On the Conversion of Functions into Series *

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§70 In the last chapter we have already partially shown the application the general expressions found there for the finite differences have for the investigation of series exhibiting the value of a certain function of x. For, if y was a given function of x, the value it has for $x = 0$, will be known; and if this value is put $= A$, it will be, as we found,

$$y - \frac{xdy}{dx} + \frac{x^2ddy}{1 \cdot 2dx^2} - \frac{x^3d^3y}{1 \cdot 2 \cdot 3} + \frac{x^4d^4y}{1 \cdot 2 \cdot 3 \cdot 4dx^4} - \text{etc.} = A.$$

Therefore, we not only have a, in most cases infinite, series, whose sum is equal to the constant quantity A, even though the variable quantity x is contained in each term, but we will also be able to express the function y by means of a series; for, it will be

$$y = A + \frac{xdy}{dx} - \frac{xxddy}{1 \cdot 2dx^2} + \frac{x^3d^3y}{1 \cdot 2 \cdot 3dx^3} - \frac{x^4dy^4}{1 \cdot 2 \cdot 3 \cdot 4dx^4} + \text{etc.},$$

several examples of which were already mentioned.

§71 But for this investigation to extend further, let us put that the function y goes over into z, if one writes $x + \omega$ instead of x everywhere, so that z is such a function of $x + \omega$ as y is of x, and we showed [§ 48] that it will be

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$$z = y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{1 \cdot 2 dx^2} + \frac{\omega^3 d^3 y}{1 \cdot 2 \cdot 3 dx^3} + \frac{\omega^4 d^4 y}{1 \cdot 2 \cdot 3 \cdot 4 dx^4} + \text{etc.}$$

Therefore, since each term of this series can be found by iterated differentiation of y, having put dx to be constant, and at the same time the value of z can actually be exhibited by substituting $x + \omega$ for x, this way one will always obtain a series equal to the value of z, which, if ω was a very small quantity, converges rapidly and, by taking many terms, will yield an approximately true value of z. Hence the use of this formula will be huge for approximations.

§72 Therefore, to proceed in order in the demonstration of the vast applicability of this formula, let us at first substitute algebraic functions of x for y. Let $y = x^n$ and, if one writes $x + \omega$ instead of x, it will be $z = (x + \omega)^n$. Therefore, since

$$\frac{dy}{dx} = nx^{n-1}, \quad \frac{ddy}{dx^2} = n(n-1)x^{n-2}, \quad \frac{d^3y}{dx^3} = n(n-1)(n-2)x^{n-3},$$
$$\frac{d^4y}{dx^4} = n(n-1)(n-2)(n-3)x^{n-4} \quad \text{etc.},$$

having substituted these values, it will be

$$(x+\omega)^n = x^n + \frac{n}{1}x^{n-1}\omega + \frac{n(n-1)}{1 \cdot 2}x^{n-2}\omega^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^{n-3}\omega^3 + \text{etc.},$$

which is the well-known Newtonian expression, by which the power of the binomial $(x + \omega)^n$ is converted into a series. And the number of terms of this series is always finite, if n was a positive integer.

§73 Hence we will also be able to find the progression expressing the value of the power of the binomial in such a way that it terminates, if the exponent of the power was a negative number. For, let us set

$$\omega = \frac{-ux}{x+u};$$

it will be

$$z = (x+\omega)^n = \left(\frac{xx}{x+u}\right)^n$$

and hence one will have

$$\frac{x^{2n}}{(x+u)^n} = x^n - \frac{nx^n u}{1(x+u)} + \frac{n(n-1)x^n u^2}{1\cdot 2(x+u)^2} - \frac{n(n-1)(n-2)x^n u^3}{1\cdot 2\cdot 3(x+u)^3} + \text{etc.}$$

Divide by x^{2n} everywhere and it will be

$$(x+u)^{-n} = x^{-n} - \frac{nx^{-n} u}{1(x+u)} + \frac{n(n-1)x^{-n} u^2}{1\cdot 2(x+u)^2} - \frac{n(n-1)(n-2)x^{-n} u^3}{1\cdot 2\cdot 3(x+u)^3} + \text{etc.}$$

Now put $-n = m$ and this equation will result

$$(x+u)^m = x^m + \frac{mx^m u}{1(x+u)} + \frac{m(m+1)x^m u^2}{1\cdot 2(x+u)^2} + \frac{m(m+1)(m+2)x^m u^3}{1\cdot 2\cdot 3(x+u)^3} + \text{etc.},$$

which series, if m is a negative integer, will consist of a finite number of terms. Therefore, this series is equal to the one found first, if one writes u and m instead of ω and n; for, hence it will be

$$(x+u)^m = x^m + \frac{mx^{m-1}u}{1} + \frac{m(m-1)x^{m-2}u^2}{1\cdot 2} + \frac{m(m-1)(m-2)x^{m-3}u^3}{1\cdot 2\cdot 3} + \text{etc.}$$

§74 This same series can also be deduced from the expression given at the beginning of § 70. For, because, if y goes over into A for $x = 0$,

$$y - \frac{xdy}{dx} + \frac{xxddy}{1\cdot 2dx^2} - \frac{x^3d^3y}{1\cdot 2\cdot 3dx^3} + \frac{x^4d^4y}{1\cdot 2\cdot 3\cdot 4dx^4} - \text{etc.} = A,$$

put $y = (x+a)^n$ and it will be $A = a^n$ and, because of

$$\frac{dy}{dx} = n(x+a)^{n-1}, \quad \frac{ddy}{dx^2} = n(n-1)(x+a)^{n-2},$$
$$\frac{d^3y}{dx^3} = n(n-1)(n-2)(x+a)^{n-3} \quad \text{etc.},$$

it will be

$$(x+a)^n - \frac{n}{1}x(x+a)^{n-1} + \frac{n(n-1)}{1\cdot 2}x^2(x+a)^{n-2} - \text{etc.} = a^n;$$

divide by $a^n(x+a)^n$ and this equation will result

$$(x+a)^{-n} = a^{-n} - \frac{na^{-n}x}{1(x+a)} + \frac{n(n-1)a^{-n}x^2}{1\cdot 2(x+a)^2} - \text{etc.},$$

which, having substituted u, x and $-m$ for x, a and n respectively, this expression will turn out to be the series found before.

§75 If one substitutes fractional numbers for m, both series will continue forever; nevertheless, if u was a very small quantity with respect to x, the series will converge to the true value rapidly. Therefore, let $m = \frac{\mu}{\nu}$ and $x = a^\nu$; from the series found first it will be

$$(a^\nu + u)^{\frac{\mu}{\nu}} = a^\mu \left(1 + \frac{\mu}{\nu}\cdot\frac{u}{a^\nu} + \frac{\mu(\mu-\nu)}{\nu\cdot 2\nu}\cdot\frac{uu}{a^{2\nu}} + \frac{\mu(\mu-\nu)(\mu-2\nu)}{\nu\cdot 2\nu\cdot 3\nu}\cdot\frac{u^3}{a^{3\nu}} + \text{etc.}\right).$$

But the series found later will give

$$(a^\nu + u)^{\frac{\mu}{\nu}} = a^\mu \left(1 + \frac{\mu u}{\nu(a^\nu+u)} + \frac{\mu(\mu+\nu)u^2}{\nu\cdot 2\nu(a^\nu+u)^2} + \frac{\mu(\mu+\nu)(\mu+2\nu)u^3}{\nu\cdot 2\nu\cdot 3\nu(a^\nu+u)^3} + \text{etc.}\right).$$

But this last series converges more rapidly than the first, since its terms also decrease, if it was $u > a^\nu$, in which case the first series even diverges.

Therefore, let $\mu = 1$, $\nu = 2$, it will be

$$\sqrt{a^2+u} = a\left(1 + \frac{1u}{2(a^2+u)} + \frac{1\cdot 3u^2}{2\cdot 4(a^2+u)^2} + \frac{1\cdot 3\cdot 5u^3}{2\cdot 4\cdot 6(a^2+u)^3 + \text{etc.}}\right).$$

In like manner, by substituting the numbers 3, 4, 5 etc. for ν, while still $\mu = 1$, it will be

$$\sqrt[3]{a^3+u} = a\left(1 + \frac{1u}{3(a^3+u)} + \frac{1\cdot 4u^2}{3\cdot 6(a^3+u)^2} + \frac{1\cdot 4\cdot 7u^3}{3\cdot 6\cdot 9(a^3+u)^3} + \text{etc.}\right)$$

$$\sqrt[4]{a^4+u} = a\left(1 + \frac{1u}{4(a^4+u)} + \frac{1\cdot 5u^2}{4\cdot 8(a^4+u)^2} + \frac{1\cdot 5\cdot 9u^3}{4\cdot 8\cdot 12(a^4+u)^3} + \text{etc.}\right)$$

$$\sqrt[5]{a^5+u} = a\left(1 + \frac{1u}{5(a^5+u)} + \frac{1\cdot 6u^2}{5\cdot 10(a^5+u)^2} + \frac{1\cdot 6\cdot 11u^3}{5\cdot 10\cdot 15(a^5+u)^3} + \text{etc.}\right)$$

etc.

§76 From these formulas one can therefore easily find the root of a certain power of any number. For, having propounded the number c, find the power closest to it, either larger or smaller; in the first case u will become a negative number, in the second a positive number. But if the resulting series does not converge fast enough, multiply the number c by any power, say by f^{ν}, if the root of the power ν has to be extracted, and find the root of the number $f^{\nu}c$, which divided by f will give the root in question of the number c. The greater the number f is assumed, the more the series will converge and that especially, if a similar power a^{ν} does not deviate much from $f^{\nu}c$.

