On Finding finite differences *

Leonhard Euler

§44 At the beginning of this book we explained, how from the finite differences of the functions their differentials can easily be found, and even derived the principle of differentials from this source. For, if the differences, if they were assumed to be finite, vanish and go over into zero, the differentials result; and because in this case many and often innumerable terms, which constitute the finite difference, are neglected, the differentials can be found a lot more easily and can be expressed both more conveniently and succinctly than the finite differences. And therefore, there seems to be no way to ascend from differentials to finite differences. Nevertheless, by the method we will use here, one will be able to determine the finite differences from the differentials of all orders of any function.

§45 Let y be an arbitrary function of x; because this function, having written $x + dx$ instead of x, goes over into $y + dy$, if one writes $x + dx$ instead of x once more, the value $y + dy$ will be increased by its differential $dy + ddy$ and it will be $= y + 2dy + ddy$ which value therefore corresponds to $x + 2dx$ written instead of x in the function y. In the same way, if we assume that x is continuously increased by its differential dx that it successively takes on the values $x + dx$, $x + 2dx$, $x + 3dx$, $x + 4dx$ etc., the corresponding values of y will be those seen in the following table.

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Values of x	Corresponding Values of the Function
x	y
$x + dx$	$y + dy$
$x + 2dx$	$y + 2dy + ddy$
$x + 3dx$	$y + 3dy + 3ddy + d^3y$
$x + 4dx$	$y + 4dy + 6ddy + 4d^3y + d^4y$
$x + 5dx$	$y + 5dy + 10ddy + 10d^3y + 5d^4y + d^5y$
$x + 6dx$	$y + 6dy + 15ddy + 20d^3y + 15d^4y + 6d^5y + d^6y$
etc.	etc.

§46 Therefore, if in general x goes over into $x + ndx$, the function y will take on this form

$$y + \frac{n}{1}dy + \frac{n(n-1)}{1 \cdot 2}ddy + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}d^3y + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}d^4y + \text{etc.};$$

even though in this expression each term is infinite times smaller than its preceding term, we nevertheless did not omit any term, so that this formula can be used for the present task. For, we will assume n to be an infinitely large number, and since we know that it can happen that the product of an infinitely large and an infinitely small quantity is equal to a finite quantity, the second term can certainly become homogeneous to the first term, i.e. the quantity ndy can represent a finite quantity. For the same reason, the third term $\frac{n(n-1)}{1 \cdot 2}ddy$, even though ddy is infinite times smaller than dy, because the one factor $\frac{n(n-1)}{1 \cdot 2}$ is infinite times larger than n, can also express a finite quantity; and hence, having put n to be an infinite number, it is not possible to neglect any term of that expression.

§47 But having put n to be an infinite number, by whatever finite number it is either increased or decreased, the resulting number will have the ratio of 1 to n and hence one can write the number n for each of the factors $n-1$, $n-2$, $n-3$, $n-4$ etc. everywhere. For, because $\frac{n(n-1)}{1 \cdot 2}ddy = \frac{1}{2}nnddy - \frac{1}{2}nddy$, the first term $\frac{1}{2}nnddy$ will have the same ratio to the second $\frac{1}{2}nddy$ as n to 1 and that second term will vanish with respect to the first; therefore, one will be able to write $\frac{1}{2}nn$ instead of $\frac{n(n-1)}{1\cdot 2}$. In like manner, the coefficient of the fourth term $\frac{n(n-1)(n-3)}{1\cdot 2\cdot 3}$ can be contracted to $\frac{n^3}{6}$ and in like manner one can neglect the numbers subtracted from n in the factors in the following. But having done this, the function y, if in one writes $x + ndx$ instead of x, while n is an infinite number, will have the following value

$$y + \frac{ndy}{1} + \frac{nnddy}{1\cdot 2} + \frac{n^3d^3y}{1\cdot 2\cdot 3} + \frac{n^4d^4y}{1\cdot 2\cdot 3\cdot 4} + \frac{n^5d^5y}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \text{etc.}$$

§48 Therefore, because having assumed n as infinitely large number, even though dx is infinitely small, the product ndx can express a finite quantity, let us put $ndx = \omega$, so that $n = \frac{\omega}{dx}$; n will certainly be an infinite number being the quotient resulting from a division of the finite quantity ω by the infinitely small quantity dx. But having used this value instead of n, if the variable quantity x is increased by a certain quantity ω or if one writes $x + \omega$ instead of x, a function y of that variable x will go over into this form

$$y + \frac{\omega dy}{1dx} + \frac{\omega^2 ddy}{1\cdot 2dx^2} + \frac{\omega^3 d^3y}{1\cdot 2\cdot 3dx^3} + \frac{\omega^4 d^4y}{1\cdot 2\cdot 3\cdot 4dx^4} + \text{etc.},$$

each term of which expression can be found by iterated differentiation of y. For, because y is a function x, we showed above that these functions $\frac{dy}{dx}$, $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3}$ etc. all exhibit finite quantities.

§49 Therefore, because, while the variable quantity x is increased by the finite quantity ω, any function y of that x is increased by its first difference, which we indicated by Δy above, where we had $\omega = \Delta x$, one will be able to find the difference of y by iterated differentiation; for, it will be

$$\Delta y = \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} + \text{etc.}$$

or

$$\Delta y = \frac{\Delta x}{1}\cdot\frac{dy}{dx} + \frac{\Delta x^2}{2}\cdot\frac{ddy}{dx^2} + \frac{\Delta x^3}{6}\cdot\frac{d^3y}{dx^3} + \frac{\Delta x^4}{24}\cdot\frac{d^4y}{dx^4} + \text{etc.}$$

And hence the finite difference Δy is expressed by a progression whose terms proceed in powers of Δx. Therefore, vice versa it is obvious, if the quantity x is increased only by an infinitely small quantity, i.e. Δx goes over into its differential dx, that all terms vanish with respect to the first term and that it will be $\Delta y = dy$; for, having set $\Delta x = dx$, the difference Δy, by definition, goes over into the differential dy.

