A further Generalization of the summation method treated in chapter V *

Leonhard Euler

§167 To cure the defect of the summation method we presented before, in this chapter we will consider series, whose general terms are more complex. Since the expression found before in the case of geometric progressions, even though by means of other methods they can be summed very easily, does not yield the true sum in terms of a finite formula, series whose terms are products of terms of a geometric series and an arbitrary other series will be contemplated here at first. Therefore, let this series be propounded

$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & & x \\ s & = ap & + bp^2 & + cp^3 & + dp^4 & + \cdots\cdots & + yp^x, \end{array}$$

which is conflated of the geometric series p, p^2, p^3 etc. and another arbitrary series $a+b+c+d+$ etc., whose general term or the one corresponding to index x is $=y$, and let us investigate the general term for the value of its sum $s = S.yp^x$.

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§168 Let us perform the calculation the same way as we did above, and let v be the term preceding y in the series $a+b+c+d+$ etc. and A the one preceding a or the one corresponding to the index 0, then vp^{x-1} will be the general term of this series

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & & x \\ A + ap & + bp^2 & + cp^3 & + & \cdots\cdots & + yp^x; \end{array}$$

if its sum is indicated by $S.vp^{x-1}$, it will be

$$S.vp^{x-1} = \frac{1}{p}S.vp^x = S.yp^x - yp^x + A.$$

But since

$$v = y - \frac{dy}{dx} + \frac{ddy}{2dx^2} - \frac{d^3y}{6dx^3} + \frac{d^4y}{24dx^4} - \frac{d^5y}{120dx^5} + \text{etc.},$$

it will be

$$S.yp^x - yp^x + A = \frac{1}{p}S.yp^x - \frac{1}{p}S.\frac{dy}{dx}p^x + \frac{1}{2p}S.\frac{ddy}{dx^2}p^x$$
$$-\frac{1}{6p}S.\frac{d^3y}{dx^3}p^x + \frac{1}{24p}S.\frac{d^4y}{dx^4}p^x - \text{etc.}$$

From this

$$S.yp^x = \frac{1}{p-1}\left(yp^{x+1} - Ap - S.\frac{dy}{dx}p^x + S.\frac{ddy}{2dx^2}p^x - S.\frac{d^3y}{6dx^3} + \text{etc.}\right)$$

Therefore, if one has the summatory terms of the series whose general terms are $\frac{dy}{dx}p^x$, $\frac{ddy}{dx^2}p^x$, $\frac{d^3y}{dx^3}p^x$ etc., from them one will be able to define the summatory term $S.yp^x$.

§169 Therefore, one will be able to find the sums of the series whose general terms are contained in the form x^np^x. For, let $y = x^n$; it will be $A = 0$, if not $n = 0$, in which case $A = 1$, and since

$$\frac{dy}{dx} = nx^{n-1}, \quad \frac{ddy}{2dx^2} = \frac{n(n-1)}{1\cdot 2}x^{n-2}, \quad \frac{d^3y}{6dx^3} = \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3} \quad \text{etc.},$$

it will be

$$S.x^n p^x = \frac{1}{p-1}\left\{\begin{array}{c} x^n p^{x+1} - Ap - nS.x^{n-1}p^x + \frac{n(n-1)}{1\cdot 2}S.x^{n-2}p^x \\ -\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}S.x^{n-3}p^x + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}S.x^{n-4}p^x - \text{etc.} \end{array}\right\}$$

From this form, successively substituting the numbers 0, 1, 2, 3 etc. for n, one will obtain the following summations; first, if $n = 0$, $A = 1$, but in the remaining cases it will be $A = 0$.

$$S.x^0 p^x = S.p^x = \frac{1}{p-1}(p^{x+1} - p) = \frac{p^{x+1} - p}{p-1} = \frac{p(p^x - 1)}{p-1},$$

which is the known sum of the geometric progression;

$$S.xp^x = \frac{1}{p-1}(xp^{x+1} - S.p^x) = \frac{xp^{x+1}}{p-1} - \frac{p^{x+1} - p}{(p-1)^2}$$

or

$$S.xp^x = \frac{pxp^x}{p-1} - \frac{p(p^x-1)}{(p-1)^2};$$

$$S.x^2 p^x = \frac{1}{p-1}(x^2 p^{x+1} - 2S.xp^x + S.p^x)$$

or

$$S.x^2 p^x = \frac{x^2 p^{x+1}}{p-1} - \frac{2xp^{x+1}}{(p-1)^2} + \frac{p(p+1)(p^x-1)}{(p-1)^3}.$$

Further,

$$S.x^3 p^x = \frac{1}{p-1}(x^3 p^{x+1} - 3S.x^2 p^x + 3S.xp^x - S.p^x)$$

or

$$S.x^3 p^x = \frac{x^3 p^{x+1}}{p-1} - \frac{3x^2 p^{x+1}}{(p-1)^2} + \frac{3(p+1)xp^{x+1}}{(p-1)^3} - \frac{p(pp+4p+1)(p^x-1)}{(p-1)^4}$$

and proceeding this way, one will be able to define the sums of the higher powers $x^4 p^x$, $x^5 p^x$, $x^6 p^x$ etc.; but this is achieved more conveniently by means of the general expression we will now investigate.