EXAMPLE 1

Let the square root of the number 2 *be in question.*

If without any further preparation one puts $a = 1$ and $u = 1$, it will be

$$\sqrt{2} = 1 + \frac{1}{2 \cdot 2} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 2^2} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 2^3} + \text{etc.};$$

even though this series already converges rapidly, it is nevertheless preferable to multiply the number 2 by a square, as 25, before, so that the product 50 deviates from another square 49 as less as possible. Therefore, find the square root of 50, which divided by 5 will give $\sqrt{2}$. But then it will be $a = 7$ and $u = 1$, whence it will be

$$\sqrt{50} = 5\sqrt{2} = 7\left(1 + \frac{1}{2 \cdot 50} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 50^2} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 50^3} + \text{etc.}\right)$$

or

$$\sqrt{2} = \frac{7}{5}\left(1 + \frac{1}{100} + \frac{1 \cdot 3}{100 \cdot 200} + \frac{1 \cdot 3 \cdot 5}{100 \cdot 200 \cdot 300} + \text{etc.}\right),$$

which is most appropriate for the calculation in decimal numbers. For, it will be

$$\begin{aligned}
\frac{7}{5} &= 1.4000000000000 \\
\frac{7}{5}\cdot\frac{1}{100} &= 140000000000 \\
\frac{7}{5}\cdot\frac{1}{100}\cdot\frac{3}{200} &= 2100000000 \\
\frac{7}{5}\cdot\frac{1}{100}\cdot\frac{3}{200}\cdot\frac{5}{300} &= 35000000 \\
\text{the preceding by } \frac{7}{400} &= 612500 \\
\text{the preceding by } \frac{9}{500} &= 11025 \\
\text{the preceding by } \frac{11}{600} &= 202 \\
\text{the preceding by } \frac{13}{400} &= 3 \\
\text{Therefore, } \sqrt{2} &= 1.4142135623730.
\end{aligned}$$

Example 2

Let the cube root of 3 *be in question.*

Multiply 3 by the cube 8 and find the cube root of 24; for, it will be $\sqrt[3]{24} = 2\sqrt[3]{3}$. Therefore, put $a = 3$ and $u = -3$ and it will be

$$\sqrt[3]{24} = 3\left(1 - \frac{1\cdot 3}{3\cdot 24} + \frac{1\cdot 4\cdot 3^2}{3\cdot 6\cdot 24^2} - \frac{1\cdot 4\cdot 7\cdot 3^2}{3\cdot 6\cdot 9\cdot 24^3} + \text{etc.}\right)$$

and

$$\sqrt[3]{3} = \frac{3}{2}\left(1 - \frac{1}{3\cdot 8} + \frac{1\cdot 4}{3\cdot 6\cdot 8^2} - \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9\cdot 8^3} + \text{etc.}\right)$$

or

$$\sqrt[3]{3} = \frac{3}{2}\left(1 - \frac{1}{24} + \frac{1}{24}\cdot\frac{4}{48} - \frac{1}{24}\cdot\frac{4}{48}\cdot\frac{7}{72} + \text{etc.}\right),$$

which series already converges rapidly, since every term is more than eight times smaller than the preceding. But if 3 is multiplied by the cube 729, it will be 2187 and $\sqrt[3]{2187} = \sqrt{13^3 - 10} = 9\sqrt[3]{3}$. Therefore, because of $a = 13$ and $u = -10$, it will be

$$\sqrt[3]{3} = \frac{13}{9}\left(1 - \frac{1 \cdot 10}{3 \cdot 2187} + \frac{1 \cdot 4 \cdot 10^2}{3 \cdot 6 \cdot 2187^2} - \frac{1 \cdot 4 \cdot 7 \cdot 10^3}{3 \cdot 6 \cdot 9 \cdot 2187^3} + \text{etc.}\right),$$

every term of which series is more than two hundred times smaller than the preceding.

§77 The expansion of the power of the binomial extends so far that all algebraic functions are comprehended by it. For, if for the sake of an example the value of this function $\sqrt{a + 2bx + cxx}$ expressed as a series is in question, this question can be answered by means of the preceding formulas considering two terms as one. Furthermore, this expansion can be done by means of the expression given first; for, if one puts $\sqrt{a + 2bx + cxx} = y$, since, having put $x = 0$, $y = a$, it will be $A = \sqrt{a}$, and since the differentials of y will be as follows

$$\frac{dy}{dx} = \frac{b + cx}{\sqrt{a + 2bx + cxx}}, \quad \frac{ddy}{dx^2} = \frac{ac - bb}{(a + 2bx + cxx)^{\frac{3}{2}}}, \quad \frac{d^3y}{dx^3} = \frac{3(bb - ac)(b + cx)}{(a + 2bx + cxx)^{\frac{5}{2}}},$$

$$\frac{d^4y}{dx^4} = \frac{3(bb - ac)(ac - 5bb - 8bcx - 4ccxx)}{(a + 2bx + cxx)^{\frac{7}{2}}} \quad \text{etc.},$$

from these one will therefore obtain

$$\sqrt{a + 2bx + cxx} - \frac{(b + cx)x}{\sqrt{a + 2bx + cxx}} - \frac{(bb - ac)xx}{2(a + 2bx + cxx)^{\frac{3}{2}}} - \frac{(bb - ac)(b + cx)x^3}{2(a + 2bx + cxx)^{\frac{5}{2}}}$$

$$- \frac{(bb - ac)(5bb - ac + 8bcx + 4ccxx)x^4}{8(a + 2bx + cxx)^{\frac{7}{2}}} - \text{etc.} = \sqrt{a}$$

Therefore, if one multiplies by $\sqrt{a + 2bx + cxx}$ everywhere, the series will be rational and it will be

$$\sqrt{a(a + 2bx + cxx)} = a + 2bx + cxx - (b + cx)x - \frac{(bb - ac)xx}{2(a + 2bx + cxx)}$$

$$-\frac{(bb-ac)(b+cx)x^3}{2(a+2bx+cxx)^2}-\frac{(bb-ac)(5bb-ac+8bcx+4ccxx)x^4}{8(a+2bx+cxx)^3}-\text{etc.}$$

or

$$\sqrt{a+2bx+cxx}=\sqrt{a}+\frac{bx}{\sqrt{a}}-\frac{(bb-ac)xx}{2(a+2bx+cxx)\sqrt{a}}-\frac{(bb-ac)(b+cx)x^3}{2(a+2bx+cxx)^2\sqrt{a}}-\text{etc.}$$

§78 Hence let us go over to transcendental functions and let us substitute them for y. Therefore, at first let $y=\log x$ and having put $x+\omega$ instead of x it will be $z=\log(x+\omega)$. But let these logarithms have a ratio of $n:1$ to the hyperbolic logarithms; and for the hyperbolic logarithms it will be $n=1$ and for the tabulated logarithms it will be $n=0.4343944819032$. Hence the differentials of $y=\log x$ will be

$$\frac{dy}{dx}=\frac{n}{x},\quad \frac{ddy}{dx^2}=-\frac{n}{x^2},\quad \frac{d^3y}{dx^3}=\frac{2n}{x^3}\quad \text{etc.,}$$

from which one concludes

$$\log(x+\omega)=\log x+\frac{n\omega}{x}-\frac{n\omega^2}{2x^2}+\frac{n\omega^3}{3x^3}-\frac{n\omega^4}{4x^4}+\text{etc.}$$

In like manner, if ω is assumed to be negative, it will be

$$\log(x-\omega)=\log x-\frac{n\omega}{x}-\frac{n\omega^2}{2x^2}-\frac{n\omega^3}{3x^3}-\frac{n\omega^4}{4x^4}-\text{etc.}$$

Therefore, if this series is subtracted from the first, it will be

$$\log\frac{x+\omega}{x-\omega}=2n\left(\frac{\omega}{x}+\frac{\omega^3}{3x^3}+\frac{\omega^5}{5x^5}+\frac{\omega^7}{7x^7}+\text{etc.}\right).$$

§79 If in the series found first

$$\log(x+\omega)=\log x+\frac{n\omega}{x}-\frac{n\omega^2}{2x^2}+\frac{n\omega^3}{3x^3}-\frac{n\omega^4}{4x^4}+\text{etc.}$$

one puts

$$\omega=\frac{xx}{u-x},$$

it will be $x+\omega=\frac{ux}{u-x}$ and

$$\log(x+\omega) = \log u + \log x - \log(u-x) = \log x + \frac{nx}{u-x} - \frac{nxx}{2(u-x)^2} + \text{etc.}$$

and

$$\log(u-x) = \log u - \frac{nx}{u-x} + \frac{nxx}{2(u-x)^2} - \frac{nx^3}{3(u-x)^3} + \text{etc.}$$

and, having taken a negative x, one will have

$$\log(u+x) = \log u + \frac{nx}{u+x} + \frac{nxx}{2(u+x)^2} + \frac{nx^3}{3(u+x)^3} + \frac{nx^4}{4(u+x)^4} + \text{etc.}$$

Therefore, the logarithms can be found by means of these series in a convenient manner, if these series converge rapidly. Indeed, the following series, which are easily deduced from those already found, will be of this kind

$$\log(x+1) = \log x + n\left(\frac{1}{x} - \frac{1}{2xx} + \frac{1}{3x^3} - \frac{1}{4x^4} + \text{etc.}\right)$$

$$\log(x-1) = \log x - n\left(\frac{1}{x} + \frac{1}{2xx} + \frac{1}{3x^3} + \frac{1}{4x^4} + \text{etc.}\right);$$

because these two series differ only in regard to the signs, if they are used for a calculation, given the logarithm of the number x, at the same time the logarithms of the two numbers $x-1$ and $x+1$ will be found. Furthermore, from the remaining series it will be

$$\log(x+1) = \log(x-1) + 2n\left(\frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \frac{1}{7x^7} + \text{etc.}\right)$$

$$\log(x-1) = \log x - n\left(\frac{1}{x-1} - \frac{1}{2(x-1)^2} + \frac{1}{3(x-1)^3} - \frac{1}{4(x-1)^4} + \text{etc.}\right)$$

$$\log(x+1) = \log x + n\left(\frac{1}{x+1} + \frac{1}{2(x+1)^2} + \frac{1}{3(x+1)^3} + \frac{1}{4(x+1)^4} + \text{etc.}\right).$$

§80 Therefore, given the logarithm of the number x, the logarithms of the contiguous numbers $x+1$ and $x-1$ can easily be found; given the logarithm of the number $x-1$, even the logarithm of the number greater by two units and vice versa will be found. Although this was shown in much detail in the *Introductio*, we will nevertheless add certain examples here.

EXAMPLE 1

Given hyperbolic logarithm of the number 10, *which is* 2.3025850919940, *to find the hyperbolic logarithms of the numbers* 11 *and* 9.