§50 Because, if one writes $x+\omega$ instead of x, any function y of that variable x then has the following value

$$y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3 y}{6dx^3} + \frac{\omega^4 d^4 y}{24dx^4} + \text{etc.};$$

the validity of this expression can checked in examples in which the higher differentials of y finally vanish; for, in these cases the number of terms of the above expression will become finite.

EXAMPLE 1

Let the value of the expression $xx - x$ be in question, if one writes $x+1$ instead of x.

Put $y = xx - x$, and because we assume that x goes over into $x+1$, it will be $\omega = 1$; now, having taken the differentials, it will be

$$\frac{dy}{dx} = 2x - 1, \quad \frac{ddy}{dx^2} = 2, \quad \frac{d^3 y}{dx^3} = 0 \quad \text{etc.}$$

Therefore, the function $y = xx - x$, having written $x+1$ instead of x, will go over into

$$xx - x + 1(2x-1) + \frac{1}{2}\cdot 2 = xx + x.$$

But if in $xx - x$ one actually writes $x+1$ instead of x,

$$\begin{array}{rcr} xx & \text{goes over into} & xx + 2x + 1 \\ x & \text{goes over into} & x + 1 \\ \hline x - xx & \text{goes over into} & xx + x\phantom{{}+1} \end{array}$$

Therefore

EXAMPLE 2

Let the value of the expression $x^3 + xx + x$ be in question, if one puts $x+2$ instead of x.

Put $y = x^3 + xx + x$ and it will be $\omega = 2$; now, since

$$y = x^3 + xx + x,$$

it will be

$$\frac{dy}{dx} = 3xx + 2x + 1, \quad \frac{ddy}{dx^2} = 6x + 2, \quad \frac{d^3y}{dx^3} = 6, \quad \frac{d^4y}{dx^4} = 0 \quad \text{etc.}$$

Hence the value of the function $y = x^3 + xx + x$, if one substitutes $x + 2$ for x, will be the following

$$x^3 + xx + x + 2(3xx + 2x + 1) + \frac{4}{2}(6x + 2) + \frac{8}{6} \cdot 6 = x^3 + 7xx + 17x + 14,$$

which same function arises, if $x + 2$ is actually substituted for x.

EXAMPLE 3

Let the value of the expression $xx + 3x + 1$ be in question, if one writes $x - 3$ instead of x.

Therefore, it will be $\omega = -3$, and having put

$$y = xx + 3x + 1,$$

it will be

$$\frac{dy}{dx} = 2x + 3, \quad \frac{ddy}{dx^2} = 2, \quad \frac{d^3y}{dx^3} = 0 \quad \text{etc.},$$

whence, written put $x - 3$ instead of x, the function $x^2 + 3x + 1$ will go over into

$$x^2 + 3x + 1 - \frac{3}{1}(2x + 3) + \frac{9}{2} \cdot 2 = x^2 - 3x + 1.$$

§51 If a negative number is taken for ω, one will find the value, which a function of x has, if the quantity x is decreased by the given quantity ω. Of course, if one writes $x - \omega$ instead of x, an arbitrary function y of x will then have this value

$$y - \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} - \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} - \text{etc.},$$

whence all variations the function y can undergo, while the quantity x is changed arbitrarily, can be found. But if y was a polynomial function of x, since in that case one eventually gets to vanishing differentials, the varied value will be expressed finitely; but if y was not a function of this kind, the varied value will be expressed by an infinite series, whose sum, since, if the substitution is actually done, the varied value is easily assigned, can be exhibited by a finite expression.

§52 But as the first difference was found, so also the following differences can be exhibited by similar expressions. For, let x successively take on the values $x+\omega$, $x+2\omega$, $x+3\omega$, $x+4\omega$ etc. and denote the corresponding values of y by y^{I}, y^{II}, y^{III}, y^{IV} etc., as we did at the beginning of this book. Therefore, since y^{I}, y^{II}, y^{III}, y^{IV} etc. are the values y will have, if one respectively writes $x+\omega$, $x+2\omega$, $x+3\omega$, $x+4\omega$ etc. instead of y, using the demonstrated method, the values of these ys will be expressed this way:

$$y^{\mathrm{I}} = y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} + \text{etc.}$$

$$y^{\mathrm{II}} = y + \frac{2\omega dy}{dx} + \frac{4\omega^2 ddy}{2dx^2} + \frac{8\omega^3 d^3y}{6dx^3} + \frac{16\omega^4 d^4y}{24dx^4} + \text{etc.}$$

$$y^{\mathrm{III}} = y + \frac{3\omega dy}{dx} + \frac{9\omega^2 ddy}{2dx^2} + \frac{27\omega^3 d^3y}{6dx^3} + \frac{81\omega^4 d^4y}{24dx^4} + \text{etc.}$$

$$y^{\mathrm{IV}} = y + \frac{4\omega dy}{dx} + \frac{16\omega^2 ddy}{2dx^2} + \frac{64\omega^3 d^3y}{6dx^3} + \frac{256\omega^4 d^4y}{24dx^4} + \text{etc.}$$

etc.