§170 Since we found that

$$S.yp^x = \frac{1}{p-1}\left(yp^{x+1} - Ap - S.\frac{dy}{dx}p^x + S.\frac{ddy}{2dx^2}p^x - S.\frac{d^3y}{6dx^3}p^x + \text{etc.}\right),$$

where A is a constant of such a kind that the sum becomes $= 0$, if one puts $x = 0$ (for, in this case $y = A$ and $yp^{x+1} = Ap$), we will be able to omit this constant, if we only always remember that to a certain sum always a constant of such a kind is to be added that having put $x = 0$ the sum vanishes or that in another case the correct sum is obtained using our summation formula. Therefore, let us write z instead of y and it will be

$$S.p^x z = \frac{p^{x+1}z}{p-1} - \frac{1}{p-1}S.p^x\frac{dz}{dx} + \frac{1}{2(p-1)}S.p^x\frac{ddz}{dx^2} - \frac{1}{6(p-1)}S.p^x\frac{d^3z}{dx^3}$$
$$+\frac{1}{24(p-1)}S.p^x\frac{d^4z}{dx^4} - \frac{1}{120(p-1)}S.p^x\frac{d^5z}{dx^5} + \text{etc.}$$

Furthermore, let us successively write $\frac{dz}{dx}, \frac{ddz}{dx^2}, \frac{d^3z}{dx^3}$ etc. instead of y and it will be

$$S.\frac{p^x dz}{dx} = \frac{p^{x+1}}{p-1}\cdot\frac{dz}{dx} - \frac{1}{p-1}S.\frac{p^x ddz}{dx^2} + \frac{1}{2(p-1)}S.\frac{p^x d^3z}{dx^3} - \text{etc.}$$

$$S.\frac{p^x ddz}{dx^2} = \frac{p^{x+1}}{p-1}\cdot\frac{ddz}{dx^2} - \frac{1}{p-1}S.\frac{p^x d^3z}{dx^3} + \frac{1}{2(p-1)}S.\frac{p^x d^4z}{dx^4} - \text{etc.}$$

$$S.\frac{p^x d^3z}{dx^3} = \frac{p^{x+1}}{p-1}\cdot\frac{d^3z}{dx^3} - \frac{1}{p-1}S.\frac{p^x d^4z}{dx^4} + \frac{1}{2(p-1)}S.\frac{p^x d^5z}{dx^5} - \text{etc.}$$

etc.

Therefore, if these values are successively substituted, $S.p^x z$ will be expressed by a form of this kind

$$S.p^x z = \frac{p^{x+1}z}{p-1} - \frac{\alpha p^{x+1}}{p-1}\cdot\frac{dz}{dx} + \frac{\beta p^{x+1}}{p-1}\cdot\frac{ddz}{dx^2} - \frac{\gamma p^{x+1}}{p-1}\cdot\frac{d^3z}{dx^3}$$
$$+\frac{\delta p^{x+1}}{p-1}\cdot\frac{d^4z}{dx^4} - \frac{\varepsilon p^{x+1}}{p-1}\cdot\frac{d^5z}{dx^5} + \text{etc.}$$

§171 To define the values of the letters α, β, γ, δ, ε etc. for each term substitute the series found before, i.e.

$$\frac{p^{x+1}z}{p-1} = S.p^x z + \frac{1}{p-1}S.\frac{p^x dz}{dx} - \frac{1}{2(p-1)}S.\frac{p^x ddz}{dx^2} + \frac{1}{6(p-1)}S.\frac{p^x d^3z}{dx^3} - \text{etc.}$$

$$\frac{p^{x+1}dz}{(p-1)dx} = S.\frac{p^x dz}{dx} + \frac{1}{p-1}S.\frac{p^x ddz}{dx^2} - \frac{1}{2(p-1)}S.\frac{p^x d^3z}{dx^3} + \text{etc.}$$

$$\frac{p^{x+1}ddz}{(p-1)dx^2} = S.\frac{p^x ddz}{dx^2} + \frac{1}{p-1}S.\frac{p^x d^3z}{dx^3} - \text{etc.}$$

