccx^2 + Ddx^3 + Eex^4 + \text{etc.},$$

if the sum S of this series was known, of course, i.e.

$$S = a + bx + cx^2 + dx^3 + dx^3 + ex^4 + fx^5 + \text{etc.}$$

and A, B, C, D etc. constitute a series eventually reduced to constant differences. For, since its general form is concluded from the preceding, assume this sum

$$\alpha S + \frac{\beta x dS}{dx} + \frac{\gamma x^2 ddS}{2dx^2} + \frac{\delta x^3 d^3 S}{6dx^3} + \frac{\varepsilon x^4 d^4 S}{24dx^4} + \text{etc.}$$

Now to find the letters α, β, γ, δ etc., expand each series and it will be

$$\begin{aligned}
\alpha S &= \alpha a + \alpha bx + \alpha cx^2 + \alpha dx^3 + \alpha ex^4 + \text{etc.} \\
\frac{\beta x dS}{dx} &= \quad + \beta bx + 2\beta cx^2 + 3\beta dx^3 + 4\beta ex^4 + \text{etc.} \\
\frac{\gamma x^2 ddS}{2dx^2} &= \quad + \gamma cx^2 + 3\gamma dx^3 + 6\gamma ex^4 + \text{etc.} \\
\frac{\delta x^3 d^3 S}{6dx^3} &= \quad + \delta dx^3 + 4\delta ex^4 + \text{etc.} \\
\frac{\varepsilon x^4 d^4 S}{24dx^4} &= \quad + \varepsilon ex^4 + \text{etc.}
\end{aligned}$$

etc.

compare this series, having arranged it according to the powers of x, to the propounded one, i.e.

$$Z = Aa + Bbx + Ccx^2 + Ddx^3 + Eex^4 + \text{etc.}$$

and having made the comparison for each term, we find

$$\begin{aligned}
\alpha &= A \\
\beta &= B - \alpha = B - A \\
\gamma &= C - 2\beta - \alpha = C - 2B + A \\
\delta &= D - 3\gamma - 3\beta - \alpha = D - 3C + 3B - A \\
&\qquad \text{etc.}
\end{aligned}$$

Having found these values, the sum in question will therefore be

$$Z = AS + \frac{(B-A)xdS}{1dx} + \frac{(C-2B+A)x^2ddS}{1\cdot 2dx^2} + \frac{(D-3C+3B-A)x^3d^3S}{1\cdot 2\cdot 3dx^3} + \text{etc.},$$

or, if the differences of the series A, B, C, D, E etc. are denoted as usual, it will be

$$Z = AS + \frac{\Delta A\cdot xdS}{1dx} + \frac{\Delta^2 A\cdot x^2d^2S}{1\cdot 2dx^2} + \frac{\Delta^3 A\cdot x^3d^3S}{1\cdot 2\cdot 3dx^3} + \text{etc.},$$

if, as we assumed, it was

$$S = a + bx + cx^2 + dx^3 + ex^4 + fx^5 + \text{etc.}$$

Therefore, if the series A, B, C, D etc. has eventually constant differences, one will be able to express the sum of the series Z finitely.

§27 Since, having taken e for the number whose hyperbolic logarithm is $= 1$,

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} + \frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \text{etc.},$$

assume this series for the first, and because $S = e^x$, it will be $\frac{dS}{dx} = e^x$, $\frac{ddS}{dx^2} = e^x$ etc. Therefore, the sum of this series composed of that one and this one: A, B, C, D etc., i.e. the series

$$A + \frac{Bx}{1} + \frac{Cx^2}{1\cdot 2} + \frac{Dx^3}{1\cdot 2\cdot 3} + \frac{Ex^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}$$

will be expressed this way

$$e^x\left(A + \frac{x\Delta A}{1} + \frac{xx\Delta^2 A}{1\cdot 2} + \frac{x^3\Delta^3 A}{1\cdot 2\cdot 3} + \frac{x^4\Delta^4 A}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}\right).$$

Hence, if this series is propounded

$$2 + \frac{5x}{1} + \frac{10x^2}{1 \cdot 2} + \frac{17x^2}{1 \cdot 2 \cdot 3} + \frac{26x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{37x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \text{etc.},$$

because of the series

$$A, \quad B, \quad C, \quad D, \quad E \quad \text{etc.},$$

$$\begin{array}{lcccccccccccc}
A & = & 2, & & 5, & & 10, & & 17, & & 26 & & \text{etc.} \\
\Delta A & = & & 3, & & 5, & & 7, & & 9 & & \text{etc.} & \\
\Delta\Delta A & = & & & 2, & & 2, & & 2 & & \text{etc.,} & &
\end{array}$$

the sum of this series

$$2 + 5x + \frac{10x^2}{2} + \frac{17x^3}{6} + \frac{26x^4}{24} + \text{etc.}$$

will be

$$= e^x(2 + 3x + xx) = e^x(1 + x)(2 + x),$$

which is immediately clear. For,

$$\begin{aligned}
2e^x &= 2 + \frac{2x}{1} + \frac{2x^2}{2} + \frac{2x^3}{6} + \frac{2x^4}{24} + \text{etc.} \\
3xe^x &= \quad + 3x + \frac{3x^2}{1} + \frac{3x^2}{2} + \frac{3x^4}{6} + \text{etc.} \\
xxe^x &= \qquad + xx + \frac{x^3}{1} + \frac{x^4}{2} + \text{etc.}
\end{aligned}$$

and in total

$$e^x(1 + 3x + xx) = 2 + 5x + \frac{10xx}{2} + \frac{17x^3}{6} + \frac{24x^4}{24} + \text{etc.}$$