Since this question concerns hyperbolic logarithms, it will be $n = 1$ and hence one will have these series

$$\log 11 = \log 10 + \frac{1}{10} - \frac{1}{2\cdot 10^2} + \frac{1}{3\cdot 10^3} - \frac{1}{4\cdot 10^4} + \frac{1}{5\cdot 10^5} - \text{etc.}$$

$$\log 9 \;= \log 10 + \frac{1}{10} + \frac{1}{2\cdot 10^2} + \frac{1}{3\cdot 10^3} + \frac{1}{4\cdot 10^4} + \frac{1}{5\cdot 10^5} - \text{etc.}$$

To find the sums of these series, collect the even and odd terms separately and it will be

$\frac{1}{10}$	=	0.1000000000000	$\frac{1}{2\cdot 10^2}$	=	0.0050000000000
$\frac{1}{3\cdot 10^3}$	=	0.0003333333333	$\frac{1}{4\cdot 10^4}$	=	0.0000250000000
$\frac{1}{5\cdot 10^5}$	=	0.0000020000000	$\frac{1}{6\cdot 10^6}$	=	0.0000001666666
$\frac{1}{7\cdot 10^7}$	=	0.0000000142857	$\frac{1}{8\cdot 10^8}$	=	0.0000000012500
$\frac{1}{3\cdot 10^9}$	=	0.0000000001111	$\frac{1}{10\cdot 10^{10}}$	=	0.0000000000100
$\frac{1}{11\cdot 10^{11}}$	=	0.0000000000009	$\frac{1}{12\cdot 10^{12}}$	=	0.0000000000001
Sum	=	0.1003353477310	Sum	=	0.05050251679267

The sum of both will be			0.1053605156577
The difference of both will be			0.0953101798043
Now	$\log 10$	$=$	2.3025850929940
Therefore, it will be	$\log 11$	$=$	2.397895272793
and	$\log 9$	$=$	2.1972245773363
Hence further	$\log 3$	$=$	1.0986122886681
and	$\log 99$	$=$	4.5951198501346

EXAMPLE 2

Using the hyperbolic logarithm of the number 99 *just found to find the logarithm of the number* 101.

For this, apply the series found above, i.e.

$$\log(x+1) = \log(x-1) + \frac{2}{x} + \frac{2}{3x^3} + \frac{2}{5x^5} + \frac{2}{7x^7} + \text{etc.},$$

in which one has to put $x = 100$, and it will be

$$\log 101 = \log 99 + \frac{2}{100} + \frac{2}{3 \cdot 100^3} + \frac{2}{5 \cdot 100^5} + \frac{2}{7 \cdot 100^7} + \text{etc.},$$

the sum of which series is calculated to be $= 0.0200006667066$ from these four terms, which number added to $\log 99$ will give $\log 101 = 4.6151205168412$.

EXAMPLE 3

Using the given tabulated logarithm of the number 10, *which is* $= 1$, *to find the logarithm of the numbers* 11 *and* 9.

Since here we look for the common logarithm, it will be

$$n = 0.434244819032;$$

therefore, having put $x = 10$, it will be

$$\log 11 = \log 10 + \frac{n}{10} - \frac{n}{2\cdot 10^2} + \frac{n}{3\cdot 10^3} - \frac{n}{4\cdot 10^4} + \text{etc.}$$
$$\log 9 \;= \log 10 - \frac{n}{10} - \frac{n}{2\cdot 10^2} - \frac{n}{3\cdot 10^3} - \frac{n}{4\cdot 10^4} - \text{etc.}$$

Therefore, collect the even and odd terms separately

$\frac{n}{10}$	=	0.0434294481903	$\frac{n}{2\cdot 10^2}$	=	0.0021714724095
$\frac{n}{3\cdot 10^3}$	=	0.0001447648273	$\frac{n}{4\cdot 10^4}$	=	0.0000108573620
$\frac{n}{5\cdot 10^5}$	=	0.0000008685889	$\frac{n}{6\cdot 10^6}$	=	0.0000000723824
$\frac{n}{7\cdot 10^7}$	=	0.0000000062042	$\frac{n}{8\cdot 10^8}$	=	0.0000000005428
$\frac{n}{3\cdot 10^9}$	=	0.0000000000482	$\frac{n}{10\cdot 10^{10}}$	=	0.0000000000043
$\frac{n}{11\cdot 10^{11}}$	=	0.0000000000004	$\frac{n}{12\cdot 10^{12}}$	=	0.0000000000000
Sum	=	0.0435750878593	Sum	=	0.0021824027010

The aggregate of both is		=	0.0457574905603
Their difference is		=	0.0413926851583
Therefore, because	$\log 10$	=	1.0000000000000
it will be	$\log 11$	=	1.0413926851582
and	$\log 9$	=	0.9542425094397
hence	$\log 3$	=	0.4771212547198
and	$\log 99$	=	1.9956351945980

EXAMPLE 4

Using the tabulated logarithm of the number 99 *found here to find the tabulated logarithm of the number* 101.

Here, by applying the same series we used in the second example, we will have

$$\log 101 = \log 99 + 2n\left(\frac{1}{100} + \frac{1}{3\cdot 100^3} + \frac{1}{5\cdot 100^5} + \text{etc.}\right),$$

the sum of which series, having substituted the corresponding value for n, will quickly be found to be

$$\begin{array}{llcl} & & = & 0.0086861791849 \\ \text{having added which to} & \log 99 & = & 1.9956351945980 \\ \hline \text{it results} & \log 101 & = & 2.0043213737829 \end{array}$$

§81 Now, in our general expression, let us attribute an exponential value to y and let $y = a^x$; having substituted $x + \omega$ for x, it will be $z = a^{x+\omega}$, whose value, because of the differentials

$$\frac{dy}{dx} = a^x \log a, \quad \frac{ddy}{dx^2} = a^x(\log a)^2, \quad \frac{d^3y}{dx^3} = a^x(\log a)^3 \quad \text{etc.},$$

will be

$$a^{x+\omega} = a^x\left(1 + \frac{\omega\log a}{1} + \frac{\omega^2(\log a)^2}{1\cdot 2} + \frac{\omega^3(\log a)^3}{1\cdot 2\cdot 3} + \text{etc.}\right);$$

if this equation is divided by a^x, the series expressing the values of an exponential quantity will result, which series we already found in the *Introductio*, i.e.

$$a^\omega = 1 + \frac{\omega\log a}{1} + \frac{\omega^2(\log a)^2}{1\cdot 2} + \frac{\omega^3(\log a)^3}{1\cdot 2\cdot 3} + \frac{\omega^4(\log a)^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}$$

In like manner, for negative ω it will be

$$a^{-\omega} = 1 - \frac{\omega\log a}{1} + \frac{\omega^2(\log a)^2}{1\cdot 2} - \frac{\omega^3(\log a)^3}{1\cdot 2\cdot 3} + \frac{\omega^4(\log a)^4}{1\cdot 2\cdot 3\cdot 4} - \text{etc.},$$

from whose combination these equations result

$$\frac{a^{\omega}+a^{-\omega}}{2}=1+\frac{\omega^2(\log a)^2}{1\cdot 2}+\frac{\omega^4(\log a)^4}{1\cdot 2\cdot 3\cdot 4}+\frac{\omega^6(\log a)^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}+\text{etc.}$$

$$\frac{a^{\omega}-a^{-\omega}}{2}=\frac{\omega\log a}{1}+\frac{\omega^3(\log a)^3}{1\cdot 2\cdot 3}+\frac{\omega^5(\log a)^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\text{etc.},$$

where it is to be noted that $\log a$ denotes the hyperbolic logarithm of the number a.

§82 By means of this formula, given a logarithm, one will be able to find the number corresponding to it. For, let any logarithm u be propounded, for which the logarithm of the number a is set $=1$. In the same base find the logarithm coming closest to u and let $u=x+\omega$, but let the number corresponding to x be $y=a^x$; the number corresponding to the logarithm $u=x+\omega$ will be $=a^{x+\omega}=z$ and it will be

$$z=y\left(1+\frac{\omega\log a}{1}+\frac{\omega^2(\log a)^2}{1\cdot 2}+\frac{\omega^3(\log a)^3}{1\cdot 2\cdot 3}+\frac{\omega^4(\log a)^4}{1\cdot 2\cdot 3\cdot 4}+\text{etc.}\right),$$

which series, because of the very small number ω, converges rapidly; we will show its use in the following example.

Example

Let the number equal to this power of two $2^{2^{24}}$ *be in question.*

Since $2^{24}=16777216$, it will be $2^{2^{24}}=2^{16777216}$ and by taking common logarithms the logarithm of this number will be $=16777216\log 2$. But since

$$\log 2=0.301029995663981195213738 89,$$

the logarithm of the number in question will be

$$5050445.259733675932039063,$$

whose characteristic indicates that the number in question has 5050446 digits; since they cannot all be exhibited, it will suffice to have assigned the first few digits, which must be investigated from the mantissa

$$,259733675932029063 = u.$$

But from tables one concludes that the number, whose logarithm comes closest to this, will be $= 1.818$, which number we want to put y; its logarithm is

$$\begin{array}{lrcl} & x & = & 0.259593878885948644 \\ \text{whence it will be} & \omega & = & 0.000139797046090419 \\ \text{Because now} & a & = & 10 \\ \text{it will be} & \log a & = & 2.302585092994045684017991 4 \\ \hline \text{and} & \omega \log a & = & 0.000321894594372400 \\ \hline \text{Further, it will be} & y & = & 1.818000000000000000 \\ & \dfrac{\omega \log a}{1} y & = & 0.000585204372569023 \quad \text{and} \\ & \dfrac{\omega^2 (\log a)^2}{1 \cdot 2} y & = & 0.000000094187062066 \\ & \dfrac{\omega^3 (\log a)^3}{1 \cdot 2 \cdot 3} y & = & 0.000000000010106102 \\ & \dfrac{\omega^4 (\log a)^4}{1 \cdot 2 \cdot 3 \cdot 4} y & = & 0.000000000000000813 \\ \hline & & & 1818585298569738004 \end{array}$$

these are the first few digits of the number in question, of which all digits, except for the maybe the last, are correct.

§83 Let us consider quantities depending on the circle and let, as usual, the radius of the circle be $= 1$ and let y denote the arc of the circle, whose sine is $= x$, or let $y = \arcsin x$. Write $x + \omega$ instead of x and it will be $z = \arcsin(x + \omega)$; to express this value, find the differentials of y [§ 200 of the first part]

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-xx}}, \quad \frac{ddy}{dx^2} = \frac{x}{(1-xx)^{\frac{3}{2}}}, \quad \frac{d^3y}{dx^3} = \frac{1+2xx}{(1-xx)^{\frac{5}{2}}}, \quad \frac{d^4y}{dx^4} = \frac{9x+6x^3}{(1-xx)^{\frac{7}{2}}},$$

$$\frac{d^5y}{dx^5} = \frac{9+72x^2+24x^4}{(1-xx)^{\frac{9}{2}}}, \quad \frac{d^6y}{dx^6} = \frac{225x+600x^3+120x^5}{(1-xx)^{\frac{11}{2}}}$$

etc.