$$\frac{p^{x+1}d^3z}{(p-1)dx^3} = S.\frac{p^x d^3z}{dx^3} + \text{etc.}$$

Therefore, we will have

$$S.p^x z = S.p^x z$$

$$\begin{array}{lllll}
+\dfrac{1}{p-1}S.\dfrac{p^x dz}{dx} & -\dfrac{1}{2(p-1)}S.\dfrac{p^x ddz}{dx^2} & +\dfrac{1}{6(p-1)}S.\dfrac{p^x d^3z}{dx^3} & -\dfrac{1}{24(p-1)}S.\dfrac{p^x d^4z}{dx^4} & +\text{etc.} \\
-\alpha & -\dfrac{\alpha}{p-1} & +\dfrac{\alpha}{2(p-1)} & -\dfrac{\alpha}{6(p-1)} & \\
 & +\beta & +\dfrac{\beta}{p-1} & -\dfrac{\beta}{2(p-1)} & \\
 & & -\gamma & -\dfrac{\gamma}{p-1} & \\
 & & & +\delta &
\end{array}$$

whence the following values of the coefficients α, β, γ, δ etc. will be obtained

$$\alpha = \frac{1}{p-1}, \quad \beta = \frac{1}{p-1}\left(\alpha + \frac{1}{2}\right), \quad \gamma = \frac{1}{p-1}\left(\beta + \frac{\alpha}{2} + \frac{1}{6}\right),$$

$$\delta = \frac{1}{p-1}\left(\gamma + \frac{\beta}{2} + \frac{\alpha}{6} + \frac{1}{24}\right), \quad \varepsilon = \frac{1}{p-1}\left(\delta + \frac{\gamma}{2} + \frac{\beta}{6} + \frac{\alpha}{24} + \frac{1}{120}\right)$$

etc.

§172 For the sake of brevity, let $\frac{1}{p-1} = q$; it will be

$$\alpha = q$$

$$\beta = \alpha q + \frac{1}{2}q = qq + \frac{1}{2}q$$

$$\gamma = \beta + \frac{1}{2}\alpha q + \frac{1}{6}q = q^3 + qq + \frac{1}{6}q$$

$$\delta = \gamma q + \frac{1}{2}\beta q + \frac{1}{6}\alpha q + \frac{1}{24}q = q^4 + \frac{3}{2}q^3 + \frac{7}{12}q^2 + \frac{1}{24}q$$

$$\varepsilon = \delta q + \frac{1}{2}\gamma q + \frac{1}{6}\beta q + \frac{1}{24}\alpha q + \frac{1}{120}q = q^5 + 2q^4 + \frac{5}{4}q^3 + \frac{1}{4}q^2 + \frac{1}{120}q$$

$$\zeta = q^6 + \frac{5}{2}q^5 + \frac{13}{6}q^4 + \frac{3}{4}q^3 + \frac{31}{360}q^2 + \frac{1}{720}q$$

etc.

or express them this way

$$\alpha = \frac{q}{1}$$

$$\beta = \frac{2qq + q}{1 \cdot 2}$$

$$\gamma = \frac{6q^3 + 6q^2 + q}{1 \cdot 2 \cdot 3}$$

$$\delta = \frac{24q^4 + 36q^3 + 14q^2 + q}{1 \cdot 2 \cdot 3 \cdot 4}$$

$$\varepsilon = \frac{120q^5 + 240q^4 + 150q^3 + 30q^2 + q}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$$

$$\zeta = \frac{720q^6 + 1800q^5 + 1560q^4 + 540q^3 + 62q^2 + q}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$$

$$\eta = \frac{5040q^7 + 15120q^6 + 16800q^5 + 8400q^4 + 1806q^3 + 126q^2 + q}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}$$

etc.,

where the coefficient 16800 results, if the sum of the two above ones, $1560 + 1800$, is multiplied by the exponent of q, which is 5 here.

§173 Let us substitute the value $\frac{1}{p-1}$ for q again:

$$\alpha = \frac{1}{1(p-1)}$$

$$\beta = \frac{p+1}{1\cdot 2(p-1)^2}$$

$$\gamma = \frac{pp+4p+1}{1\cdot 2\cdot 3(p-1)^3}$$

$$\delta = \frac{p^3+11p^2+11p+1}{1\cdot 2\cdot 3\cdot 4(p-1)^4}$$

$$\varepsilon = \frac{p^4+26p^3+66p^2+26p^2+1}{1\cdot 2\cdot 3\cdot 4\cdot 5(p-1)^5}$$

$$\zeta = \frac{p^5+57p^4+302p^3+302p^2+57p+1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6(p-1)^6}$$

$$\eta = \frac{p^6+120p^5+1191p^4+2416p^3+1191p^2+120p+1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7(p-1)^7}$$

etc.