§28 The things discussed up to this point apply not only to infinite series, but also to sums of series consisting of a finite number of terms; for, the coefficients a, b, c, d etc. can either be continued to infinity or can terminate at any arbitrary point. But because this does not require any further explanation, let us consider in more detail what follows from the results found up to this point. Therefore, having propounded any arbitrary series, whose terms each consist of two factors, the one group of which factors constitutes a series leading to constant differences, one will be able to assign the sum of this series, as long as, having omitted these factors, the sum was summable. Of course, if this series is propounded

$$Z = Aa + Bbx + Ccx^2 + Ddx^3 + Eex^4 + \text{etc.}$$

in which the quantities A, B, C, D, E etc. constitute a series eventually reduced to constant differences, one will be able to exhibit the sum of this series, if the sum S of the following series is known

$$S = a + bx + cx^2 + dx^3 + ex^4 + \text{etc.}$$

For, having calculated the iterated differences of the progression A, B, C, D, E etc., as we showed at the beginning of this book,

$$\begin{array}{ccccccccccccc}
A, & & B, & & C, & & D, & & E, & & F, & & \text{etc.} \\
& \Delta A & & \Delta B, & & \Delta C, & & \Delta D, & & \Delta E & & \text{etc.} & \\
& & \Delta^2 A & & \Delta^2 B, & & \Delta^2 C, & & \Delta^2 D & & \text{etc.} & & \\
& & & \Delta^3 A & & \Delta^3 B, & & \Delta^3 C, & & \text{etc.} & & & \\
& & & & \Delta^4 A & & \Delta^4 B, & & \text{etc.} & & & & \\
& & & & & \Delta^5 A & & \text{etc.} & & & & & \\
& & & & & & \text{etc.} & & & & & &
\end{array}$$

the sum of the propounded series will be

$$Z = SA + \frac{xdS}{1dx}\Delta A + \frac{x^2 ddS}{1 \cdot 2dx^2}\Delta^2 A + \frac{x^3 d^3 S}{1 \cdot 2 \cdot 3dx^3}\Delta^3 A + \text{etc.},$$

having put dx to be constant in the higher powers of S, of course.

§29 Therefore, if the series A, B, C, D etc. never leads to constant differences, the sum of the series Z will be expressed by means of a new infinite series converging more rapidly than the initial one, and hence this series will be transformed into another one equal to it. To illustrate this, let this series be propounded

$$Y = y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} + \frac{y^5}{5} + \frac{y^6}{6} + \text{etc.},$$

which is known to express $\log \frac{1}{1-y}$ such that $Y = -\log(1-y)$. Divide the series by y and put $y = x$ and $Y = yZ$, so that

$$Z = -\frac{1}{y}\log(1-y) = -\frac{1}{x}\log(1-x);$$

it will be

$$Z = 1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \frac{x^5}{6} + \text{etc.},$$

which series compared to

$$S = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \text{etc.} = \frac{1}{1-x}$$

will give these values for the series A, B, C, D, E etc.

$$\begin{array}{ccccccccc}
1, & & \frac{1}{2}, & & \frac{1}{3}, & & \frac{1}{4}, & & \frac{1}{5} \quad \text{etc.} \\
& -\frac{1}{1\cdot 2}, & & -\frac{1}{2\cdot 3}, & & -\frac{1}{3\cdot 4}, & & -\frac{1}{4\cdot 5} \quad \text{etc.} & \\
& & \frac{1\cdot 2}{1\cdot 2\cdot 3}, & & \frac{1\cdot 2}{2\cdot 3\cdot 4}, & & \frac{1\cdot 2}{3\cdot 4\cdot 5} \quad \text{etc.} & & \\
& & & -\frac{1\cdot 2\cdot 3}{1\cdot 2\cdot 3\cdot 4}, & & -\frac{1\cdot 2\cdot 3}{2\cdot 3\cdot 4\cdot 5} \quad \text{etc.} & & & \\
& & & & \text{etc.} & & & &
\end{array}$$

Therefore, it will be

$$A = 1, \quad \Delta A = -\frac{1}{2}, \quad \Delta^2 A = \frac{1}{3}, \quad \Delta^3 A = -\frac{1}{4} \quad \text{etc.}$$

Further, because $S = \frac{1}{1-x}$, it will be

$$\frac{dS}{dx} = \frac{1}{(1-x)^2}, \quad \frac{ddS}{1 \cdot 2dx^2} = \frac{1}{(1-x)^3}, \quad \frac{d^3S}{1 \cdot 2 \cdot 3dx^3} = \frac{1}{(1-x)^4} \quad \text{etc.}$$

Having substituted these values, this sum will result

$$Z = \frac{1}{1-x} - \frac{x}{2(1-x)^2} + \frac{x^2}{3(1-x)^3} - \frac{x^3}{4(1-x)^4} + \frac{x^4}{5(1-x)^5} - \text{etc.}$$

Therefore, because $x = y$ and $Y = -\log(1-y) = yZ$, it will be

$$-\log(1-y) = \frac{y}{1-y} - \frac{y^2}{2(1-y)^2} + \frac{y^3}{3(1-y)^3} - \frac{y^4}{4(1-y)^4} + \text{etc.},$$

which series obviously expresses $\log\left(1 + \frac{y}{1-y}\right) = \log\frac{1}{1-y} = -\log(1-y)$, the validity of which is even immediate considering the results demonstrated before.