§53 Therefore, because, if Δy, $\Delta^2 y$, $\Delta^3 y$, $\Delta^4 y$ etc. denote the first, second, third, fourth etc. differences,

$$
\begin{aligned}
\Delta y\ &= y^{\mathrm{I}} - y\\
\Delta^2 y &= y^{\mathrm{II}} - 2y^{\mathrm{I}} + y\\
\Delta^3 y &= y^{\mathrm{III}} - 3y^{\mathrm{II}} + 3y^{\mathrm{I}} - y\\
\Delta^4 y &= y^{\mathrm{IV}} - 4y^{\mathrm{III}} + 6y^{\mathrm{II}} - 4y^{\mathrm{I}} + y\\
&\text{etc.,}
\end{aligned}
$$

these differences will be expressed this way by means of differentials:

$$
\begin{aligned}
\Delta y\ &= \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} + \text{etc.}\\
\Delta^2 y &= \frac{(2^2 - 2\cdot 1)\omega^2 ddy}{2dx^2} + \frac{(2^3 - 2\cdot 1)\omega^3 d^3y}{6dx^3} + \frac{(2^4 - 2\cdot 1)\omega^4 d^4y}{24dx^4} + \text{etc.}\\
\Delta^3 y &= \frac{(3^3 - 3\cdot 2^3 + 3\cdot 1)\omega^3 d^3y}{6dx^3} + \frac{(3^4 - 3\cdot 2^4 + 3\cdot 1)\omega^4 d^4y}{24dx^4} + \text{etc.}\\
\Delta^4 y &= \frac{(4^4 - 4\cdot 3^4 + 6\cdot 2^4 - 4\cdot 1)\omega^4 d^4y}{24dx^4} + \frac{(4^5 - 4\cdot 3^5 + 6\cdot 2^5 - 4\cdot 1)\omega^5 d^5y}{120dx^5} + \text{etc.}\\
&\text{etc.}
\end{aligned}
$$

§54 It is immediately clear, of how much use these expressions of differences are in the doctrine of series and progressions, and we will explain it in greater detail later parts of this chapter. Meanwhile, we want to consider applications following immediately follows from this for the understanding of series. Although the indices of the terms of a certain series are usually assumed to constitute an arithmetic progression whose difference is 1, for the sake of broader and easier applicability, let us set the difference $= \omega$, such that, if the general term or that corresponding to the index x, was y, the following terms correspond to the indices $x+\omega$, $x+2\omega$, $x+3\omega$ etc. Therefore, if the following terms correspond to these indices

$$
\begin{array}{cccccc}
x, & x+\omega, & x+2\omega, & x+3\omega, & x+4\omega & \text{etc.}\\
y, & P, & Q, & R, & S, & \text{etc.}
\end{array}
$$

each term will be defined from y and its differentials this way:

$$P = y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} + \text{etc.}$$

$$Q = y + \frac{2\omega dy}{dx} + \frac{4\omega^2 ddy}{2dx^2} + \frac{9\omega^3 d^3y}{6dx^3} + \frac{16\omega^4 d^4y}{24dx^4} + \text{etc.}$$

$$R = y + \frac{3\omega dy}{dx} + \frac{9\omega^2 ddy}{2dx^2} + \frac{27\omega^3 d^3y}{6dx^3} + \frac{81\omega^4 d^4y}{24dx^4} + \text{etc.}$$

$$S = y + \frac{4\omega dy}{dx} + \frac{16\omega^2 ddy}{2dx^2} + \frac{64\omega^3 d^3y}{6dx^3} + \frac{256\omega^4 d^4y}{24dx^4} + \text{etc.}$$

$$T = y + \frac{5\omega dy}{dx} + \frac{25\omega^2 ddy}{2dx^2} + \frac{125\omega^3 d^3y}{6dx^3} + \frac{625\omega^4 d^4y}{24dx^4} + \text{etc.}$$

etc.

§55 If these expressions are subtracted from each other, y will no longer enter the differences and it will be

$$P - y = \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} + \text{etc.}$$

$$Q - P = \frac{\omega dy}{dx} + \frac{3\omega^2 ddy}{2dx^2} + \frac{7\omega^3 d^3y}{6dx^3} + \frac{15\omega^4 d^4y}{24dx^4} + \text{etc.}$$

$$R - Q = \frac{\omega dy}{dx} + \frac{5\omega^2 ddy}{2dx^2} + \frac{19\omega^3 d^3y}{6dx^3} + \frac{65\omega^4 d^4y}{24dx^4} + \text{etc.}$$

$$S - R = \frac{\omega dy}{dx} + \frac{7\omega^2 ddy}{2dx^2} + \frac{37\omega^3 d^3y}{6dx^3} + \frac{175\omega^4 d^4y}{24dx^4} + \text{etc.}$$

$$T - S = \frac{\omega dy}{dx} + \frac{9\omega^2 ddy}{2dx^2} + \frac{61\omega^3 d^3y}{6dx^3} + \frac{369\omega^4 d^4y}{24dx^4} + \text{etc.}$$

etc.

If these expressions are again subtracted from each other, the first differentials will also cancel each other and it will be

$$Q - 2P + y = \frac{2\omega^2 ddy}{2dx^2} + \frac{6\omega^3 d^3 y}{6dx^3} + \frac{14\omega^4 d^4 y}{24dx^4} + \text{etc.}$$

$$R - 2Q + P = \frac{2\omega^2 ddy}{2dx^2} + \frac{12\omega^3 d^3 y}{6dx^3} + \frac{50\omega^4 d^4 y}{24dx^4} + \text{etc.}$$

$$S - 2R + Q = \frac{2\omega^2 ddy}{2dx^2} + \frac{18\omega^3 d^3 y}{6dx^3} + \frac{110\omega^4 d^4 y}{24dx^4} + \text{etc.}$$

$$T - 2S + R = \frac{2\omega^2 ddy}{2dx^2} + \frac{24\omega^3 d^3 y}{6dx^3} + \frac{194\omega^4 d^4 y}{24dx^4} + \text{etc.}$$

etc.