The structure of these formulas is that, if any term is set

$$= \frac{p^{n-2}+Ap^{n-3}+Bp^{n-4}+Cp^{n-5}+Dp^{n-6}+\text{etc.}}{1\cdot 2\cdot 3\cdots(n-1)(p-1)^{n-1}},$$

it will be

$$A = 2^{n-1} - n$$

$$B = 3^{n-1} - n \cdot 2^{n-1} + \frac{n(n-1)}{1 \cdot 2}$$

$$C = 4^{n-1} - n \cdot 3^{n-1} + \frac{n(n-1)}{1 \cdot 2} 2^{n-1} - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$$

$$D = 5^{n-1} - n \cdot 4^{n-1} + \frac{n(n-1)}{1 \cdot 2} 3^{n-1} - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} 2^{n-1} + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}$$

etc.,

whence these coefficients α, β, γ, δ etc. can be continued arbitrarily far.

§174 But if on the other hand we consider the law, according to which these coefficients depend on each other, it will easily become clear that they constitute a recurring series and result, if this fraction is expanded

$$\frac{1}{1 - \frac{u}{p-1} - \frac{u^2}{2(p-1)} - \frac{u^3}{6(p-1)} - \frac{u^4}{24(p-1)} - \text{etc.}};$$

for, this series will result

$$1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \zeta u^6 + \text{etc.};$$

put that fraction $= V$, and since

$$V = \frac{p-1}{p - 1 - u - \frac{u^2}{2} - \frac{u^3}{6} - \frac{u^4}{24} - \text{etc.}},$$

it will be

$$V = \frac{p-1}{p - e^u},$$

where e is the number whose hyperbolic logarithm is $= 1$. And if the value of V is expressed by means of a power series in u, this equation will result

$$V = 1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \zeta u^6 + \text{etc.},$$

whose coefficients α, β, γ, δ etc. will be those, we need in the present task. Therefore, having found those, it will be

$$S.p^x z = \frac{p^{x+1}}{p-1}\left(z - \frac{\alpha dz}{dx} + \frac{\beta ddz}{dx^2} - \frac{\gamma d^3 z}{dx^3} + \frac{\delta d^4 z}{dx^4} - \text{etc.}\right) \pm \text{Const.},$$

which expression therefore is the summatory of this series

$$ap + bp^2 + cp^3 + \cdots + p^x z,$$

whose general term is $= p^x z$.

§175 Since we found that $V = \frac{p-1}{p-e^u}$, it will be

$$e^u = \frac{pV - p + 1}{V}$$

and, by taking logarithms, it will be

$$u = \log(pV - p + 1) - \log V$$

and hence by differentiating

$$du = \frac{(p-1)dV}{pV^2 - (p-1)V},$$

whence it will be

$$pV^2 = (p-1)V + \frac{(p-1)dV}{du}.$$

Therefore, since

$$V = 1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \text{etc.},$$

it will be

$$\begin{aligned} pV^2 = p + 2\alpha u &+ 2\beta p u^2 + 2\gamma p u^3 + 2\delta p u^4 + 2\varepsilon p u^5 + \text{etc.} \\ &+ \alpha^2 p u^2 + 2\alpha\beta p u^3 + 2\alpha\gamma p u^4 + 2\alpha\delta p u^5 + \text{etc.} \\ &+ \beta\beta p u^4 + 2\beta\gamma p u^5 + \text{etc.} \end{aligned}$$

$$(p-1)V = (p-1) + \alpha(p-1)u + \beta(p-1)u^2 + \gamma(p-1)u^3$$
$$+\delta(p-1)u^4 + \varepsilon(p-1)u^5 + \text{etc.}$$
$$\frac{(p-1)dV}{du} = (p-1)\alpha + 2(p-1)\beta u + 2(p-1)\gamma u^2 + 4(p-1)\delta u^3$$
$$+5(p-1)\varepsilon u^4 + 6(p-1)\zeta u^5 + \text{etc.},$$

equating which expressions one will find

$$\begin{aligned}
1(p-1)\alpha &= \ 1 \\
2(p-1)\beta &= \alpha(p+1) \\
3(p-1)\gamma &= \beta(p+1) + \alpha^2 p \\
4(p-1)\delta &= \gamma(p+1) + 2\alpha\beta p \\
5(p-1)\varepsilon &= \delta(p+1) + 2\alpha\gamma p + \beta\beta p \\
6(p-1)\zeta &= \varepsilon(p+1) + 2\alpha\delta p + 2\beta\gamma p \\
7(p-1)\eta &= \zeta(p+1) + 2\alpha\varepsilon p + 2\beta\delta p + \gamma\gamma p \\
&\quad \text{etc.}
\end{aligned}$$

from which formulas, if given the number p, the values of the coefficients α, β, γ, δ etc. can be determined more easily than from the rule we found first.