§30 To illustrate the application for a case in which only odd powers occur and the signs alternate, let this series be propounded

$$Y = y - \frac{y^3}{3} + \frac{y^5}{5} - \frac{y^7}{7} + \frac{y^9}{9} - \frac{y^{11}}{11} + \text{etc.},$$

which equation is equivalent to $Y = \arctan y$.

Divide this series by y and put $\frac{Y}{y} = Z$ and $yy = x$; it will be

$$Z = 1 - \frac{x}{3} + \frac{xx}{5} - \frac{x^3}{7} + \frac{x^4}{9} - \frac{x^5}{11} + \text{etc.}$$

If it is compared to the following

$$S = 1 - x + xx - x^3 + x^4 - x^5 + \text{etc.},$$

it will be $S = \frac{1}{1+x}$ and the series of coefficients A, B, C, D etc. will become

$$\begin{array}{llllllll}
A & = 1, & \frac{1}{3}, & & \frac{1}{5}, & & \frac{1}{7}, & \frac{1}{9}, \quad \text{etc.} \\
\Delta A & = & -\frac{2}{3}, & -\frac{2}{3\cdot 5}, & & -\frac{2}{5\cdot 7}, & & -\frac{2}{7\cdot 9} \quad \text{etc.} \\
\Delta^2 A & = & \frac{2\cdot 4}{3\cdot 5}, & & \frac{2\cdot 4}{3\cdot 5\cdot 7}, & & \frac{2\cdot 4}{5\cdot 7\cdot 9} & \text{etc.} \\
\Delta^3 A & = & & -\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}, & & -\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7\cdot 9} & \text{etc.} & \\
\Delta^4 A & = & & & \frac{2\cdot 4\cdot 6\cdot 8}{3\cdot 5\cdot 7\cdot 9} \quad \text{etc.} & & & \\
& & & & \text{etc.} & & &
\end{array}$$

But, since $S = \frac{1}{1+x}$, it will be

$$\frac{dS}{1dx} = -\frac{1}{(1+x)^2}, \quad \frac{ddS}{1\cdot 2dx^2} = \frac{1}{(1+x)^3}, \quad \frac{d^3S}{1\cdot 2\cdot 3dx^3} = -\frac{1}{(1+x)^4} \quad \text{etc.}$$

Hence, having substituted these values, the form will become

$$Z = \frac{1}{1+x} + \frac{2x}{3(1+x)^2} + \frac{2\cdot 4x^2}{3\cdot 5(1+x)^3} + \frac{2\cdot 4\cdot 6x^3}{3\cdot 5\cdot 7(1+x)^4} + \text{etc.};$$

having substituted $x = yy$ again and multiplied by y, it will be

$$Y = \arctan y = \frac{y}{1+yy} + \frac{2y^3}{3(1+yy)^2} + \frac{2\cdot 4y^5}{3\cdot 5(1+yy)^3} + \frac{2\cdot 4\cdot 6y^7}{3\cdot 5\cdot 7(1+yy)^4} + \text{etc.}$$

§31 One can also transform the above series expressing the arc of a circle in another way by comparing it to the logarithmic series.

For, let us consider the series

$$Z = 1 - \frac{x}{3} + \frac{xx}{5} - \frac{x^3}{7} + \frac{x^4}{9} - \frac{x^5}{11} + \text{etc.},$$

which we want to compare to this one

$$S = \frac{1}{0} - \frac{x}{2} + \frac{xx}{4} - \frac{x^3}{6} + \frac{x^4}{8} - \text{etc.} = \frac{1}{0} - \frac{1}{2}\log(1+x),$$

and the values of the letters A, B, C, D etc. will be

$$\begin{array}{lllllllllll}
A & = & \frac{0}{1}, & & \frac{2}{3}, & & \frac{4}{5}, & & \frac{6}{7}, & & \frac{8}{9} \quad \text{etc.} \\
\Delta A & = & & \frac{2}{3}, & & \frac{+2}{3\cdot 5}, & & \frac{+2}{5\cdot 7}, & & \frac{+2}{7\cdot 9} & \text{etc.} \\
\Delta^2 A & = & & & \frac{-2\cdot 4}{3\cdot 5}, & & \frac{-2\cdot 4}{3\cdot 5\cdot 7}, & & \frac{-2\cdot 4}{5\cdot 7\cdot 9} & & \text{etc.} \\
\Delta^3 A & = & & & & \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7} & \text{etc.} \\
& & & & & \text{etc.}
\end{array}$$

Further, since $S = \frac{1}{0} - \frac{1}{2}\log(1+x)$, it will be

$$\frac{dS}{1dx} = -\frac{1}{2(1+x)}, \quad \frac{ddS}{1\cdot 2dx^2} = \frac{1}{4(1+x)^2},$$

$$\frac{d^3S}{1\cdot 2\cdot 3dx^3} = -\frac{1}{6(1+x)^3}, \quad \frac{d^4S}{1\cdot 2\cdot 3\cdot 4dx^4} = \frac{1}{8(1+x)^4} \quad \text{etc.}$$