Therefore, from these one finds

$$\arcsin(x+\omega) = \arcsin x + \frac{\omega}{\sqrt{1-xx}} + \frac{\omega^2 x}{2(1-xx)^{\frac{3}{2}}} + \frac{\omega^3(1+2xx)}{6(1-xx)^{\frac{5}{2}}}$$

$$+\frac{\omega^4(9x+6x^3)}{24(1-xx)^{\frac{7}{2}}} + \frac{\omega^5(9+72x^2+24x^4)}{120(1-xx)^{\frac{9}{2}}} + \text{etc.}$$

§84 Therefore, if the arc, whose sine is $= x$, was known, then, by means of this formula one will be able to find the arc, whose sine is $x+\omega$, if ω was a very small quantity. But the series, whose sum must be added, will be expressed in parts of the radius, which will easily be reduced to an arc, as it will be understood from this example.

EXAMPLE

Let the arc of the circle, whose sine is $= \frac{1}{3} = 0.3333333333$, *be in question.*

From tables find the arc, whose sine is approximately $\frac{1}{3}$ but a little bit smaller, which arc will be $19^\circ 28^{\mathrm{I}}$, whose sine is $= 0.3332584$. Therefore, put $19^\circ 28^{\mathrm{I}} = \arcsin x = y$; it will be $x = 0.3332584$ and $\omega = 0.000749$ and from tables $\sqrt{1-xx} = \cos y = 0.9428356$. Therefore, the arc in question z, whose sine $= \frac{1}{3}$ is propounded, will be

$$= 19^\circ 28^{\mathrm{I}} + \frac{\omega}{\cos y} + \frac{\omega\omega \sin y}{2\cos^3 y},$$

which expression already suffices; therefore, using logarithms in the calculation, it will be

$$
\begin{array}{rcl}
\log\omega & = & 5.8744818 \\
\log\cos y & = & \underline{9.9744359} \\
\log\dfrac{\omega}{\cos y} & = & 5.9000459 \qquad \dfrac{\omega}{\cos y} = 0.0000794412 \\
\log\dfrac{\omega^2}{\cos^2 y} & = & 1.8000918 \\
\log\dfrac{\omega^3}{\cos^3 y} & = & \underline{9.5483452} \\
 & & 1.3484370 \\
\log 2 & = & 0.3010300 \\
\log\dfrac{\omega^2\sin y}{2\cos^2 y} & = & 1.0474070 \qquad \dfrac{\omega^2\sin y}{2\cos^3 y} = \underline{0.0000000011} \\
 & & \qquad\qquad \text{sum} = 0.000794442,
\end{array}
$$

which is the value of the arc to be added to $19^\circ 28^{\mathrm{I}}$, to express which in minutes and seconds let us take its logarithm, which is

$$
\begin{array}{ll}
 & 5.9000518 \\
\text{from which subtract} & \underline{4.6855749} \\
 & 1.2144769 \\
\text{to which logarithm corresponds the number} & = 16.38615,
\end{array}
$$

which is the number of minutes and seconds; but, expressing this fraction in thirds and quarters, the arc in question will be

$$= 19^\circ 28^{\mathrm{I}} 16^{\mathrm{II}} 23^{\mathrm{III}} 10^{\mathrm{IV}} 8^{\mathrm{V}} 14^{\mathrm{VI}}.$$

§85 In like manner, the expression for the cosine will be found; for, having put $y = \arccos x$, since $dy = \frac{-dx}{\sqrt{1-xx}}$, the series we found remains unchanged, as long as its signs are changed. Therefore, it will be

$$\arccos(x+\omega) = \arccos x - \frac{\omega}{\sqrt{1-xx}} - \frac{\omega^2 x}{2(1-xx)^{\frac{3}{2}}} - \frac{\omega^3(1+2xx)}{6(1-xx)^{\frac{5}{2}}}$$
$$- \frac{\omega^4(9x+6x^3)}{24(1-xx)^{\frac{7}{2}}} - \frac{\omega^5(9+72x^2+24x^4)}{120(1-xx)^{\frac{9}{2}}} - \text{etc.},$$

which series, as the preceding, will always converge rapidly, if, consulting tables of sines, angles close to the true one are chosen, such that in most cases the first term $\frac{\omega}{\sqrt{1-xx}}$ alone suffices. Nevertheless, if x is approximately equal to 1 or to a sinus totus, then, because of the very small denominators, the series will only converge slowly. Therefore, in these cases, in which x does not deviate much from 1, since the differences become very small, it will be more convenient to use only the usual method of interpolation.

§86 Therefore, let us also substitute the arc, whose tangent is given, for y and let $y = \arctan x$ and $z = \arctan(x+\omega)$, so that

$$z = y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \text{etc.}$$

To investigate the terms, find each differential of y

$$\frac{dy}{dx} = \frac{1}{1+xx}, \quad \frac{ddy}{dx^2} = \frac{-2x}{(1+xx)^2}, \quad \frac{d^3y}{dx^3} = \frac{-2+6xx}{(1+xx)^3}, \quad \frac{d^4y}{dx^4} = \frac{24x-24x^3}{(1+xx)^4},$$

$$\frac{d^5y}{dx^5} = \frac{24.240x^2+120x^4}{(1+xx)^5}, \quad \frac{d^5y}{dx^5} = \frac{-720x+2400x^3-720x^5}{(1+xx)^5}$$

$$\text{etc.},$$

whence one concludes that

$$\begin{aligned} \arctan(x+\omega) &= \arctan x \\ &+\frac{\omega}{1+xx} - \frac{\omega^2 x}{(1+xx)^2} + \frac{\omega^3}{(1+xx)^3}\left(xx-\frac{1}{3}\right) - \frac{\omega^4}{(1+xx)^4}(x^3-x) \\ &+\frac{\omega^5}{(1+xx)^5}\left(x^4-2x^2+\frac{1}{5}\right) - \frac{\omega^6}{(1+xx)^6}\left(x^5-\frac{10}{3}x^3+x\right) + \text{etc.} \end{aligned}$$

§87 This series, whose law of progression is not that obvious, can be transformed into another form, whose structure is immediately clear. For this aim, put $\arctan x = 90^\circ - u$, so that $x = \cot u = \frac{\cos u}{\sin u}$; it will be $1+xx = \frac{1}{\sin^2 u}$, whence $\frac{dy}{dx} = \frac{1}{1+xx} = \sin^2 u$. Further, since $dx = \frac{-du}{\sin^2 u}$ or $du = -dx\sin^2 u$, taking more differentials, it will be

$$\frac{ddy}{dx} = 2du \sin u \cdot \cos u = du \sin 2u = -dx \sin^2 u \cdot \sin 2u$$

and hence

$$\frac{ddy}{1dx^2} = -\sin^2 u \cdot \sin 2u,$$

$$\frac{d^3y}{2dx^2} = -du \sin u \cdot \cos u \cdot \sin 2u - du \sin^2 u \cdot \cos 2u = -du \sin u \cdot \sin 3u$$
$$= dx \sin^3 u \cdot \sin 3u$$

and hence

$$\frac{d^3y}{1 \cdot 2dx^3} = +\sin^3 u \cdot \sin 3u,$$

$$\frac{d^4y}{1 \cdot 2 \cdot 3dx^3} = du \sin^2 u(\cos u \cdot \sin 3u + \sin u \cdot \cos 3u) = du \sin^2 u \cdot \sin 4u$$
$$= -dx \sin^4 \cdot \sin 4u$$

and hence

$$\frac{d^4y}{1 \cdot 2 \cdot 3dx^4} = -\sin^4 \cdot \sin 4u,$$

$$\frac{d^5y}{1 \cdot 2 \cdot 3 \cdot 4dx^4} = -du \sin^3 u(\cos u \cdot \sin 4u + \sin u \cdot \cos 4u) = -du \cdot \sin^3 u \cdot \sin 5u$$
$$= +dx \sin^5 u \cdot 5u$$

and hence

$$\frac{d^5y}{1 \cdot 2 \cdot 3 \cdot 4dx^5} = +\sin^5 u \cdot \sin 5u$$

etc.

From these one concludes that it will be

$$\arctan(x+\omega) = \arctan x + \frac{\omega}{1} \sin u \cdot \sin u - \frac{\omega^2}{2} \sin^2 u \cdot \sin 2u + \frac{\omega^3}{3} \sin^3 u \cdot \sin 3u$$
$$-\frac{\omega^4}{4} \sin^4 u \cdot \sin 4u + \frac{\omega^5}{5} \sin^5 u \cdot \sin 5u - \frac{\omega^6}{6} \sin^6 \cdot \sin 6u + \text{etc.};$$

because here $\arctan x = y$ and $\arctan x = 90^\circ - u$, it will be $y = 90^\circ - u$.

§88 If one puts $\operatorname{arccot} x = y$ and $\operatorname{arccot}(x+\omega) = z$, it will be

$$z = y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{1 \cdot 2dx^2} + \frac{\omega^3 d^3 y}{1 \cdot 2 \cdot 3dx^3} + \frac{\omega^4 d^4 y}{1 \cdot 2 \cdot 3 \cdot 4dx^4} + \text{etc.}$$

But, since $dy = \frac{-dx}{1+xx}$, all terms with exception of the the first of this series agree with the ones found before, just with different signs. Hence, if, as before, one puts $\arctan x = 90^\circ - u$ or $\operatorname{arccot} x = u$, that $u = y$, it will be

$$\operatorname{arccot}(x+\omega) = \operatorname{arccot} x - \frac{\omega}{1}\sin u \cdot \sin u + \frac{\omega^2}{2}\sin^2 u \cdot \sin 2u - \frac{\omega^3}{3}\sin^3 u \cdot \sin 3u$$

$$+ \frac{\omega^4}{4}\sin^4 u \cdot \sin 4u - \frac{\omega^5}{5}\sin^5 u \cdot \sin 5u + \text{etc.},$$

which expression follows from the preceding immediately; for, since

$$\operatorname{arccot}(x+\omega) = 90^\circ - \arctan(x+\omega) \quad \text{and} \quad \operatorname{arccot} x = 90^\circ - \arctan x,$$

it will be

$$\operatorname{arccot}(x+\omega) - \operatorname{arccot} x = -\arctan(x+\omega) + \arctan x.$$

§89 From these expressions many extraordinary corollaries follow, depending on which values are substituted for the given x and ω. Therefore, first let $x = 0$, and because $u = 90^\circ - \arctan x$, it will be $u = 90^\circ$ and $\sin u = 1$, $\sin 2u = 0$, $\sin 3u = -1$, $\sin 4u = 0$, $\sin 5u = 1$, $\sin 6u = 0$, $\sin 7u = -1$ etc., whence it will be

$$\arctan \omega = \frac{\omega}{1} - \frac{\omega^3}{3} + \frac{\omega^5}{5} - \frac{\omega^7}{7} + \frac{\omega^9}{9} - \frac{\omega^{11}}{11} + \text{etc.},$$

which is the well-known series expressing the arc whose tangent is $= \omega$.