Having subtracted them from each other again, the second differentials will also go out of the computation:

$$R - 3Q + 3P - y = \frac{6\omega^3 d^3 y}{6dx^3} + \frac{36\omega^4 d^4 y}{24dx^4} + \text{etc.}$$

$$S - 3R + 3Q - P = \frac{6\omega^3 d^3 y}{6dx^3} + \frac{60\omega^4 d^4 y}{24dx^4} + \text{etc.}$$

$$T - 3S + 3R - Q = \frac{6\omega^3 d^3 y}{6dx^3} + \frac{84\omega^4 d^4 y}{24dx^4} + \text{etc.}$$

etc.

By continuing the subtraction it will be

$$S - 4R + 6Q - 4P + y = \frac{24\omega^4 d^4 y}{24dx^4} + \text{etc.}$$

$$T - 4S + 6R - 4Q + P = \frac{24\omega^4 d^4 y}{24dx^4} + \text{etc.}$$

etc.

and

$$T - 5S + 10R - 10Q + 5P - y = \frac{120\omega^5 d^5 y}{120dx^5} + \text{etc.}$$

etc.

§56 Therefore, if y was a polynomial function of x, since its higher differentials will vanish eventually, proceeding this way, one will reach vanishing expressions. Therefore, because these expressions are differences of y, let us consider their forms and coefficients more diligently.

$$y = y$$

$$\Delta y = \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} + \frac{\omega^5 d^5y}{120dx^5} + \text{etc.}$$

$$\Delta^2 y = \frac{\omega^2 ddy}{dx^2} + \frac{3\omega^3 d^3y}{3dx^3} + \frac{7\omega^4 d^4y}{3\cdot 4dx^4} + \frac{15\omega^5 d^5y}{3\cdot 4\cdot 5dx^5} + \frac{31\omega^6 d^6y}{3\cdot 4\cdot 5\cdot 6dx^6} + \text{etc.}$$

$$\Delta^3 y = \frac{\omega^3 d^3y}{dx^3} + \frac{6\omega^4 d^4y}{4dx^4} + \frac{25\omega^5 d^5y}{4\cdot 5dx^5} + \frac{90\omega^6 d^6y}{4\cdot 5\cdot 6dx^6} + \frac{301\omega^7 d^7y}{4\cdot 5\cdot 6\cdot 7dx^7} + \text{etc.}$$

$$\Delta^4 y = \frac{\omega^4 d^4y}{dx^4} + \frac{10\omega^5 d^5y}{5dx^5} + \frac{65\omega^6 d^6y}{5\cdot 6dx^6} + \frac{350\omega^7 d^7y}{5\cdot 6\cdot 7dx^7} + \frac{1701\omega^8 d^8y}{5\cdot 6\cdot 7\cdot 8dx^8} + \text{etc.}$$

$$\Delta^5 y = \frac{\omega^5 d^5y}{dx^5} + \frac{15\omega^6 d^6y}{6dx^6} + \frac{140\omega^7 d^7y}{6\cdot 7dx^7} + \frac{1050\omega^8 d^8y}{6\cdot 7\cdot 8dx^8} + \frac{6951\omega^9 d^9y}{6\cdot 7\cdot 8\cdot 9dx^9} + \text{etc.}$$

$$\Delta^6 y = \frac{\omega^6 d^6y}{dx^6} + \frac{21\omega^7 d^7y}{7dx^7} + \frac{266\omega^8 d^8y}{7\cdot 8dx^8} + \frac{2646\omega^9 d^9y}{7\cdot 8\cdot 9dx^9} + \frac{22827\omega^{10} d^{10}y}{7\cdot 8\cdot 9\cdot 10dx^{10}} + \text{etc.}$$

etc.

§57 Let us also consider the same series continued backwards at the same time, which contains the terms corresponding to the indices $x-\omega$, $x-2\omega$, $x-3\omega$ etc.

$x-4\omega$	$x-3\omega$,	$x-2\omega$,	$x-\omega$,	x,	$x+\omega$,	$x+2\omega$,	$x+3\omega$,	$x+4\omega$	etc.
s,	r,	q,	p,	y,	P,	Q,	R,	S	etc.

Therefore, since

$$p = y - \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} - \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} - \text{etc.}$$

$$q = y - \frac{2\omega dy}{dx} + \frac{4\omega^2 ddy}{2dx^2} - \frac{8\omega^3 d^3y}{6dx^3} + \frac{16\omega^4 d^4y}{24dx^4} - \text{etc.}$$

$$r = y - \frac{3\omega dy}{dx} + \frac{9\omega^2 ddy}{2dx^2} - \frac{27\omega^3 d^3y}{6dx^3} + \frac{81\omega^4 d^4y}{24dx^4} - \text{etc.}$$

$$s = y - \frac{4\omega dy}{dx} + \frac{16\omega^2 ddy}{2dx^2} - \frac{64\omega^3 d^3y}{6dx^3} + \frac{256\omega^4 d^4y}{24dx^4} - \text{etc.}$$

etc.,

by subtracting these values from the above ones, i.e. from P, Q, R, S etc., it will be

$$\frac{P-p}{2} = \frac{\omega dy}{dx} + \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^5 d^5y}{120dx^5} + \text{etc.}$$

$$\frac{Q-q}{2} = \frac{2\omega dy}{dx} + \frac{8\omega^3 d^3y}{6dx^3} + \frac{32\omega^5 d^5y}{120dx^5} + \text{etc.}$$

$$\frac{R-r}{2} = \frac{3\omega dy}{dx} + \frac{27\omega^3 d^3y}{6dx^3} + \frac{243\omega^5 d^5y}{120dx^5} + \text{etc.}$$

$$\frac{S-s}{2} = \frac{4\omega dy}{dx} + \frac{64\omega^3 d^3y}{6dx^3} + \frac{1024\omega^5 d^5y}{120dx^5} + \text{etc.}$$

etc.