§176 Before considering special cases of the value p, let us put $z = x^n$, so that this series has to be summed

$$s = p + 2^n p^2 + 3^n p^3 + 4^n p^4 + \cdots + x^n p^x,$$

and by the expression found before it will be

$$s = p^x \left\{ \begin{array}{c} \dfrac{p}{p-1} \cdot x^n - \dfrac{p}{(p-1)^2} n x^{n-1} + \dfrac{pp+p}{(p-1)^3} \cdot \dfrac{n(n-1)}{1 \cdot 2} x^{n-2} \\ - \dfrac{p^3+4p^2+p}{(p-1)^4} \cdot \dfrac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^{n-3} + \text{etc.} \end{array} \right\}$$

$\pm C$, which renders $s = 0$, if one puts $x = 0$.

Therefore, successively substituting the numbers 0, 1, 2, 3, 4 etc. for n, it will be

$$S.x^0p^x = p^x\frac{p}{p-1} - \frac{p}{p-1}$$

$$S.x^1p^x = p^x\left(\frac{px}{p-1} - \frac{p}{(p-1)^2}\right) + \frac{p}{(p-1)^2}$$

$$S.x^2p^x = p^x\left(\frac{px^2}{p-1} - \frac{2px}{(p-1)^2} + \frac{p(p+1)}{(p-1)^3}\right) - \frac{p(p+1)}{(p-1)^3}$$

$$S.x^3p^x = p^x\left(\frac{px^3}{p-1} - \frac{3px^2}{(p-1)^2} + \frac{3p(p+1)x}{(p-1)^3} - \frac{p(p^2+4p+1)}{(p-1)^4}\right) + \frac{p(p^2+4p+1)}{(p-1)^4}$$

$$S.x^4p^x = p^x\left(\frac{px^4}{p-1} - \frac{4px^3}{(p-1)^2} + \frac{6p(p+1)x^2}{(p-1)^3} - \frac{4p(p^2+4p+1)x}{(p-1)^4} + \frac{p(p^3+11p^2+11p+1)}{(p-1)^5}\right)$$
$$- \frac{p(p^3+11p^2+11p+1}{(p-1)^5}$$

$$S.x^5p^x = \frac{p^{x+1}x^5}{p-1} - \frac{5p^{x+1}x^4}{(p-1)^2} + \frac{10(p+1)p^{x+1}x^3}{(p-1)^3} - \frac{10(p^2+4p+1)p^{x+1}x^2}{(p-1)^4}$$
$$+ \frac{5(p^3+11p^2+11p+1)p^{x+1}x}{(p-1)^5} - \frac{(p^4+26p^3+66p^2+26p+1)(p^{x+1}-p)}{(p-1)^6}$$

$$S.x^6p^x = \frac{p^{x+1}x^6}{p-1} - \frac{6p^{x+1}x^5}{(p-1)^2} + \frac{15(p+1)p^{x+1}x^4}{(p-1)^3} - \frac{20(p^2+4p+1)(p^{x+1}-p)}{(p-1)^4}$$
$$+ \frac{15(p^3+11p^2+11p+1)p^{x+1}x^2}{(p-1)^5} - \frac{6(p^4+26p^3+66p^2+26p+1)p^{x+1}x}{(p-1)^6}$$
$$+ \frac{(p^5+57p^4+302p^3+302p^2+57p+1)(p^{x+1}-p)}{(p-1)}$$

etc.

§177 Therefore, it is understood, if z was a polynomial function of x, that the sum of the series whose general term is p^xz can be exhibited, since by taking the differentials of z one finally gets to vanishing ones. If this series is propounded

$$p + 3p^2 + 6p^3 + 10p^4 + \cdots + \frac{xx+x}{2}p^x,$$

because of

$$z = \frac{xx+x}{2} \quad \text{and} \quad \frac{dz}{dx} = x + \frac{1}{2} \quad \text{and} \quad \frac{ddz}{dx^2} = 1,$$

the summatory term will be

$$s = \frac{p^{x+1}}{p-1}\left(\frac{1}{2}xx + \frac{1}{2}x - \frac{2x+1}{2(p-1)} + \frac{p+1}{2(p-1)^2}\right) - \frac{p}{p-1}\left(\frac{p+1}{2(p-1)^2} - \frac{1}{2(p-1)}\right)$$

or

$$s = p^{x+1}\left(\frac{xx}{2(p-1)} + \frac{(p-3)}{2(p-1)^2} + \frac{1}{(p-1)^3}\right) - \frac{p}{p-1}.$$