Therefore, it will be $SA = S\frac{0}{1} = 1$ and from the remaining terms it will be

$$Z = 1 - \frac{x}{3(1+x)} - \frac{2xx}{3\cdot 5(1+x)^2} - \frac{2\cdot 4x^3}{3\cdot 5\cdot 7(1+x)^3} - \text{etc.}$$

Now, let us put $x = yy$ and multiply by y; it will be

$$Y = \arctan y = y - \frac{y^3}{3(1+yy)} - \frac{2y^5}{3\cdot 5(1+yy)^2} - \frac{2\cdot 4y^7}{3\cdot 5\cdot 7(1+yy)^3} - \text{etc.}$$

This transformation will therefore not be obstructed by the infinite term $\frac{1}{0}$ entering the series S. But if there remains any doubt, just expand each but the first term into power series in y and one will discover that indeed the series propounded initially results.

§32 Up to this point we considered only series in which all powers of the variable occurred. Now, we want proceed to other series which in each term contain the same power of the variable; the following series is of this kind

$$S = \frac{1}{a+x} + \frac{1}{b+x} + \frac{1}{c+x} + \frac{1}{d+x} + \text{etc.}$$

For, if the sum S of this series was known and is expressed by a certain function of x, by differentiating and by dividing by $-dx$, it will be

$$\frac{-dS}{dx} = \frac{1}{(a+x)^2} + \frac{1}{(b+x)^2} + \frac{1}{(c+x)^2} + \frac{1}{(d+x)^2} + \text{etc.}$$

If this series is differentiated again and divided by $-2dx$, one will recognize the series of the cubes

$$\frac{ddS}{2dx^2} = \frac{1}{(a+x)^3} + \frac{1}{(b+x)^3} + \frac{1}{(c+x)^3} + \frac{1}{(d+x)^3} + \text{etc.}$$

and this series, differentiated again and divided by $-3dx$, will give

$$\frac{-d^3S}{dx^3} = \frac{1}{(a+x)^4} + \frac{1}{(b+x)^4} + \frac{1}{(c+x)^4} + \frac{1}{(d+x)^4} + \text{etc.}$$

And in the same way, the sum of all following powers will be found, if the sum of the first series was known.

§33 But we found series of fractions of this kind involving a variable quantity in the *Introductio*, where we showed, if the half of the circumference of the circle, whose radius is $=1$, is set $=\pi$, that

$$\frac{\pi}{n \sin \frac{m}{n}\pi} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \text{etc.}$$

$$\frac{\pi \cos \frac{m}{n}\pi}{n \sin \frac{m}{n}\pi} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \text{etc.}$$

Therefore, because it is possible to assume any arbitrary numbers for m and n, let us set $n=1$ and $m=x$ so that we obtain a series similar to that one we propounded in the preceding paragraph; having done this, it will be

$$\frac{\pi}{\sin \pi x} = \frac{1}{x} + \frac{1}{1-x} - \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} + \frac{1}{3-x} - \text{etc.}$$

$$\frac{\pi \cos \pi}{\sin \pi x} = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \frac{1}{3-x} + \text{etc.}$$

Therefore, one will be able to exhibit the sums of any powers of fractions resulting from these fractions by means of differentiations.

§34 Let us consider the first series and for the sake of brevity put $\frac{\pi}{\sin \pi x} = S$; take its higher differentials alwas assuming dx to be constant, and it will be

$$\begin{aligned}
S &= \frac{1}{x} + \frac{1}{1-x} - \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} + \frac{1}{3-x} - \text{etc.}\\
\frac{-dS}{dx} &= \frac{1}{xx} - \frac{1}{(1-x)^2} - \frac{1}{(1+x)^2} + \frac{1}{(2-x)^2} + \frac{1}{(3+x)^2} - \frac{1}{(3-x)^2} - \text{etc.}\\
\frac{ddS}{2d^2x} &= \frac{1}{x^3} + \frac{1}{(1-x)^3} - \frac{1}{(1+x)^3} - \frac{1}{(2-x)^3} + \frac{1}{(3+x)^3} + \frac{1}{(3-x)^3} - \text{etc.}\\
\frac{-d^3S}{6d^3x} &= \frac{1}{x^4} - \frac{1}{(1-x)^4} - \frac{1}{(1+x)^4} + \frac{1}{(2-x)^4} + \frac{1}{(3+x)^4} - \frac{1}{(3-x)^4} - \text{etc.}\\
\frac{d^4S}{24d^4x} &= \frac{1}{x^5} + \frac{1}{(1-x)^5} - \frac{1}{(1+x)^5} - \frac{1}{(2-x)^5} + \frac{1}{(3+x)^5} + \frac{1}{(3-x)^5} - \text{etc.}\\
\frac{-d^5S}{120d^3x} &= \frac{1}{x^6} - \frac{1}{(1-x)^6} - \frac{1}{(1+x)^6} + \frac{1}{(2-x)^6} + \frac{1}{(3+x)^6} - \frac{1}{(3-x)^6} - \text{etc.}\\
&\text{etc.}
\end{aligned}$$

where it is to be noted that in the series of even powers the signs follow the same law and in like manner in the series of odd powers the structure of the signs is always the same. Therefore, the sums of all these series are found from the differentials of the expression $S = \frac{\pi}{\sin \pi x}$.