Let $x = 1$; it will be $\arctan x = 45^\circ$ and hence $u = 45^\circ$, hence $\sin u = \frac{1}{\sqrt{2}}$, $\sin 2u = 1$, $\sin 3u = \frac{1}{\sqrt{2}}$, $\sin 4u = 0$, $\sin 5u = -\frac{1}{\sqrt{2}}$, $\sin 6u = -1$, $\sin 7u = -\frac{1}{\sqrt{2}}$, $\sin 8u = 0$, $\sin 9u = \frac{1}{\sqrt{2}}$ etc. Hence

$$\arctan(1+\omega) = 45^\circ + \frac{\omega}{2} - \frac{\omega^2}{2 \cdot 2} + \frac{\omega^3}{3 \cdot 4} - \frac{\omega^5}{5 \cdot 8} + \frac{\omega^6}{6 \cdot 8} - \frac{\omega^7}{7 \cdot 16} + \frac{\omega^9}{9 \cdot 32} - \frac{\omega^{10}}{10 \cdot 32}$$

$$+\frac{\omega^{11}}{11\cdot 64}-\frac{\omega^{13}}{13\cdot 128}+\frac{\omega^{14}}{14\cdot 128}-\text{etc.}$$

Therefore, if $\omega=-1$, because of $\arctan(1+\omega)=0$ and $45^\circ=\frac{\pi}{4}$, it will be

$$\frac{\pi}{4}=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2}+\frac{1}{3\cdot 2^3}-\frac{1}{6\cdot 2^3}-\frac{1}{7\cdot 2^4}+\frac{1}{9\cdot 2^5}+\frac{1}{10\cdot 2^5}+\frac{1}{11\cdot 2^6}-\text{etc.};$$

If this value is substituted for 45° in that expression, it will be

$$\arctan(1+\omega)=\frac{\omega+1}{1\cdot 2}-\frac{\omega^2-1}{2\cdot 2}+\frac{\omega^3+1}{3\cdot 2^2}-\frac{\omega^5+1}{5\cdot 2^3}+\frac{\omega^6-1}{6\cdot 2^3}-\frac{\omega^7+1}{7\cdot 2^4}+\text{etc.}$$

But that series is most appropriate to find the value of $\frac{\pi}{4}$ approximately.

§90 Because

$$\frac{\pi}{4}=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2}+\frac{1}{3\cdot 2^2}-\frac{1}{5\cdot 2^3}-\frac{1}{6\cdot 2^3}-\frac{1}{7\cdot 2^4}+\text{etc.},$$

but the terms containing 2, 6, 10 etc. in the denominators, i.e.

$$\frac{1}{2\cdot 2}-\frac{1}{6\cdot 2^3}+\frac{1}{10\cdot 2^5}-\frac{1}{14\cdot 2^7}+\text{etc.},$$

express $\frac{1}{2}\arctan\frac{1}{2}$, it will be

$$\frac{\pi}{4}=\frac{1}{2}\arctan\frac{1}{2}+\frac{1}{1\cdot 2}+\frac{1}{3\cdot 2^2}-\frac{1}{5\cdot 2^3}-\frac{1}{7\cdot 2^4}+\frac{1}{9\cdot 2^5}+\frac{1}{11\cdot 2^6}-\text{etc.}$$

But because in the other formula for negative ω

$$\begin{aligned}\arctan(1-\omega)&=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2}+\frac{1}{3\cdot 2^2}-\frac{1}{5\cdot 2^3}-\frac{1}{6\cdot 2^3}-\frac{1}{7\cdot 2^4}+\text{etc.}\\&\quad-\frac{\omega}{1\cdot 2}-\frac{\omega^2}{2\cdot 2}-\frac{\omega^3}{3\cdot 2^2}+\frac{\omega^5}{5\cdot 2^3}+\frac{\omega^6}{6\cdot 2^3}+\frac{\omega^7}{7\cdot 2^4}-\text{etc.},\end{aligned}$$

if $\omega=\frac{1}{2}$, it will be

$$\arctan(1-\omega) = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 2^2} - \frac{1}{5\cdot 2^3} - \frac{1}{6\cdot 2^3} - \frac{1}{7\cdot 2^4} + \text{etc.}$$
$$- \frac{1}{1\cdot 2} - \frac{1}{2\cdot 2} - \frac{1}{3\cdot 2^2} + \frac{1}{5\cdot 2^3} + \frac{1}{6\cdot 2^3} + \frac{1}{7\cdot 2^4} - \text{etc.}$$

and, having taken the terms divided by 2, 6, 10 etc., it will be

$$\arctan\frac{1}{2} = \frac{1}{2}\arctan\frac{1}{2} + \frac{1}{1\cdot 2} + \frac{1}{3\cdot 2^3} - \frac{1}{5\cdot 2^3} - \frac{1}{7\cdot 2^4} + \frac{1}{9\cdot 2^5} + \text{etc.}$$
$$- \frac{1}{2}\arctan\frac{1}{2} - \frac{1}{1\cdot 2^2} - \frac{1}{3\cdot 2^5} + \frac{1}{5\cdot 2^8} + \frac{1}{7\cdot 2^{11}} - \frac{1}{9\cdot 2^{14}} - \text{etc.}$$

and hence

$$\frac{1}{2}\arctan\frac{1}{2} = \qquad + \frac{1}{1\cdot 2} + \frac{1}{3\cdot 2^2} - \frac{1}{5\cdot 2^3} - \frac{1}{7\cdot 2^4} + \text{etc.}$$
$$- \frac{1}{2}\arctan\frac{1}{8} - \frac{1}{1\cdot 2^2} - \frac{1}{3\cdot 2^5} + \frac{1}{5\cdot 5^8} + \frac{1}{7\cdot 2^{11}} - \text{etc.};$$

if this value is substituted in the above series and $\arctan\frac{1}{8}$ is converted into a series, one will find

$$\frac{\pi}{4} = \begin{cases} \qquad 1 + \frac{1}{3\cdot 2^1} - \frac{1}{5\cdot 2^2} - \frac{1}{7\cdot 2^3} + \frac{1}{9\cdot 2^4} + \text{etc.} \\ - \frac{1}{1\cdot 2^2} - \frac{1}{3\cdot 2^5} + \frac{1}{5\cdot 2^8} + \frac{1}{7\cdot 2^{11}} - \frac{1}{9\cdot 2^{14}} - \text{etc.} \\ - \frac{1}{1\cdot 2^4} - \frac{1}{3\cdot 2^{10}} - \frac{1}{5\cdot 2^{16}} + \frac{1}{7\cdot 2^{22}} - \frac{1}{9\cdot 2^{28}} + \text{etc.} \end{cases}$$

§ 90a These and many others follow, if one puts $x = 1$; but if we put $x = \sqrt{3}$ that $\arctan x = 60^\circ$, it will be $u = 30^\circ$ and $\sin u = \frac{1}{2}$, $\sin 2u = \frac{\sqrt{3}}{2}$, $\sin 3u = 1$, $\sin 4u = \frac{\sqrt{3}}{2}$, $\sin 5u = \frac{1}{2}$, $\sin 6u = 0$, $\sin 7u = -\frac{1}{2}$ etc., whence it will be

$$\arctan(\sqrt{3}+\omega) = 60^\circ + \frac{\omega}{1\cdot 2^2} - \frac{\omega^2\sqrt{3}}{2\cdot 2^3} + \frac{\omega^3}{3\cdot 2^3} - \frac{\omega^4\sqrt{3}}{4\cdot 2^5} + \frac{\omega^5}{5\cdot 2^6} - \frac{\omega^7}{7\cdot 2^8}$$
$$+\frac{\omega^8\sqrt{3}}{8\cdot 2^9} - \frac{\omega^9}{9\cdot 2^9} + \frac{\omega^{10}\sqrt{3}}{10\cdot 2^{11}} - \frac{\omega^{11}}{11\cdot 2^{12}} + \text{etc.}$$

But if one puts $x = \frac{1}{\sqrt{3}}$, so that $\arctan x = 30^\circ$, it will be $u = 60^\circ$ and $\sin u = \frac{\sqrt{3}}{2}$, $\sin 2u = \frac{\sqrt{3}}{2}$, $\sin 3u = 0$, $\sin 4u = -\frac{\sqrt{3}}{2}$, $\sin 6u = 0$, $\sin 7u = \frac{\sqrt{3}}{2}$ etc., having substituted which values it will be

$$\arctan\left(\frac{1}{\sqrt{3}}+\omega\right) = 30^\circ + \frac{3\omega}{1\cdot 2^2} - \frac{3\omega^2\sqrt{3}}{2\cdot 2^3} + \frac{3^2\omega^4\sqrt{3}}{4\cdot 2^5} - \frac{3^3\omega^5}{5\cdot 2^5} + \text{etc.};$$

therefore, if $\omega = -\frac{1}{\sqrt{3}}$, because of $30^\circ = \frac{\pi}{6}$, it will be

$$\frac{\pi}{6\sqrt{3}} = \frac{1}{1\cdot 2^2} + \frac{1}{2\cdot 2^3} - \frac{1}{4\cdot 2^5} - \frac{1}{5\cdot 2^6} + \frac{1}{7\cdot 2^8} + \frac{1}{8\cdot 2^9} - \text{etc.}$$

§91 Let us return to the general expression we found, i.e.

$$\arctan(x+\omega)$$
$$= \arctan x + \frac{\omega}{1}\sin u \cdot \sin u - \frac{\omega^2}{2}\sin^2 u \cdot \sin 2u + \frac{\omega^3}{3}\sin^3 u \cdot \sin 3u - \text{etc.}$$

and let us put $\omega = -x$, so that $\arctan(x+\omega) = 0$ and it will be

$$\arctan x = \frac{x}{1}\sin u \cdot \sin u + \frac{x^2}{2}\sin^2 u \cdot \sin 2u + \frac{x^3}{3}\sin^3 u \cdot \sin 3u + \text{etc.}$$

But because $\arctan x = 90^\circ - u = \frac{\pi}{2} - u$, it will be $x = \cot u = \frac{\cos u}{\sin u}$. Therefore, it will be

$$\frac{\pi}{2} = u + \cos u \cdot \sin u + \frac{1}{2}\cos^2 u \cdot \sin 2u + \frac{1}{3}\cos^3 u \cdot \sin 3u + \frac{1}{4}\cos^4 u \cdot \sin 4u + \text{etc.},$$

which series is even more remarkable, since, whatever arc is taken for u, the value of the series always turns out to be the same, $= \frac{\pi}{2}$.