But if these terms are added to the above ones, then, as the differentials of even orders are missing here, the odd differentials will go out of the computation in this case. For, it will be

$$\frac{P+p}{2} = y + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^4 d^4y}{24dx^4} + \frac{\omega^6 d^6y}{720dx^6} + \text{etc.}$$

$$\frac{Q+q}{2} = y + \frac{4\omega^2 ddy}{2dx^2} + \frac{16\omega^4 d^4y}{24dx^4} + \frac{64\omega^6 d^6y}{720dx^6} + \text{etc.}$$

$$\frac{R+r}{2} = y + \frac{9\omega^2 ddy}{2dx^2} + \frac{81\omega^4 d^4y}{24dx^4} + \frac{729\omega^6 d^6y}{720dx^6} + \text{etc.}$$

$$\frac{S+s}{2} = y + \frac{16\omega^2 ddy}{2dx^2} + \frac{256\omega^4 d^4y}{24dx^4} + \frac{4096\omega^6 d^6y}{720dx^6} + \text{etc.}$$

etc.

§58 Since the preceding terms can all be expressed, if they are collected into one sum, the summatory term of the propounded series will result. Let the first term correspond to the index $x - n\omega$ and the first term itself will be

$$= y - \frac{n\omega dy}{dx} + \frac{n^2\omega^2}{2dx^2} - \frac{n^3\omega^3 d^3y}{6dx^3} + \frac{n^4\omega^4 d^4y}{24dx^4} + \text{etc.}$$

Therefore, since the term corresponding to the index x will be $= y$ and the number of all terms is $= n+1$, the sum of all, from the first up to y, i.e. the summatory term will be

$$\begin{aligned} = (n+1)y - & \frac{\omega dy}{dx}(1+2+3+\cdots+n) \\ + & \frac{\omega^2 ddy}{2dx^2}(1+2^2+3^2+\cdots+n^2) \\ - & \frac{\omega^3 d^3y}{6dx^3}(1+2^3+3^3+\cdots+n^3) \\ + & \frac{\omega^4 d^4y}{24dx^4}(1+2^4+3^4+\cdots+n^4) \\ - & \frac{\omega^5 d^5y}{120dx^5}(1+2^5+3^5+\cdots+n^5) \\ + & \text{etc.} \end{aligned}$$

§59 Above we exhibited the sums of each of these series [§ 62 of the first part]; if these are substituted here, the sum of the propounded series will be

$$
\begin{aligned}
= (n+1)y - \; & \frac{\omega dy}{dx} \left(\frac{1}{2}nn + \frac{1}{2}n\right) \\
+ \; & \frac{\omega^2 ddy}{2dx^2}\left(\frac{1}{3}n^3 + \frac{1}{2}nn + \frac{1}{6}n\right) \\
- \; & \frac{\omega^3 d^3y}{6dx^3}\left(\frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2\right) \\
+ \; & \frac{\omega^4 d^4y}{24dx^4}\left(\frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n\right) \\
- \; & \frac{\omega^5 d^5y}{120dx^5}\left(\frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2\right) \\
& \text{etc.,}
\end{aligned}
$$

where n will be given from the index of the first term which is the initial term of the sum. If one puts $\omega = 1$ and the index of the first term is $= 1$, the index of the second $= 2$ and of the last $= x$ such that this series is propounded

$$
\begin{aligned}
&1, \quad 2, \quad 3, \quad 4, \cdots\cdots x \\
&a, \quad b, \quad c, \quad d, \cdots\cdots y,
\end{aligned}
$$

because of $x - n = 1$ and $n = x - 1$, the sum of this series will be

$$
\begin{aligned}
= xy - \quad & \frac{dy}{dx} \left(\frac{1}{2}xx - \frac{1}{2}x\right) \\
+ \quad & \frac{ddy}{2dx^2} \left(\frac{1}{3}x^3 - \frac{1}{2}xx + \frac{1}{6}x\right) \\
- \quad & \frac{d^3y}{6dx^3} \left(\frac{1}{4}x^4 - \frac{1}{2}x^3 + \frac{1}{4}xx\right) \\
+ \quad & \frac{d^4y}{24dx^4} \left(\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}xx - \frac{1}{30}x\right) \\
+ \quad & \frac{d^5y}{120dx^4} \left(\frac{1}{6}x^6 - \frac{1}{2}x^5 + \frac{5}{12}x^4 - \frac{1}{12}x^2\right) \\
+ \quad & \frac{d^6y}{720dx^4} \left(\frac{1}{7}x^7 - \frac{1}{2}x^6 + \frac{1}{2}x^5 - \frac{1}{6}x^3 + \frac{1}{42}x\right) \\
& \text{etc.}
\end{aligned}
$$

§60 From this expression, since the number of coefficients will increase immensely, if x was a large number, hardly anything of use for the doctrine of series follows; nevertheless, it will be helpful to have mentioned other properties following from these considerations. Let the general term be x^n and indicate the summatory term by $S.y$ or $S.x$. Having used this notation everywhere, it will be

$$
\begin{aligned}
&\frac{1}{2}xx - \frac{1}{2}x = S.x - x \\
&\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{6}x = S.x^2 - x^2 \\
&\frac{1}{4}x^4 - \frac{1}{2}x^3 + \frac{1}{4}xx = S.x^3 - x^3 \\
&\text{etc.}
\end{aligned}
$$

Therefore, from the above expression one will obtain

$$S.x^n = x^{n+1} - nx^{n+1}S.x + nx^n$$
$$+\frac{n(n-1)}{1\cdot 2}x^{n-2}S.x^2 - \frac{n(n-1)}{1\cdot 2}x^n - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}S.x^3 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^n + \text{etc.}$$