§35 To express this differentials more conveniently, let us put

$$\sin \pi = p \quad \text{and} \quad \cos \pi = q;$$

it will be

$$dp = \pi dx \cos \pi x = \pi q dx \quad \text{and} \quad dq = -\pi p dx.$$

Therefore, because $S = \frac{\pi}{p}$, it will be

$$\begin{aligned}
\frac{-dS}{dx} &= \frac{\pi^2 q}{pp} \\
\frac{ddS}{dx^2} &= \frac{\pi^3(pp+2qq)}{p^3} = \frac{\pi^3(qq+1)}{p^3} \quad \text{since} \quad pp+qq=1 \\
\frac{-d^3S}{dx^3} &= \pi^4\left(\frac{5q}{pp}+\frac{6q^3}{p^4}\right) = \frac{\pi^4(q^3+5q)}{p^4} \\
\frac{d^4S}{dx^4} &= \pi^5\left(\frac{24q^4}{p^5}+\frac{28q^2}{p^3}+\frac{5}{p}\right) = \frac{\pi^5(q^4+18q^2+5)}{p^5} \\
\frac{-d^5S}{dx^5} &= \pi^6\left(\frac{120q^5}{p^6}+\frac{180q^3}{p^4}+\frac{61q}{pp}\right) = \frac{\pi^6(q^5+58q^3+61q)}{p^6} \\
\frac{d^6S}{dx^6} &= \pi^7\left(\frac{720q^6}{p^7}+\frac{1320q^4}{p^5}+\frac{662q^2}{p^3}+\frac{61}{p}\right) = \frac{\pi^7(q^6+179q^4+479q^2+61)}{p^7} \\
\frac{-d^7S}{dx^7} &= \pi^8\left(\frac{5040q^7}{p^8}+\frac{10920q^5}{p^6}+\frac{7266q^3}{p^4}+\frac{1385q}{p^2}\right)
\end{aligned}$$

or

$$\begin{aligned}
&= \frac{\pi^8}{p^8}(q^7+543q^5+3111q^3+1385q) \\
\frac{d^8S}{dx^8} &= \pi^9\left(\frac{40320q^8}{p^9}+\frac{100800q^6}{p^7}+\frac{83664q^4}{p^5}+\frac{24568q^2}{p^3}+\frac{1385}{p}\right)
\end{aligned}$$

or

$$= \frac{\pi^9}{p^9}(q^8+1636q^6+18270q^4+19028q^2+1385)$$

etc.

These expressions are easily continued arbitrarily far; for, if it was

$$\pm\frac{d^nS}{dx^n} = \pi^{n+1}\left(\frac{\alpha q^n}{p^{n+1}}+\frac{\beta q^{n-2}}{p^{n-1}}+\frac{\gamma q^{n-4}}{p^{n-3}}+\frac{\delta q^{n-6}}{p^{n-5}}+\text{etc.}\right),$$

then its differential, having changed the signs, will be

$$\mp\frac{d^{n+1}S}{dx^{n+1}}\left\{\begin{array}{l}(n+1)\alpha\dfrac{q^{n+1}}{p^{n+2}}+(n\alpha+(n-1)\beta)\dfrac{q^{n-1}}{p^{n}}+((n-2)\beta+(n-3)\gamma)\dfrac{q^{n-3}}{p^{n-2}}\\ +((n-4)\gamma+(n-5)\delta)\dfrac{q^{n-5}}{p^{n-4}}+\text{etc.}\end{array}\right\}$$

§36 Therefore, from these series one will obtain the following sums of the series exhibited in § 34

$$\begin{aligned}
S &= \pi\cdot\frac{1}{p}\\
\frac{-dS}{dx} &= \frac{\pi^2}{1}\cdot\frac{q}{p^2}\\
\frac{ddS}{24dx^2} &= \frac{\pi^3}{2}\left(\frac{2q^2}{p^3}+\frac{1}{p}\right)\\
\frac{-d^3S}{6dx^3} &= \frac{\pi^4}{6}\left(\frac{6q^3}{p^4}+\frac{5q}{p^2}\right)\\
\frac{d^4S}{24dx^4} &= \frac{\pi^5}{24}\left(\frac{24q^4}{p^5}+\frac{28q^2}{p^3}+\frac{5}{p}\right)\\
\frac{-d^5S}{120dx^5} &= \frac{\pi^6}{120}\left(\frac{120q^5}{p^6}+\frac{180q^3}{p^4}+\frac{61q}{p^2}\right)\\
\frac{d^6S}{720dx^6} &= \frac{\pi^7}{720}\left(\frac{720q^6}{p^7}+\frac{1320q^4}{p^5}+\frac{662q^2}{p^3}+\frac{61}{p}\right)\\
\frac{-d^7S}{720dx^6} &= \frac{\pi^8}{5040}\left(\frac{5040q^7}{p^8}+\frac{10920q^5}{p^6}+\frac{7266q^3}{p^4}+\frac{1385q}{p^2}\right)\\
\frac{d^8S}{40320dx^8} &= \frac{\pi^9}{40320}\left(\frac{40320q^8}{p^9}+\frac{100800q^6}{p^7}+\frac{83664q^4}{p^5}+\frac{24568q^2}{p^3}+\frac{1385}{p}\right)\\
&\text{etc.}
\end{aligned}$$