But if $\omega = -2x$, because of $\arctan(-x) = -\arctan x$, it will be

$$2\arctan x = \frac{2x}{1}\sin u \cdot \sin u + \frac{4x^2}{2}\sin^2 u \cdot \sin 2u + \frac{8}{3}\sin^3 u \cdot \sin 3u + \text{etc.}$$

But because $\arctan x = \frac{\pi}{2} - u$ and $x = \frac{\cos u}{\sin u}$, it will be

$$\pi = 2u + \frac{2}{1}\cos u \cdot \sin u + \frac{2^2}{2}\cos^2 u \cdot \sin 2u + \frac{2^3}{3}\cos^3 u \cdot \sin 3u + \text{etc.}$$

Let $u = 45^\circ$; it will be $\cos u = \frac{1}{\sqrt{2}}$, $\sin u = \frac{1}{\sqrt{2}}$, $\sin 2u = 1$, $\sin 3u = \frac{1}{\sqrt{2}}$, $\sin 4u = 0$, $\sin 5u = \frac{-1}{\sqrt{2}}$, $\sin 6u = -1$, $\sin 7u = \frac{-1}{\sqrt{2}}$, $\sin 8u = 0$, $\sin 9u = \frac{1}{\sqrt{2}}$ etc. and it will be

$$\frac{\pi}{2} = \frac{1}{1} + \frac{2}{2} + \frac{2}{3} - \frac{2^5}{5} - \frac{2^3}{6} - \frac{2^3}{7} + \frac{2^4}{9} + \frac{2^5}{10} + \frac{2^5}{11} - \text{etc.},$$

which series, even though it diverges, nevertheless is remarkable for its simplicity.

§92 In the general expression we found put

$$\omega = -x - \frac{1}{x} = \frac{-1}{\sin u \cdot \cos u};$$

because of $x = \frac{\cos u}{\sin u}$, it will be

$$\arctan(x+\omega) = \arctan\left(-\frac{1}{x}\right) = -\arctan\frac{1}{x} = -\frac{\pi}{2} + \arctan x.$$

Therefore, one will hence obtain the following expression

$$\frac{\pi}{2} = \frac{\sin u}{1\cos u} + \frac{\sin 2u}{2\cos^2 u} + \frac{\sin 3u}{3\cos^3 u} + \frac{\sin 4u}{4\cos^4 u} + \frac{\sin 5u}{5\cos^5 u} + \text{etc.},$$

which, having put $u = 45^\circ$, gives the same series we found last.

But if we put $\omega = -\sqrt{1+xx}$, because of $x = \frac{\cos u}{\sin u}$, it will be

$$\omega = -\frac{1}{\sin u}$$

and

$$\arctan(x-\sqrt{1+xx}) = -\arctan(\sqrt{1+xx}-x)$$
$$= -\frac{1}{2}\arctan\frac{1}{x} = -\frac{1}{2}\left(\frac{\pi}{2}-\arctan x\right) = -\frac{1}{2}u$$

and

$$\arctan x = \frac{\pi}{2} - u.$$

Therefore, it will be

$$\frac{\pi}{2} = \frac{1}{2}u + \frac{1}{1}\sin u + \frac{1}{2}\sin 2u + \frac{1}{3}\sin 3u + \frac{1}{4}\sin 4u + \text{etc.}$$

Therefore, if this equation is differentiated, it will be

$$0 = \frac{1}{2} + \cos u + \cos 2u + \cos 3u + \cos 4u + \cos 5u + \text{etc.},$$

whose correctness is seen from the nature of recurring series.

§93 In like manner, if the series found before are differentiated, new summable series will be found. First, from the series

$$\arctan(1+\omega) = \frac{\pi}{4} + \frac{\omega}{2} - \frac{\omega^2}{2\cdot 2} + \frac{\omega^3}{3\cdot 4} - \frac{\omega^5}{5\cdot 8} + \frac{\omega^6}{6\cdot 8} - \text{etc.}$$

it follows

$$\frac{1}{2+2\omega+\omega^2} = \frac{1}{2} - \frac{\omega}{2} + \frac{\omega^2}{4} - \frac{\omega^4}{8} + \frac{\omega^5}{8} - \frac{\omega^6}{16} + \frac{\omega^8}{32} - \text{etc.},$$

which results from the expansion of this fraction $\frac{2-2\omega+\omega^2}{4+\omega^4} = \frac{1}{2+2\omega+\omega^2}$.
Further, this series

$$\frac{\pi}{2} = u + \cos u\cdot\sin u + \frac{1}{2}\cos^2 u\cdot\sin 2u + \frac{1}{3}\cos^3 u\cdot\sin 3u + \frac{1}{4}\cos^4\cdot\sin 4u + \text{etc.},$$

by means of differentiation, will give

$$0 = 1 + \cos 2u + \cos\cdot\cos 3u + \cos^2 u\cdot\cos 4u + \cos^3 u\cdot\cos 5u + \text{etc.}$$

Finally, the series

$$\frac{\pi}{2} = \frac{\sin u}{\cos u} + \frac{\sin 2u}{2\cos^2 u} + \frac{\sin 3u}{3\cos^3 u} + \frac{\sin 4u}{4\cos^4 u} + \text{etc.}$$

gives

$$0 = \frac{1}{\cos^2 u} + \frac{\cos u}{\cos^3 u} + \frac{\cos 2u}{\cos^4 u} + \frac{\cos 3u}{\cos^5 u} + \frac{\cos 4u}{\cos^6 u} + \text{etc.}$$

or

$$0 = 1 + \frac{\cos u}{\cos u} + \frac{\cos 2u}{\cos^2 u} + \frac{\cos 3u}{\cos^3 u} + \frac{\cos 4u}{\cos^4 u} + \frac{\cos 5u}{\cos^5 u} + \text{etc.}$$

§94 But especially the expression we found

$$\begin{aligned}&\arctan(x+\omega)\\ &= \arctan x + \frac{\omega}{1}\sin u \cdot \sin u - \frac{\omega^2}{2}\sin^2 u \cdot \sin 2u + \frac{\omega^3}{3}\sin^3 u \cdot \sin 3u - \text{etc.},\end{aligned}$$

while

$$x = \cot u \quad \text{or} \quad u = \operatorname{arccot} x = 90^\circ - \arctan x,$$

can be applied to the angle or the arc corresponding to a given certain tangent. For, let the tangent $= t$ be propounded and, consulting tables, find the tangent coming closet to this, call it x, to which the arc $= y$ corresponds, and it will be $u = 90^\circ - y$. Then, put $x + \omega = t$ or $\omega = t - x$ and the arc in question will be

$$= y + \frac{\omega}{1}\sin u \cdot \sin u - \frac{\omega^2}{2} \cdot \sin 2u + \text{etc.},$$

which rule is especially useful, when the propounded tangent was very large and therefore the arc in question hardly deviates from 90°. For, in these cases, because of the rapidly increasing tangents, the usual method of interpolation leads too far away from the true value. Therefore, let this example be propounded.

Example

Let the arc be in question, whose tangent is $= 100$, *having put the radius* $= 1$, *of course.*

The arc approximately equal to the one in question is $89^\circ 25^{\mathrm{I}}$, whose tangent is

$$
\begin{array}{rrcr}
 & x & = & 98.217943 \\
\text{subtract this from} & t & = & \underline{100.000000} \\
\text{it will remain} & \omega & = & 1.782057
\end{array}
$$

Further, because $y = 89^\circ 25^{\mathrm{I}}$, it will be $u = 35^{\mathrm{I}}$, $2u = 1^\circ 10^{\mathrm{I}}$, $3u = 1^\circ 45^{\mathrm{I}}$ etc. Now investigate each term by means of logarithms.

$$
\begin{array}{llcrl}
\text{To} & \log\omega & = & 0.2509125 & \\
\text{add} & \log\sin u & = & 8.0077867 & \\
 & \log\sin u & = & \underline{8.0077867} & \\
\text{subtract} & \log\omega\sin u\cdot\sin u & = & 6.2664949 & \\
 & & & 4.6855749 & \\
 & & = & 1.5809200 & \\
\text{Therefore} & \omega\sin u\cdot\sin u & = & 38.09956 & \text{seconds} \\
\text{To} & \log\omega\sin^2 u & = & 6.2664949 & \\
\text{add} & \log\omega & = & 0.2509215 & \\
 & \log 2u & = & \underline{8.3087941} & \\
\text{subtract} & & & 4.8262105 & \\
 & \log 2 & = & \underline{0.3010300} & \\
 & \log\frac{1}{2}\omega^2\sin^2 u\cdot 2u & = & 4.5251805 & \\
\text{subtract} & & & \underline{4.6855749} & \\
\text{it remains} & & & 9.8396056 & \\
\text{Therefore} & \frac{1}{2}\omega^2\sin^2 u\cdot\sin 2u & = & \underline{0.6912000} & \text{seconds} \\
\text{Further to} & \log\omega^3 & = & 0.7527645 & \\
 & \log\sin^3 u & = & 4.0233601 & \\
\text{subtract} & \log\sin 3u & = & 3.2609725 & \\
 & \log 3 & = & \underline{0.4771213} & \\
 & & & 2.7838512 & \\
\text{subtract} & & & \underline{4.6855749} & \\
 & & & \underline{8.0982763} & \\
\text{Therefore} & \frac{1}{3}\omega^3\sin^3 u\cdot\sin 3u & = & 0.0125400 & \text{seconds.}
\end{array}
$$

Finally, to	$\log \omega^4$	$=$	1.0036860
add	$\log \sin^4 u$	$=$	2.0311468
	$\log \sin 4u$	$=$	8.6097341
			1.6445669
subtract	$\log 4$	$=$	0.6020600
			1.0425069
subtract			4.6855749
			6.3569320

Therefore,

$$\frac{1}{4}\omega^4 \sin^4 u = 0.00023 \quad \text{seconds}$$

Hence

	Terms to be added	Terms to be subtracted
	38.09956	0.69120
	0.01254	0.00023
subtract	0.69143	

Hence in total

$$37.4067 = 37^{\mathrm{II}}25^{\mathrm{III}}14^{\mathrm{IV}}24^{\mathrm{V}}36^{\mathrm{VI}}.$$

Therefore, the arc, whose tangent is hundred times the radius, will be

$$89^\circ 25^{\mathrm{I}}37^{\mathrm{II}}25^{\mathrm{III}}14^{\mathrm{IV}}24^{\mathrm{V}}36^{\mathrm{VI}}$$

and the error does not affect the fourth, but can only occur the fifth, whence we will be able to confirm that this angle is almost $= 89^\circ 25^{\mathrm{I}}37^{\mathrm{II}}25^{\mathrm{III}}14^{\mathrm{IV}}$. If an even greater tangent is propounded, even though ω might turn out to be larger, because of the still small angle u, one will nevertheless be able to define the arc in a convenient way.