But, because

$$(1-1)^n = 0 = 1 - n + \frac{n(n-1)}{1\cdot 2} - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} + \text{etc.},$$

it will be

$$n - \frac{n(n-1)}{1\cdot 2} + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} - \text{etc.} = 1$$

and hence, having excluded the case $n = 0$ in which this expression becomes $= 0$,

$$S.x^n = x^{n+1} + x^n - nx^{n-1}S.x + \frac{n(n-1)}{1\cdot 2}x^{n-2}S.x^2 - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}S.x^3$$
$$+\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}x^{n-4}S.x^4 - \text{etc.}$$

§61 To see both the validity and the applicability of this formula more clearly, let us expand some special cases and at first let $n = 1$ and it will be $S.x = x^2 + x - S.x$ and hence $S.x = \frac{xx+x}{2}$, which is well-known, of course. Therefore, let us put $n = 2$ and it will be

$$S.x^2 = x^3 + xx - 2xS.x + S.x^2,$$

which equation, since the terms $S.x^2$ appearing on both sides cancel, give the same equation, i.e. $S.x = \frac{xx+x}{2}$. But if $n = 3$, it will be

$$S.x^3 = \frac{3}{2}xS.x^2 - \frac{3}{2}x^2S.x + \frac{1}{2}x^3(x+1);$$

if one puts $n = 4$, this expression will result

$$S.x^4 = x^5 + x^4 - 4x^3S.x + 6x^2S.x^2 - 4xS.x^3 + S.x^4,$$

whence, because of the terms $S.x^4$ cancelling each other, it will be

$$S.x^3 = \frac{3}{2}xS.x^2 - x^2S.x + \frac{1}{4}x^3(x+1);$$

if from the triple of this sum the double of the preceding sum is subtracted, it will remain

$$S.x^3 = \frac{3}{2}xS.x^2 - \frac{1}{4}x^3(x+1).$$

If one puts $n = 5$, it will be

$$S.x^5 = x^5 + x^5 - 5x^4S.x + 10x^3S.x^2 - 10x^2S.x^3 + 5xS.x^4 - S.x^5$$

or

$$S.x^5 = \frac{5}{2}xS.x^4 - 5x^2S.x^3 + 5.x^3S.x^2 - \frac{5}{2}x^4S.x + \frac{1}{2}x^5(x+1)$$

and from $n = 6$ it follows

$$S.x^6 = x^7 + x^6 - 6x^5S.x + 15x^4S.x^2 - 20x^3S.x^3 + 15x^2S.x^4 - 6xS.x^5 + S.x^6$$

or

$$S.x^5 = \frac{5}{2}xS.x^4 - \frac{10}{3}x^2S.x^3 + \frac{5}{2}x^3S.x^2 - x^4S.x + \frac{1}{6}x^5(x+1).$$

§62 From these results we therefore conclude, if $n = 2m+1$, that in general it will be

$$S.x^{2m+1} = \frac{2m+1}{2}xS.x^{2m} - \frac{(2m+1)2m}{2\cdot 1\cdot 2}x^2S.x^{2m-1}$$
$$+\frac{(2m+1)2m(2m-1)}{2\cdot 1\cdot 2\cdot 3}x^3S.x^{2m-2} - \cdots - \frac{2m+1}{2}x^{2m}S.x + \frac{1}{2}x^{2m+1}(x+1).$$

But if it was $n = 2m+2$, since the terms $S.x^{2m+2}$ cancel each other, one will find

$$S.x^{2m+1} = \frac{2m+1}{2}xS.x^{2m} - \frac{(2m+1)2m}{2\cdot 3}x^2S.x^{2m-1}$$
$$+\frac{(2m-1)2m(2m+1)}{2\cdot 3\cdot 4}x^3S.x^{2m-2} - \cdots - x^{2m}S.x + \frac{1}{2m+2}x^{2m+1}(x+1).$$

Therefore, the sum of the odd powers can be determined from the sums of the lower powers in two ways and from the various combinations of these formulas infinitely many others can be formed.

§63 But the sum of the odd powers can be determined a lot more easily from the preceding ones and for this it suffices to know only the sum of the preceding even power. For, from the sums of powers exhibited above [§ 62 of the first part] it is known that the number of terms constituting the sums is only increased in the sum of the even powers, such that the sum of the odd powers consists of as many terms as the sum of the preceding even power. So, if the sum of the even power x^{2n} is

$$S.x^{2n} = \alpha x^{2n+1} + \beta x^{2n} + \gamma x^{2n-1} - \delta x^{2n-3} + \varepsilon x^{2n-5} - \text{etc.}$$

(for, we saw that after the third term each second term is missing and at the same time the signs alternate), hence the sum of the following power x^{2n+1} will be found, if each term is respectively multiplied by these numbers

$$\frac{2n+1}{2n+2}x, \quad \frac{2n+1}{2n+1}x, \quad \frac{2n+1}{2n}x, \quad \frac{2n+1}{2n-1}x, \quad \frac{2n+1}{2n-2}x \quad \text{etc.}$$

not omitting the missing terms; therefore, it will be

$$S.x^{2n+1} = \frac{2n+1}{2n+2}\alpha x^{2n+2} + \frac{2n+1}{2n+1}\beta x^{2n+1} + \frac{2n+1}{2n}\gamma x^{2n} - \frac{2n+1}{2n-1}\delta x^{2n-2}$$
$$+\frac{2n+1}{2n-4}\varepsilon x^{2n-4} - \frac{2n+1}{2n-6}\zeta x^{2n-6} + \text{etc.}$$

Therefore, if the sum of the power x^{2n} is known, from it the sum of the following power x^{2n+1} can be constructed conveniently.