§37 Let us treat the other series found above [§ 33] in the same way, i.e. the series

$$\frac{\pi\cos\pi x}{\sin\pi x}=\frac{1}{x}-\frac{1}{1-x}+\frac{1}{1+x}-\frac{1}{2-x}+\frac{1}{2+x}-\frac{1}{3-x}+\text{etc.}$$

and, for the sake of brevity having put $\frac{\pi\cos\pi x}{\sin\pi x} = T$, the following summations will result

$$\begin{array}{rcl}
T & = & \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \text{etc.} \\
\frac{-dT}{dx} & = & \frac{1}{x^2} + \frac{1}{(1-x)^2} + \frac{1}{(1+x)^2} + \frac{1}{(2-x)^2} + \frac{1}{(2+x)^2} + \text{etc.} \\
\frac{ddT}{2dx^2} & = & \frac{1}{x^3} - \frac{1}{(1-x)^3} + \frac{1}{(1+x)^3} - \frac{1}{(2-x)^3} + \frac{1}{(2+x)^2} - \text{etc.} \\
\frac{-d^3T}{6d^3x} & = & \frac{1}{x^4} + \frac{1}{(1-x)^4} + \frac{1}{(1+x)^4} + \frac{1}{(2-x)^4} + \frac{1}{(2+x)^4} + \text{etc.} \\
\frac{d^4T}{24dx^4} & = & \frac{1}{x^5} - \frac{1}{(1-x)^5} + \frac{1}{(1+x)^5} - \frac{1}{(2-x)^5} + \frac{1}{(2+x)^5} - \text{etc.} \\
\frac{-d^5T}{120d^5x} & = & \frac{1}{x^6} + \frac{1}{(1-x)^6} + \frac{1}{(1+x)^6} + \frac{1}{(2-x)^6} + \frac{1}{(2+x)^6} + \text{etc.} \\
 & & \text{etc.,}
\end{array}$$

where in the series of even powers all terms are positive, but in the series of odd powers the signs $+$ and $-$ alternate.

§38 To find the values of these differentials, as before, let us put

$$\sin\pi x = p \quad \text{and} \quad dq = -\pi p dx$$

that $pp + qq = 1$; it will be

$$dp = \pi q dx \quad \text{and} \quad dq = -\pi p dx.$$

Having added these values, it will be

$$
\begin{aligned}
T &= \pi \cdot \frac{q}{p} \\
\frac{-dT}{dx} &= \pi^2 \left(\frac{qq}{pp} + 1\right) = \frac{\pi^2}{pp} \\
\frac{ddT}{dx^2} &= \pi^3 \left(\frac{2q^3}{p^3} + \frac{2q}{p}\right) = \frac{2\pi^3 q}{p^3} \\
\frac{-d^3T}{dx^3} &= \pi^4 \left(\frac{6q^4}{p^4} + \frac{8qq}{pp} + 2\right) = \pi^4 \left(\frac{6qq}{p^4} + \frac{2}{pp}\right) \\
\frac{d^4T}{dx^4} &= \pi^5 \left(\frac{24q^3}{p^5} + \frac{16q}{p^3}\right) \\
\frac{-d^5T}{dx^5} &= \pi^6 \left(\frac{120q^4}{p^6} + \frac{120qq}{p^4} + \frac{16}{pp}\right) \\
\frac{d^6T}{dx^6} &= \pi^7 \left(\frac{720q^5}{p^7} + \frac{960q^3}{p^5} + \frac{272q}{p^3}\right) \\
\frac{-d^7T}{dx^7} &= \pi^8 \left(\frac{5040q^6}{p^8} + \frac{8400q^4}{p^6} + \frac{3696q^2}{q^4} + \frac{272}{p^2}\right) \\
\frac{d^8T}{dx^8} &= \pi^9 \left(\frac{40320q^7}{p^9} + \frac{80640q^5}{p^7} + \frac{48384q^3}{p^5} + \frac{7936q}{p^3}\right) \\
&\text{etc.}
\end{aligned}
$$

These formulas can easily be continued arbitrarily far. For, if

$$\pm\frac{d^nT}{dx^n} = \pi^{n+1}\left(\frac{\alpha q^{n-1}}{p^{n+1}} + \frac{\beta q^{n-3}}{p^{n-1}} + \frac{\gamma q^{n-5}}{p^{n-3}} + \frac{\delta q^{n-7}}{p^{n-5}} + \text{etc.}\right),$$

the expression for the following differential will be

$$\mp\frac{d^{n+1}T}{dx^{n+1}} = \pi^{n+2}\left(\frac{(n+1)\alpha q^n}{p^{n+2}} + \frac{(n-1)(\alpha+\beta)q^{n-2}}{p^n} + \frac{(n-3)(\beta+\gamma)q^{n-4}}{p^{n-2}} + \text{etc.}\right)$$