§95 Since here we substituted an arc of the circle for y, let us now substitute the inverse functions for y, i.e. $\sin x$, $\cos x$, $\tan x$, $\cot x$ etc. Therefore, let $y = \sin x$, and, having written $x + \omega$ instead of x, it will be $z = \sin(x + \omega)$ and the equation

$$z = y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} + \text{etc.},$$

because of

$$\frac{dy}{dx} = \cos x, \quad \frac{ddy}{dx^2} = -\sin x, \quad \frac{d^3y}{dx^3} = -\cos x, \quad \frac{d^4y}{dx^4} = \sin x \quad \text{etc.},$$

will give

$$\sin(x+\omega) = \sin x + \omega \cos x - \frac{1}{2}\omega^2 \sin x - \frac{1}{6}\omega^3 \cos x + \frac{1}{24}\omega^4 \sin x + \text{etc.}$$

and, having taken a negative ω, it will be

$$\sin(x-\omega) = \sin x - \omega \cos x - \frac{1}{2}\omega^2 \sin x + \frac{1}{6}\omega^3 \cos x + \frac{1}{24}\omega^4 \sin x - \text{etc.}$$

So, if one sets $y = \cos x$, because of

$$\frac{dy}{dx} = -\sin x, \quad \frac{ddy}{dx^2} = -\cos x, \quad \frac{d^3y}{dx^3} = \sin x, \quad \frac{d^4y}{dx^4} = \cos x \quad \text{etc.},$$

it will be

$$\cos(x+\omega) = \cos x - \omega \sin x - \frac{1}{2}\omega^2 \cos x + \frac{1}{6}\omega^3 \sin x + \frac{1}{24}\omega^4 \cos x - \text{etc.}$$

and for a negative ω it will be

$$\cos(x-\omega) = \cos x + \omega \sin x - \frac{1}{2}\omega^2 \cos x - \frac{1}{6}\omega^3 \sin x + \frac{1}{24}\omega^4 \cos x + \text{etc.}$$

§96 The use of these formulas is immense both for the construction and interpolation of tables of sines and cosines. For, if the sines and cosines of a certain arc x were known, from them the sines and cosines of the angles $x+\omega$ and $x-\omega$ can be easily found, if the difference ω was sufficiently small; for, in this case the found series converge rapidly. For this it is necessary that the arc ω is expressed in parts of the radius; this, because the arc of 180° is

$$3.14159265358979323846,$$

is easily done; for, after a division by 180 it will be

$$\begin{aligned}
\text{arc of } 1^{\circ} &= 0.01745329251994329576 9\\
\text{arc of } 1^{\mathrm{I}} &= 0.00029088820866572159 6\\
\text{arc of } 1^{\mathrm{II}} &= 0.00000484813681109535 99.
\end{aligned}$$

EXAMPLE 1

To find the sine and the cosine of the angles $45^{\circ}1^{I}$ and $44^{\circ}59^{I}$, the given sine and cosine of the angle 45°, both of which are $=\frac{1}{\sqrt{2}}=0.707167811865$.

Therefore, since

$$\sin x = \cos x = 0.7071067811865$$

and

$$\omega = 0.0002908882086,$$

in order to perform the multiplication more easily, note that it will be

$$
\begin{aligned}
2\omega &= 0.0005817764173\\
3\omega &= 0.0008726646259\\
4\omega &= 0.0011635528346\\
5\omega &= 0.0014544410433\\
6\omega &= 0.0017453292519\\
7\omega &= 0.0020362174606\\
8\omega &= 0.0023271056693\\
9\omega &= 0.0026179938779
\end{aligned}
$$

Therefore, $\omega \sin x$ and $\omega \cos x$ will be found this way:

$$
\begin{array}{ccr}
7 & \cdot & 0.00020362174606\\
0 & \cdot & \\
7 & \cdot & 0.00000203621746\\
1 & \cdot & 2908882\\
0 & \cdot & \\
6 & \cdot & 174532\\
7 & \cdot & 20362\\
8 & \cdot & 2327\\
1 & \cdot & 29\\
1 & \cdot & 2\\
8 & \cdot & 2\\
6 & \cdot & 0\\
\hline
\end{array}
$$

In total

$$\omega \sin x = \omega \cos x = 0.00020568902488$$

Therefore,

$$
\begin{array}{lrcr}
 & \frac{1}{2}\omega\cos x & = & 0.00010284451244 \\
\hline
\text{by } \omega & 1 & \cdot & 0.00000002908882 \\
 & 0 & \cdot & \\
 & 2 & \cdot & 58177 \\
 & 8 & \cdot & 23271 \\
 & 4 & \cdot & 1163 \\
 & 4 & \cdot & 116 \\
 & 5 & \cdot & 14 \\
\hline
 & \frac{1}{2}\omega^2\cos x & = & 0.00000002991623 \\
 & \frac{1}{6}\omega^3\cos x & = & 0.00000000997208 \\
\hline
\text{by } \omega & 9 & \cdot & 0.0000000000261 \\
 & 9 & \cdot & 26 \\
 & 7 & \cdot & 2 \\
\hline
 & \frac{1}{6}\omega^3\cos x & = & 0.0000000000289
\end{array}
$$

Therefore, to find $\sin 45^\circ 1^{\mathrm{I}}$ to

$$
\begin{array}{lrcrcl}
 & \sin x & = & 0.7071067811865 & & \\
\text{add} & \omega\cos x & = & 2056890249 & & \\
\hline
 & & & 0.7073124702114 & & \\
\text{subtract} & \frac{1}{2}\omega^2\sin x & = & 299162 & & \\
\hline
 & & & 0.7073124402952 & & \\
\text{subtract} & \frac{1}{6}\omega^3\cos x & = & 29 & & \\
\hline
 & \sin 45^\circ 1^{\mathrm{I}} & = & 0.7073124402923 & = & \cos 44^\circ 59^{\mathrm{I}}
\end{array}
$$

But to find $\cos 45^\circ 1^{\mathrm{I}}$ from

$$\begin{array}{rrcr}
 & \cos x & = & 0.7071067811865 \\
\text{subtract} & \omega \sin x & = & 2056890249 \\
\hline
 & & & 0.70769010921616 \\
\text{subtract} & \frac{1}{2}\omega^2 \cos x & = & 299162 \\
\hline
 & & & 0.7069010622454 \\
\text{add} & \frac{1}{6}\omega^3 \cos x & = & 29 \\
\hline
 & \cos 45°1^{\mathrm{I}} & = & 0.7069010622483 \quad = \quad \sin 44°59^{\mathrm{I}}
\end{array}$$

EXAMPLE 2

Given sine and cosine of the arc $67°30^I$, to find the sine and the cosine of the arcs $67°31^I$ and $67°29^I$.

Let us perform this calculation in decimal fractions up to 7 digits, as the common tables are usually constructed, and hence the task will easily be solved applying logarithms. Because $x = 67°30^{\mathrm{I}}$ and $\omega = 0.000290888$, it will be

$$\log \omega = 6.4637259$$

$$\begin{array}{rrcrrcr}
\text{and} & \log \sin x & = & 9.9656153 & \log \cos x & = & 9.5828397 \\
 & \log \omega & = & 6.6437259 & \log \omega & = & 6.6437259 \\
\hline
 & \log \omega \sin x & = & 6.4293412 & \log \omega \cos x & = & 6.0465656 \\
 & \log \frac{1}{2}\omega & = & 6.1626959 & \log \frac{1}{2}\omega^2 & = & 6.1626959 \\
\hline
 & \log \frac{1}{2}\omega^2 \sin x & = & 2.59200371 & \log \frac{1}{2}\omega^2 \cos x & = & 2.2092615 \\
\text{Therefore} & \omega \sin x & = & 0.00026874 & \omega \cos x & = & 0.00011132 \\
 & \frac{1}{2}\omega^2 \sin x & = & 0.00000004 & \frac{1}{2}\omega^2 \cos x & = & 0.00000001 \\
\text{whence} & \sin 67°31^{\mathrm{I}} & = & 0.9239908 & \cos 67°31^{\mathrm{I}} & = & 0.3824147 \\
 & \sin 67°29^{\mathrm{I}} & = & 0.9237681 & \cos 67°29^{\mathrm{I}} & = & 0.3829522
\end{array}$$

where not even the terms $\frac{1}{2}\omega^2 \sin x$ and $\frac{1}{2}\omega^2 \cos x$ were necessary.

§97 From the series we found above,

$$\sin(x+\omega) = \sin x + \omega\cos x - \frac{1}{2}\omega^2 \sin x - \frac{1}{6}\omega^3 \cos x + \frac{1}{24}\omega^4 \sin x + \text{etc.}$$

$$\cos(x+\omega) = \cos x - \omega \sin x - \frac{1}{2}\omega^2 \cos x + \frac{1}{6}\omega^3 \sin x + \frac{1}{24}\omega^4 \cos x - \text{etc.}$$

$$\sin(x-\omega) = \sin x - \omega\cos x - \frac{1}{2}\omega^2 \sin x + \frac{1}{6}\omega^3 \cos x + \frac{1}{24}\omega^4 \sin x - \text{etc.}$$

$$\cos(x-\omega) = \cos x + \omega \sin x - \frac{1}{2}\omega^2 \cos x - \frac{1}{6}\omega^3 \sin x + \frac{1}{24}\omega^4 \cos x + \text{etc.},$$

combining them we will find

$$\frac{\sin(x+\omega)+\sin(x-\omega)}{2}$$
$$= \sin x - \frac{1}{2}\omega^2 \sin x + \frac{1}{24}\sin x - \frac{1}{720}\omega^6 \sin x + \text{etc.} = \sin x \cdot \cos\omega$$

and

$$\frac{\sin(x+\omega)-\sin(x-\omega)}{2}$$
$$= \omega\cos x - \frac{1}{6}\omega^3 \cos x + \frac{1}{120}\omega^5 \cos x - \frac{1}{5040}\omega^7 \cos x + \text{etc.} = \cos x \cdot \sin\omega,$$

whence the series found above for the sines and cosines result to be

$$\cos\omega = 1 - \frac{1}{2}\omega^2 + \frac{1}{24}\omega^4 - \frac{1}{720}\omega^6 + \text{etc.}$$
$$\sin\omega = \omega - \frac{1}{6}\omega^3 + \frac{1}{120}\omega^5 - \frac{1}{5040}\omega^7 + \text{etc.},$$

which same series follow from the first for $x = 0$; for, because $\cos x = 1$ and $\sin x = 0$, the first series will exhibit $\sin\omega$, the second on the other hand $\cos\omega$.