But if z was not a polynomial function, then this expression of the summatory term runs to infinity. So, if $z = \frac{1}{x}$, that this series is to be summed

$$s = p + \frac{1}{2}p^2 + \frac{1}{3}p^3 + \frac{1}{4}p^4 + \cdots + \frac{1}{x}p^x,$$

because of

$$\frac{dz}{dx} = -\frac{1}{xx}, \quad \frac{ddz}{dx^2} = \frac{2}{x^3}, \quad \frac{d^3z}{dx^3} = -\frac{2\cdot 3}{x^4}, \quad \frac{d^4z}{dx^4} = \frac{2\cdot 3\cdot 4}{x^5} \quad \text{etc.},$$

this summatory term will result

$$s = \frac{p^{x+1}}{p-1}\left(\frac{1}{x} + \frac{1}{(p-1)x^2} + \frac{p+1}{(p-1)^2x^3} + \frac{pp+4p+1}{(p-1)^3x^4} + \frac{p^3+11p^2+11p+1}{(p-1)^4x^5} + \text{etc.}\right) + C.$$

In this case, the constant C can therefore not be defined from the case $x = 0$; to define it, put $x = 1$, and since $s = p$, it will be

$$C = p - \frac{pp}{p-1}\left(1 + \frac{1}{p-1} + \frac{p+1}{(p-1)^2} + \frac{pp+4p+1}{(p-1)^3} + \text{etc.}\right).$$

§178 From these results it is perspicuous, if p does not denote a specified number, that hardly anything of use to exhibit the sums of series approximately follows from this. But first it is plain that one cannot write 1 for p, since all coefficients α, β, γ, δ etc. would become infinitely large. Therefore, because the series we are considering now, goes over into that one we already contemplated before, if one puts $p = 1$, it is surprising that this case cannot be found from this general expression, although it is the simplest case. Then, on the other hand it is also notable that in the case $p = 1$ the summation requires the integral $\int zdx$, although in general the sum can be exhibited without a single integral. So it happens that, while all the coefficients α, β, γ, δ etc. grow to infinity, at the same time that integral formula is enters the expression. And this case, in which $p = 1$, is the only one to which this general expression found here cannot be applied. But even in this case the general formula is not to be considered to be wrong; for, even though each term becomes infinite, nevertheless all the infinities indeed cancel each other and a finite quantity remains which is equal to the one we found by means of the first method; we will elaborate on this in more detail below.

§179 Therefore, let $p = -1$ and the signs in the series to be summed will alternate

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & & x \\ -\,a & +\,b & -\,c & +\,d & -\cdots & \pm\,z, \end{array}$$

where z will be positive, if x was an even number, but negative, if x is an odd number. Therefore, having put

$$-a + b - c + d - \cdots \pm z = s,$$

it will be

$$s = \pm\frac{1}{2}\left(z - \frac{\alpha dz}{dx} + \frac{\beta ddz}{dx^2} - \frac{\gamma d^3z}{dx^3} + \frac{\delta d^4z}{dx^4} - \text{etc.}\right) + C,$$

where the upper of the ambiguous signs hold, if x is an even number, the other, if x is an odd number. Therefore, by changing the sings, it will be

$$a - b + c - d + e - f + \cdots \mp z = \mp\frac{1}{2}\left(z - \frac{\alpha dz}{dx} + \frac{\beta ddz}{dx^2} - \frac{\gamma d^3z}{dx^3} + \frac{\delta d^4z}{dx^4} - \text{etc.}\right) + C,$$

where the ambiguity of the signs follows the same rule.

§180 In this case the coefficients α, β, γ, δ, ε, ζ etc. can be found from the values given before by putting $p = -1$ everywhere. But they will be found more easily from the general formulas given in § 175, whence at the same time it is understood that each second of these coefficients vanishes. For, having put $p = -1$, these formulas will go over into

$$-2\alpha = 0, \quad -4\beta = 0, \quad -6\gamma = 0 - \alpha^2, \quad -8\delta = 0 - 2\alpha\beta,$$
$$-10\varepsilon = 0 - 2\alpha\gamma - \beta\beta, \quad -12\zeta = 0 - 2\alpha\delta - 2\beta\gamma \quad \text{etc.}$$

whence, because $\beta = 0$, it also will be $\delta = 0$ and further $\zeta = 0$, $\theta = 0$ and the remaining letters will be determined as follows

$$\alpha = -\frac{1}{2}, \quad \gamma = \frac{\alpha^2}{6}, \quad \varepsilon = \frac{2\alpha\gamma}{10}, \quad \eta = \frac{2\alpha\varepsilon + \gamma\gamma}{14}, \quad \iota = \frac{2\alpha\eta + 2\gamma\varepsilon}{18} \quad \text{etc.}$$