§39 Therefore, having put $\sin \pi x = p$ and $\cos \pi x = q$, the series of powers given in § 37 will have the following sums

$$
\begin{aligned}
T &= \pi \cdot \frac{q}{p} \\
\frac{-dT}{dx} &= \pi^2 \frac{1}{pp} \\
\frac{ddT}{2dx^2} &= \pi^3 \frac{q}{p^3} \\
\frac{-d^3T}{6dx^3} &= \pi^4 \left(\frac{qq}{\pi^4} + \frac{1}{3pp} \right) \\
\frac{-d^4T}{24dx^4} &= \pi^5 \left(\frac{q^3}{p^5} + \frac{2q}{3p^3} \right) \\
\frac{-d^5T}{120dx^5} &= \pi^6 \left(\frac{q^4}{p^6} + \frac{3qq}{p^4} + \frac{2}{15pp} \right) \\
\frac{d^6T}{720dx^6} &= \pi^7 \left(\frac{q^5}{p^7} + \frac{4q^3}{3p^5} + \frac{17q}{45p^3} \right) \\
\frac{-d^7T}{5040dx^7} &= \pi^8 \left(\frac{q^6}{p^8} + \frac{5q^4}{3p^6} + \frac{11q^2}{15p^4} + \frac{17}{315pp} \right) \\
\frac{d^8T}{40320dx^8} &= \pi^9 \left(\frac{q^7}{p^9} + \frac{6q^5}{3p^7} + \frac{6q^3}{5p^5} + \frac{62q}{315p^3} \right) \\
&\text{etc.}
\end{aligned}
$$

§40 Aside from these series, we found several others in the *Introductio* from which others can be derived by means of differentiation in the same way.

For, we showed that

$$\frac{1}{2x} - \frac{\pi\sqrt{x}}{2x \tan \pi\sqrt{x}} = \frac{1}{1-x} + \frac{1}{4-x} + \frac{1}{9-x} + \frac{1}{16-x} + \frac{1}{25-x} + \text{etc.}$$

Let us put that the sum of this series is $= S$, so that

$$S = \frac{1}{2x} - \frac{\pi}{2\sqrt{x}} \cdot \frac{\cos \pi\sqrt{x}}{\sin \pi\sqrt{x}};$$

it will be

$$\frac{dS}{dx} = -\frac{1}{2xx} + \frac{\pi}{4x\sqrt{x}} \cdot \frac{\cos \pi\sqrt{x}}{\sin \pi\sqrt{x}} + \frac{\pi\pi}{4x(\sin \pi\sqrt{x})^2},$$

which expression therefore yields the sum of this series

$$\frac{1}{(1-x)^2}+\frac{1}{(4-x)^2}+\frac{1}{(9-x)^2}+\frac{1}{(16-x)^2}+\frac{1}{(25-x)^2}+\text{etc.}$$

Further, we also showed that

$$\frac{\pi}{2\sqrt{x}}\cdot\frac{e^{2\pi\sqrt{x}}+1}{e^{2\pi\sqrt{x}}-1}-\frac{1}{2x}=\frac{1}{1+x}+\frac{1}{4+x}+\frac{1}{9+x}+\frac{1}{16+x}+\text{etc.}$$

Therefore, if this sum is put $= S$, it will be

$$\frac{-dS}{dx}=\frac{1}{(1+x)^2}+\frac{1}{(4+x)^2}+\frac{1}{(9+x)^2}+\frac{1}{(16+x)^2}+\text{etc.}$$

But

$$\frac{dS}{dx}=\frac{-\pi}{4x\sqrt{x}}\cdot\frac{e^{2\pi\sqrt{x}}+1}{e^{2\pi\sqrt{x}}-1}-\frac{\pi\pi}{x}\cdot\frac{e^{2\pi\sqrt{x}}}{(e^{2\pi\sqrt{x}}-1)^2}+\frac{1}{2xx}.$$

Therefore, the sum of this series will be

$$\frac{-dS}{dx}=\frac{\pi}{4x\sqrt{x}}\cdot\frac{e^{2\pi\sqrt{x}}+1}{e^{2\pi\sqrt{x}}-1}+\frac{\pi\pi}{x}\cdot\frac{e^{2\pi\sqrt{x}}}{(e^{2\pi\sqrt{x}}-1)^2}-\frac{1}{2xx}.$$

And in like manner the sums of the following powers will be found by means of further differentiation.

§41 If the value of a certain product composed of factors involving the variable letter x was known, one will be able to find innumerable summable series from it by means of the same method. For, let the value of this product

$$(1+\alpha x)(1+\beta x)(1+\gamma x)(1+\delta x)(1+\varepsilon x)\text{etc.}$$

be $= S$, a function of x, of course; taking logarithms it will be

$$\log S=\log(1+\alpha x)+\log(1+\beta x)+\log(1+\gamma x)+\log(1+\delta x)+\text{etc.}$$

Now take the differentials; after division by dx it will be

$$\frac{dS}{Sdx}=\frac{\alpha}{1+\alpha x}+\frac{\beta}{1+\beta x}+\frac{\gamma}{1+\gamma x}+\frac{\delta}{1+\delta x}+\text{etc.},$$

from further differentiation of which series the sums of any powers of these fractions will be found, precisely as we explained it in more detail in the preceding examples.