§98 Now let us also put $y = \tan x$, so that $z = \tan(x+\omega)$; because of $y = \frac{\sin x}{\cos x}$ [§ 206 of the first part]

$$\frac{dy}{dx} = \frac{1}{\cos^2 x}, \quad \frac{ddy}{2dx^2} = \frac{\sin x}{\cos^3 x}, \quad \frac{d^3y}{2dx^3} = \frac{1}{\cos^2 x} + \frac{3\sin^2 x}{\cos^4 x} = \frac{3}{\cos^4 x} - \frac{2}{\cos^2 x},$$

$$\frac{d^4y}{2\cdot 4dx^4} = \frac{3\sin x}{\cos^5 x} - \frac{\sin x}{\cos^3 x}, \quad \frac{d^5y}{2\cdot 4} = \frac{15}{\cos^6 x} - \frac{15}{\cos^4 x} + \frac{2}{\cos^2 x},$$

whence it follows that

$$\tan(x+\omega) = \tan x + \begin{cases} \dfrac{\omega}{\cos^2 x} + \dfrac{\omega^2 \sin x}{\cos^3 x} + \dfrac{\omega^3}{\cos^4 x} + \dfrac{\omega^4 \sin x}{\cos^5 x} + \text{etc.} \\ \qquad\qquad\qquad - \dfrac{2\omega^3}{3\cos^2 x} - \dfrac{\omega^4 \sin x}{3\cos^3 x} - \text{etc.} \end{cases}$$

by means of which formula, given tangent of any angle, one can find the tangents of angles very close to it. Since the above series is a geometric one, having collected it into one sum, it will be

$$\tan(x+\omega) = \tan x + \frac{\omega + \omega^2 \tan x}{\cos^2 x - \omega^2} - \frac{2\omega^3}{3\cos^2 x} - \frac{\omega^4 \sin x}{3\cos^3 x} - \text{etc.}$$

or

$$\tan(x+\omega) = \frac{\sin x \cdot \cos x + \omega}{\cos^2 x - \omega^2} - \frac{2\omega^3}{3\cos^2 x} - \frac{\omega^4 \sin x}{3\cos^3 x} - \text{etc.},$$

which formula is applied more conveniently for this aim.

§99 Similar expressions can also be found for the logarithms of sines, cosines and tangents. For, let $y =$ a logarithm of the sine of the angle x, which we want to express as $y = \log \sin x$, and $z = \log \sin(x+\omega)$; because of $\frac{dy}{dx} = \frac{n \cos x}{\sin x}$, it will be $\frac{ddy}{dx^2} = \frac{-n}{\sin^2 x}$, $\frac{d^3y}{dx^3} = \frac{+2n\cos x}{\sin^3 x}$ etc., whence it will be

$$z = \log\sin(x+\omega) = \log \sin x + \frac{n\omega \cos x}{\sin x} - \frac{n\omega^2}{2\sin^2 x} + \frac{n\omega^3 \cos x}{3\sin^3 x} - \text{etc.},$$

where n denotes the number, by which the hyperbolic must be multiplied that the propounded logarithms result. But if $y = \tan x$ and $z = \log\tan(x+\omega)$, it will be $\frac{dy}{dx} = \frac{n}{\sin x \cdot \cos x} = \frac{2n}{\sin 2x}$, $\frac{ddy}{2dx^2} = \frac{-2n\cos 2x}{(\sin 2x)^2}$ etc. and hence

$$\log\tan(x+\omega) = \log\tan x + \frac{2n\omega}{\sin 2x} - \frac{2n\omega^2 \cos 2x}{(\sin 2x)^2} + \text{etc.},$$

by means of which formulas the logarithms of sines and tangents can be interpolated.

§100 Let us put that y denotes the arc, whose logarithm of the sine is $= x$, or that $y = \mathrm{A}\,.\log x$, and that z is the arc, whose logarithm of the sine we want to put $= x+\omega$ or $z = \mathrm{A}\,.\log\sin(x+\omega)$; it will be $x = \log\sin y$ and

$$\frac{dx}{dy} = \frac{n\cos y}{\sin y}, \quad \text{whence} \quad \frac{dy}{dx} = \frac{\sin y}{n\cos y};$$

it will be

$$\frac{ddy}{dx} = \frac{dy}{n\cos^2 y} = \frac{dx\sin y}{n^2\cos^3 y}, \quad \text{therefore} \quad \frac{ddy}{dx^2} = \frac{\sin y}{n^2\cos^3 y}.$$

As a logical consequence

$$z = y + \frac{\omega\sin y}{n\cos y} + \frac{\omega^2\sin y}{2n^2\cos^3 y} + \text{etc.}$$

In like manner, given the logarithm of a cosine, the expression will be found. But if $y = \mathrm{A}\,.\log\tan x$ and $z = \mathrm{A}\,.\log(x+\omega)$, since $x = \log\tan y$, it will be

$$\frac{dx}{dy} = \frac{n}{\sin y\cdot\cos y} \quad \text{and} \quad \frac{dy}{dx} = \frac{\sin y\cdot\cos y}{n} = \frac{\sin 2y}{2n},$$

whence

$$\frac{ddy}{dx} = \frac{2dy\cos 2y}{2n} = \frac{dx\sin 2y\cdot\cos 2y}{2nn}$$

and

$$\frac{ddy}{dx^2} = \frac{\sin 2y\cdot\cos 2y}{2nn} = \frac{\sin 4y}{4nn}, \quad \frac{d^3y}{dx^3} = \frac{\sin 2y\cdot 4y}{2n^3} \quad \text{etc.};$$

hence

$$z = y + \frac{\omega \sin 2y}{2n} + \frac{\omega^2 \sin 2y \cdot \cos 2y}{4nn} + \frac{\omega^3 \sin 2y \cdot \cos 4y}{12n^3} + \text{etc.}$$

§101 Since the use of these expressions for the construction of tables of logarithms of sines and cosines can easily be seen from the preceding paragraphs, we will not treat this here any longer. Therefore, lastly let us consider the value $y = e^x \sin nx$ and let $z = e^{x+\omega} \sin n(x+\omega)$; since

$$\begin{aligned}
\frac{dy}{dx} &= e^x(\sin nx + n\cos nx) \\
\frac{ddy}{dx^2} &= e^x((1-nn)\sin nx + 2n\cos nx) \\
\frac{d^3y}{dx^3} &= e^x((1-3nn)\sin nx + n(3-nn)\cos nx) \\
\frac{d^4y}{dx^4} &= e^x((1-6nn+n^4)\sin nx + n(4-4nn)\cos nx) \\
\frac{d^5y}{dx^5} &= e^x((1-10nn+5n^4)\sin nx + n(5-10nn+n^4)\cos nx),
\end{aligned}$$

$$\text{etc.;}$$

having substituted these values and having divided by e^x, it will be

$$e^\omega \sin n(x+\omega) = \sin nx$$

$$+\omega \sin nx + \frac{1-nn}{2}\omega^2 \sin nx + \frac{1-3nn}{6}\omega^3 \sin nx + \frac{1-6nn+n^4}{24}\omega^4 \sin nx + \text{etc.}$$

$$+n\omega \cos nx + \frac{2n}{2}\omega^2 \cos nx + \frac{n(3-nn)}{6}\omega^3 \cos nx + \frac{n(4-4nn)}{24}\omega^4 \cos nx + \text{etc.}$$

§102 Hence many extraordinary corollaries can be deduced; but it suffices for us to have mentioned the following things here. If it was $x = 0$, it will be

$$e^\omega \sin n\omega = n\omega + \frac{2n}{2}\omega^2 + \frac{n(3-nn)}{6}\omega^3 + \frac{n(4-nn)}{24}\omega^4 + \frac{n(5-10n^2+n^4)}{120}\omega^5 + \text{etc.}$$

If $\omega = -x$, because of $\sin n(x+\omega) = 0$, it will be

$$\tan nx = \frac{nx - \frac{2n}{2}x^2 + \frac{n(3-nn)}{6}x^3 - \frac{n(4-4nn)}{24}x^4 + \frac{n(5-10n^2+n^4)}{120}x^5 - \text{etc.}}{1 - x + \frac{1-nn}{2}x^2 - \frac{1-3nn}{6}x^3 + \frac{1-6nn+n^4}{24}x^4 - \text{etc.}}$$

But in general, if $n = 1$, one will have

$$e^{\omega}\sin(x+\omega) = \sin x\left(1 + \omega - \frac{1}{3}\omega^3 - \frac{1}{6}\omega^4 - \frac{1}{30}\omega^5 + \frac{1}{630}\omega^7 + \text{etc.}\right)$$
$$+\omega\cos x\left(1 + \omega + \frac{1}{3}\omega^2 - \frac{1}{30}\omega^4 - \frac{1}{90}\omega^5 - \frac{1}{630}\omega^6 + \text{etc.}\right)$$

But if it is $n = 0$, because $\sin n(x+\omega) = n(x+\omega)$ and $\sin nx = nx$ and $\cos nx = 1$, and if one divides by n everywhere, we have

$$e^{\omega}(x+\omega) = x + \omega x + \frac{1}{2}\omega^2 x + \frac{1}{6}\omega^3 x + \frac{1}{24}\omega^4 x + \text{etc.}$$
$$+\omega + \omega^2 + \frac{1}{2}\omega^3 + \frac{1}{6}\omega^4 + \frac{1}{24}\omega^5 + \text{etc.},$$

the validity of which equation is obvious.