§64 This investigation of the following sums is also extended to the even powers; but since the sums of these receive a new term, this term is not found by this method; nevertheless, it can always be found from the nature of the series, from which it is clear, if one puts $x = 1$, that the sum has also to become $= 1$. But vice versa from a sum of certain power one will always be able to find the sum of the preceding power. For, if it was

$$S.x^{n} = \alpha x^{n+1} + \beta x^{n} + \gamma x^{n-1} - \delta x^{n-3} + \varepsilon x^{n-5} - \zeta x^{n-7} + \text{etc.},$$

for the preceding power it will be

$$S.x^{n-1} = \frac{n+1}{n}\alpha x^n + \frac{n}{n}\beta x^{n-1} + \frac{n-1}{n}\gamma x^{n-2} - \frac{n-3}{n}\delta x^{n-4} + \text{etc.}$$

and hence one can go backwards arbitrarily far. But it is to be noted that always $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2}$ as is it already clear from the formulas given above.

§65 The attentive reader will immediately see that the sum of x^{n-1} results, if the sum of the powers x^n is differentiated and its differential is divided by ndx; and it will be $d.S.x^n = ndx \cdot S.x^{n-1}$, and since $d.x^n = nx^{n-1}dx$, it will be

$$d.S.x^n = S.nx^{n-1}dx = S.d.x^n;$$

hence the differential of the sum is equal to the sum of the differentials; so, if the general term of a certain series was $= y$ and $S.y$ was its summatory term, it will also be $S.dy = d.S.y$ in general, i.e. the sum of all differentials is equal to the differential of the sum of the terms. The validity of this equality is easily seen from those results derived above on the differentiation of series. For, because

$$S.x^n = x^n + (x-1)^n + (x-2)^n + (x-3)^n + (x-4)^n + \text{etc.},$$

it will be

$$\frac{d.S.x^n}{ndx} = x^{n-1} + (x-1)^{n-1} + (x-2)^{n-2} + (x-3)^{n-1} + \text{etc.} = S.x^{n-1},$$

which proof extends to all other series.

§66 But let us return to our starting point, i.e. to the differences of functions, on which still several things are to be remarked. Because we saw, if y was any function of x and one writes $x \pm \omega$ instead of x everywhere, that the function y will have the following value

$$y \pm \frac{\omega dy}{1dx} + \frac{\omega^2 ddy}{1 \cdot 2dx^2} \pm \frac{\omega^3 d^3y}{1 \cdot 2 \cdot 3dx^3} + \frac{\omega^4 d^4y}{1 \cdot 2 \cdot 3 \cdot 4dx^4} \pm \frac{\omega^5 d^5y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5dx^5} + \text{etc.},$$

this expression will be valid, no matter whether for ω any constant quantity or even a variable value depending on x is taken. For, having found the values of $\frac{dy}{dx}$, $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3}$ etc. by differentiation, the variability in the factors ω, ω^2, ω^3 etc. is not considered and hence it does not matter, whether ω denotes a constant quantity or a variable quantity depending on x.

§67 Therefore, let us put that $\omega = x$ and in the function y the value $x - x = 0$ is written instead of x. Therefore, if in any function of x one writes 0 instead of x everywhere, the value of the function will be

$$y - \frac{xdy}{1dx} + \frac{x^2ddy}{1 \cdot 2dx^2} - \frac{x^3d^3}{1 \cdot 2 \cdot 3dx^3} + \frac{x^4d^4y}{1 \cdot 2 \cdot 3 \cdot 4dx^4} - \text{etc.}$$

Therefore, this expression always indicates the value the function y has for $x = 0$, the validity of which statement will be illustrated by the following examples.

Example 1

Let $y = xx + ax + ab$, whose value, if one puts $x = 0$, is in question, which value is known to be $= ab$, of course.

Because $y = xx + ax + ab$, it will be

$$\frac{dy}{1dx} = 2x + a, \quad \frac{ddy}{1 \cdot 2dx^2} = 1$$

and hence the value in question results as

$$= xx + ax + ab - x(2x + a) + xx \cdot 1 = ab.$$

Example 2

Let $y = x^3 - 2x + 3$, whose value, having put $x = 0$, is in question, which value is known to be $= 3$.

Because $y = x^3 - 2x + 3$, it will be

$$\frac{dy}{dx} = 3xx - 2, \quad \frac{ddy}{1 \cdot 2dx^2} = 3x, \quad \frac{d^3y}{1 \cdot 2 \cdot 3dx^3} = 1;$$

the value in question will be found to be

$$= x^3 - 2x + 3 - x(3xx - 2) + xx \cdot 3x - x^3 \cdot 1 = 3.$$

Example 3

Let $y = \frac{x}{1-x}$, whose value, having put $x = 0$, is in question, which is known to be $= 0$.

Because $y = \frac{x}{1-x}$, it will be

$$\frac{dy}{dx} = \frac{1}{(1-x)^2}, \quad \frac{ddy}{1 \cdot 2dx^2} = \frac{1}{(1-x)^3}, \quad \frac{d^3y}{1 \cdot 2 \cdot 3dx^3} = \frac{1}{(1-x)^4} \quad \text{etc.}$$

Hence the value in question will be

$$= \frac{x}{1-x} - \frac{x}{(1-x)^2} + \frac{xx}{(1-x)^3} - \frac{x^3}{(1-x)^4} + \frac{x^4}{(1-x)^5} - \text{etc.}$$

and therefore the value of this series is $= 0$. This is also obvious from the fact that this series without the first term, i.e. $\frac{x}{(1-x)^2} - \frac{xx}{(1-x)^3} + \frac{x^3}{(1-x)^4} - \text{etc.}$ is a geometric series and its sum is $= \frac{x}{(1-x)^2 + x(1-x)} = \frac{x}{1-x}$, whence the value found will be

$$= \frac{x}{1-x} - \frac{x}{1-x} = 0.$$

Example 4

Let $y = e^x$, while e denotes the number whose hyperbolic logarithm is 1, and let the value of y be in question, if one puts $x = 0$, which value is known to be $= 1$.