§181 That this calculation can be performed in a more convenient way, let us introduce new letters and let

$$\alpha = -\frac{A}{1 \cdot 2}, \quad \gamma = \frac{B}{1 \cdot 2 \cdot 3 \cdot 4}, \quad \varepsilon = -\frac{C}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6},$$
$$\eta = \frac{D}{1 \cdot 2 \cdot 3 \cdots 8}, \quad \iota = -\frac{E}{1 \cdot 2 \cdot 3 \cdots 10}$$
$$\text{etc.}$$

And the sum exhibited before will be

$$\mp\frac{1}{2}\left(z + \frac{Adz}{1 \cdot 2dx} - \frac{Bd^3z}{1 \cdot 2 \cdot 3 \cdot 4dx^3} + \frac{Cd^5z}{1 \cdot 2 \cdots 6dx^5} - \frac{Dd^7z}{1 \cdot 2 \cdots 8dx^7} + \text{etc.}\right) + C.$$

But the coefficients will be defined from the following formulas

$$
\begin{aligned}
1A &= 1\\
3B &= \frac{4\cdot 3}{1\cdot 2}\cdot\frac{AA}{2}\\
5C &= \frac{6\cdot 5}{1\cdot 2}\cdot AB\\
7D &= \frac{8\cdot 7}{1\cdot 2}\cdot AC+\frac{8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{BB}{2}\\
9E &= \frac{10\cdot 9}{1\cdot 2}\cdot AD+\frac{10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4}\cdot BC\\
11F &= \frac{12\cdot 11}{1\cdot 2}\cdot AE+\frac{12\cdot 11\cdot 10\cdot 9}{1\cdot 2\cdot 3\cdot 4}\cdot BD+\frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\cdot\frac{CC}{2}\\
&\text{etc.,}
\end{aligned}
$$

which can be represented more easily and are more accommodated to calculation purposes this way

$$
\begin{aligned}
A &= 1\\
B &= 2\cdot\frac{AA}{2}\\
C &= 3\cdot AB\\
D &= 4\cdot AC+4\cdot\frac{6\cdot 5}{3\cdot 4}\cdot\frac{BB}{2}\\
E &= 5\cdot AC+5\cdot\frac{8\cdot 7}{3\cdot 4}\cdot BC\\
F &= 6\cdot AD+6\cdot\frac{10\cdot 9}{3\cdot 4}\cdot BD+6\cdot\frac{10\cdot 9\cdot 8\cdot 7}{\cdot 3\cdot 4\cdot 5\cdot 6}\cdot\frac{CC}{2}\\
G &= 7\cdot AE+6\cdot\frac{12\cdot 11}{3\cdot 4}\cdot BE+7\cdot\frac{12\cdot 11\cdot 10\cdot 9}{\cdot 3\cdot 4\cdot 5\cdot 6}\cdot CD\\
&\text{etc.}
\end{aligned}
$$

Therefore, having done the calculation, one will find

$$\begin{aligned}
A &= 1 \\
B &= 1 \\
C &= 3 \\
D &= 17 \\
E &= 155 &&= 5 \cdot 31 \\
F &= 2073 &&= 691 \cdot 3 \\
G &= 38227 &&= 7 \cdot 5461 &&= 7 \cdot \frac{127 \cdot 129}{3} \\
H &= 929569 &&= 3617 \cdot 257 \\
I &= 28820619 &&= 43867 \cdot 9 \cdot 73 \\
& \text{etc.}
\end{aligned}$$

§182 If we consider these numbers with more attention, from the factors 691, 3617, 43867 one can easily conclude that these numbers have a connection to the Bernoulli numbers exhibited above [§ 122] and can hence be determined. Therefore, anyone investigating this relation will immediately see that these numbers can be formed from the Bernoulli numbers $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$ etc. the following way:

$$\begin{aligned}
A &= 2 \cdot 1 \cdot 3\,\mathfrak{A} &&= 2(2^2 - 1)\mathfrak{A} \\
B &= 2 \cdot 3 \cdot 5\,\mathfrak{B} &&= 2(2^4 - 1)\mathfrak{B} \\
C &= 2 \cdot 7 \cdot 9\,\mathfrak{C} &&= 2(2^6 - 1)\mathfrak{C} \\
D &= 2 \cdot 15 \cdot 17\,\mathfrak{D} &&= 2(2^8 - 1)\mathfrak{D} \\
E &= 2 \cdot 31 \cdot 33\,\mathfrak{E} &&= 2(2^{10} - 1)\mathfrak{E} \\
F &= 2 \cdot 63 \cdot 67\,\mathfrak{F} &&= 2(2^{12} - 1)\mathfrak{F} \\
G &= 2 \cdot 127 \cdot 129\,\mathfrak{G} &&= 2(2^{14} - 1)\mathfrak{G} \\
H &= 2 \cdot 255 \cdot 257\,\mathfrak{H} &&= 2(2^{16} - 1)\mathfrak{H} \\
& \text{etc.}
\end{aligned}$$