§42 But, in the *Introductio* we exhibited several expressions we want to apply this method to. If π is the arc of 180° of the circle whose radius is $=1$, we showed that

$$\sin\frac{m\pi}{2n} = \frac{m\pi}{2n}\cdot\frac{4nn-mm}{4nn}\cdot\frac{16nn-mm}{16nn}\cdot\frac{26nn-mm}{36nn}\cdot\text{etc.}$$

$$\cos\frac{m\pi}{2n} = \frac{nn-mm}{nn}\cdot\frac{9nn-mm}{9nn}\cdot\frac{25nn-mm}{25nn}\cdot\frac{49nn-mm}{49nn}\cdot\text{etc.}$$

Let us put $n=1$ and $m=2x$, so that

$$\sin\pi x = \pi x\cdot\frac{1-xx}{1}\cdot\frac{4-xx}{4}\cdot\frac{9-xx}{9}\cdot\frac{16-xx}{16}\cdot\text{etc.}$$

or

$$\sin\pi x = \pi x\cdot\frac{1-x}{1}\cdot\frac{1+x}{1}\cdot\frac{2-x}{2}\cdot\frac{2+x}{2}\cdot\frac{3-x}{3}\cdot\frac{3+x}{3}\cdot\frac{4-x}{4}\cdot\text{etc.}$$

and

$$\cos\pi x = \frac{1-4xx}{1}\cdot\frac{9-4xx}{9}\cdot\frac{25-4xx}{25}\cdot\frac{49-4xx}{49}\cdot\text{etc.}$$

or

$$\cos\pi x = \frac{1-2x}{1}\cdot\frac{1+2x}{1}\cdot\frac{3-2x}{3}\cdot\frac{3+2x}{3}\cdot\frac{5-2x}{5}\cdot\frac{5+2x}{5}\cdot\text{etc.}$$

Therefore, from these expressions, if one takes logarithms, it will be

$$\log\sin\pi x = \log\pi x+\log\frac{1-x}{1}+\log\frac{1+x}{1}+\log\frac{2-x}{2}+\log\frac{2+x}{2}+\log\frac{3-x}{3}+\text{etc.}$$

$$\log\cos\pi x = \log\frac{1-2x}{1}+\log\frac{1+2x}{1}+\log\frac{3-2x}{3}+\log\frac{3+2x}{3}+\log\frac{5-2x}{5}+\text{etc.}$$

§43 Now let us take the differentials of these series of logarithms and, having divided by dx everywhere, the first series will give

$$\frac{\pi\cos\pi x}{\sin\pi x}=\frac{1}{x}-\frac{1}{1-x}+\frac{1}{1+x}-\frac{1}{2-x}+\frac{1}{2+x}-\frac{1}{3-x}+\text{etc.}$$

which is the series we discussed in § 37. The other series on the other hand will give

$$\frac{-\pi\sin\pi x}{\cos\pi x}=-\frac{2}{1-2x}+\frac{2}{1+2x}-\frac{2}{3-2x}+\frac{2}{3+2x}-\frac{2}{5-2x}+\text{etc.}$$

Let us put $2x=z$, so that $x=\frac{z}{2}$, and divide by -2; it will be

$$\frac{\pi\sin\frac{1}{2}\pi z}{2\cos\frac{1}{2}\pi z}=\frac{1}{1-z}-\frac{1}{1+z}+\frac{1}{3-z}-\frac{1}{3+z}+\frac{1}{5-z}-\text{etc.}$$

But, since

$$\sin\frac{1}{2}\pi z=\sqrt{\frac{1-\cos\pi z}{2}}\quad\text{and}\quad\cos\frac{1}{2}\pi z=\sqrt{\frac{1+\cos\pi z}{2}},$$

it will be

$$\frac{\pi\sqrt{1-\cos\pi z}}{\sqrt{1+\cos\pi z}}=\frac{2}{1-z}-\frac{2}{1+z}+\frac{2}{3-z}+\frac{2}{5-z}-\text{etc.}$$

or, writing x instead of z,

$$\frac{\pi\sqrt{1-\cos\pi x}}{\sqrt{1+\cos\pi x}}=\frac{2}{1-x}-\frac{2}{1+x}+\frac{2}{3-x}-\frac{2}{3+x}+\frac{2}{5-x}-\text{etc.}$$

Add this series to the one found first

$$\frac{\pi\cos\pi x}{\sin\pi x}=\frac{1}{x}-\frac{1}{1-x}+\frac{1}{1+x}-\frac{1}{2-x}+\frac{1}{2+x}-\frac{1}{3-x}+\text{etc.}$$

and one will find the sum of this series

$$\frac{1}{x}+\frac{1}{1-x}-\frac{1}{1+x}-\frac{1}{2-x}+\frac{1}{2+x}+\frac{1}{3-x}-\frac{1}{3+x}-\text{etc.}$$

to be $=\frac{\pi\sqrt{1-\cos\pi x}}{\sqrt{1+\cos\pi}}+\frac{\pi\cos\pi x}{\sin\pi x}$. But this fraction $\frac{\sqrt{1-\cos\pi x}}{\sqrt{1+\cos\pi x}}$, if the numerator and denominator are multiplied by $\sqrt{1-\cos\pi x}$, goes over into $\frac{1-\cos\pi x}{\sin\pi}$. Therefore, the sum of the series will be $=\frac{\pi}{\sin\pi x}$, which is the series we considered in § 34; therefore, we will not prosecute this any further.