Because $y = e^x$, it will be

$$\frac{dy}{dx} = e^x, \quad \frac{ddy}{dx^2} = e^x \quad \text{etc.}$$

and hence the value in question will be

$$= e^x - \frac{e^x}{1} + \frac{e^x xx}{1 \cdot 2} - \frac{e^x x^3}{1 \cdot 2 \cdot 3} + \frac{e^x x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \text{etc.}$$

$$= e^x \left(1 - \frac{x}{1} + \frac{xx}{1 \cdot 2} - \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \text{etc.}\right).$$

But above we saw that the series

$$1 - \frac{x}{1} + \frac{xx}{1 \cdot 2} - \frac{x^3}{1 \cdot 2 \cdot 3} + \text{etc.}$$

expresses the value e^{-x}; therefore, the value in question will be $e^x \cdot e^{-x} = \frac{e^x}{e^x} = 1$, of course.

Example 5

Let $y = \sin x$ and having put $x = 0$ it is obvious that it will be $y = 0$, what also the general formula will indicate.

For, if $y = \sin x$, it will be

$$\frac{dy}{dx} = \cos x, \quad \frac{ddy}{dx^2} = -\sin x, \quad \frac{d^3y}{dx^3} = -\cos x, \quad \frac{d^4y}{dx^4} = \sin x \quad \text{etc.}$$

Having put $x = 0$, the value of y will be

$$\sin x - \frac{x}{1}\cos x - \frac{xx}{1 \cdot 2}\sin x + \frac{x^3}{1 \cdot 2 \cdot 3}\cos x + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4}\sin x - \text{etc.}$$

which is

$$\begin{aligned} &= \sin x \left(1 - \frac{xx}{1 \cdot 2} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{x^6}{1 \cdot 2 \cdot 3 \cdots 6} + \text{etc.}\right) \\ &- \cos x \left(\frac{x}{1} - \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{x^7}{1 \cdot 2 \cdot 3 \cdots 7} + \text{etc.}\right) \end{aligned}$$

But the upper of these series expresses $\cos x$, the lower expresses $\sin x$, whence the value in question will be

$$= \sin x \cos x - \cos x \cdot \sin x = 0.$$

§68 Hence, vice versa we can conclude, if y was a function of x vanishing for $x = 0$, that it will be

$$y - \frac{xdy}{1dx} + \frac{xxddy}{1 \cdot 2dx^2} - \frac{x^3d^3y}{1 \cdot 2 \cdot 3dx^3} + \frac{x^4d^4y}{1 \cdot 2 \cdot 3 \cdot 4dx^4} - \text{etc.} = 0.$$

Hence this is the general equation of completely all functions of x, which, if $x = 0$, also become zero. And therefore, this equation is of such a nature, that, no matter which function of x, as long as it vanishes for vanishing x, is substituted for y, it is always satisfied. But if y was a function of x which, having put $x = 0$, has a given value $= A$, it will be

$$= y - \frac{xdy}{1dx} + \frac{x^2ddy}{1\cdot 2dx^2} - \frac{x^3d^3y}{1\cdot 2\cdot 3dx^3} + \frac{x^4d^4y}{1\cdot 2\cdot 3\cdot 4dx^4} - \text{etc.} = A,$$

which equation contains all functions of x which go over into A for $x = 0$.

§69 If one writes $2x$ or $x + x$ instead of x, any function of x, which we want to denote by y, will have this value

$$y + \frac{xdy}{1dx} + \frac{x^2ddy}{1\cdot 2dx^2} + \frac{x^3d^3y}{1\cdot 2\cdot 3dx^3} + \frac{x^4d^4y}{1\cdot 2\cdot 3\cdot 4dx^4} + \text{etc.}$$

And if we write nx instead of x, i.e. $x + (n-1)x$, the function y will have the following value

$$y + \frac{(n-1)xdy}{1dx} + \frac{(n-1)^2xxddy}{1\cdot 2dx^2} + \frac{(n-1)^3x^3d^3y}{1\cdot 2\cdot 3dx^3} + \text{etc.}$$

but if in general we write t for x, because of $t = x + t - x$, any function y of x will be transformed into the following form

$$y + \frac{(t-x)dy}{1dx} + \frac{(t-x)^2ddy}{1\cdot 2dx^2} + \frac{(t-x)^3d^3y}{1\cdot 2\cdot 3dx^3} + \text{etc.}$$

Therefore, if v was such a function of t as y is of x, since v results from y by writing t instead of x, it will be

$$v = y + \frac{(t-x)dy}{1dx} + \frac{(t-x)^2ddy}{1\cdot 2dx^2} + \frac{(t-x)^3d^3y}{1\cdot 2\cdot 3dx^3} + \text{etc.},$$

the validity of which formula can be checked in any arbitrary example.

Example

For, let $y = xx - x$; having written t instead of x, it will be obviously be $v = tt - t$, which same equation the expression we found will also reveal.

For, because of $y = xx - x$, it will be

$$\frac{dy}{dx} = 2x - 1 \quad \text{and} \quad \frac{ddy}{2dx^2} = 1;$$

therefore, it will be

$$v = xx - x + (t-x)(2x-1) + (t-x)^2$$

$$= xx - x + 2tx - 2xx - t + x + tt - 2tx + xx = tt - t.$$

Therefore, if y was a function of x, which goes over into A for $x = a$, because of $t = a$ and $v = A$, it will be

$$A = y + \frac{(a-x)dy}{1dx} + \frac{(a-x)^2 ddy}{1 \cdot 2dx^2} + \frac{(a-x)^3 d^3y}{1 \cdot 2 \cdot 3dx^3} + \text{etc.}$$

and hence all functions of x, which go over into A for $x = a$, satisfy this equation.