Since the Bernoulli numbers are fractions, but our coefficients on the other hand are integers, it is plain that these factors always cancel the fractions and they will therefore be

$$\begin{aligned}
A &= 1\\
B &= 1\\
C &= 3\\
D &= 17\\
E &= 5\cdot 31 = 155\\
F &= 3\cdot 691 = 2073\\
G &= 7\cdot 43\cdot 127 = 38277\\
H &= 257\cdot 3617 = 929569\\
I &= 9\cdot 73\cdot 43867 = 28820619\\
K &= 5\cdot 31\cdot 41\cdot 174611 = 1109652905\\
L &= 89\cdot 683\cdot 854513 = 51943281713\\
M &= 3\cdot 4097\cdot 236364091 = 2905151042481\\
N &= 2731\cdot 8191\cdot 8553103 = 191329672483963\\
&\quad\quad \text{etc.}
\end{aligned}$$

Therefore, using these numbers, one will vice versa be able to find the Bernoulli numbers.

§183 Therefore, by applying these Bernoulli numbers to the propounded series

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & & & x \\ a & -\,b & +\,c & -\,d & +\,e & - & \cdots & \mp z \end{array}$$

the sum will be

$$\mp\left(\frac{1}{2}z+\frac{(2^2-1)\mathfrak{A}dz}{1\cdot 2dx}-\frac{(2^4-1)\mathfrak{B}d^3z}{1\cdot 2\cdot 3\cdot 4dx^3}+\frac{(2^6-1)\mathfrak{C}d^5z}{1\cdot 2\cdots 6dx^5}-\frac{(2^8-1)\mathfrak{D}d^7z}{1\cdot 2\cdots 8dx^7}+\text{etc.}\right)+\text{Const.}$$

But hence it is understood that these numbers do not enter this expression accidentally; for, as the propounded series results, if from this one

$$a+b+c+d+\cdots+z,$$

where all terms have the sign $+$, the sum of all second terms $b+d+f+$ etc. is subtracted twice, so also the found expression can be resolved into two parts of which the one is the sum of all terms affected with the sign $+$ which will be

$$\int zdx + \frac{1}{2}z + \frac{\mathfrak{A}dz}{1\cdot 2dx} - \frac{\mathfrak{B}d^3z}{1\cdot 2\cdot 3\cdot 4dx^3} + \frac{\mathfrak{C}d^5z}{1\cdot 2\cdots 6dx^5} - \text{etc.}$$

But the sum of all second terms is found the same way we did it above [chap. V]. Since the last term z corresponds to the index x, the term corresponding to the index $x-2$ will be

$$z - \frac{2dz}{dx} + \frac{2^2ddz}{1\cdot 2dx^2} - \frac{2^3d^3z}{1\cdot 2\cdot 3dx^3} + \frac{2^4d^4z}{1\cdot 2\cdot 3\cdot 4dx^4} - \text{etc. },$$

which form results from the other expressing the preceding term, if one writes $\frac{x}{2}$ instead of x. Therefore, one will have the sum of all the second terms, if in the sum of all terms one writes $\frac{x}{2}$ instead of x everywhere; therefore, this sum will be

$$\frac{1}{2}\int zdx + \frac{1}{2}z + \frac{2\mathfrak{A}dz}{1\cdot 2dx} - \frac{2^3\mathfrak{B}d^3z}{1\cdot 2\cdot 3\cdot 4dx^3} + \frac{2^5\mathfrak{C}d^5z}{1\cdot 2\cdots 6dx^5} - \text{etc.};$$

if its double is subtracted from the preceding while x is an even number or if the preceding sum is subtracted from the double of this one, if x is an odd number, the residue will show the sum of the series

$$\begin{array}{ccccccccccc} 1 & & 2 & & 3 & & 4 & & 5 & & x \\ a & - & b & + & c & - & d & + & e & - \cdots & \mp z, \end{array}$$

which will therefore be

$$\mp\left(\frac{1}{2}z + \frac{(2^2-1)\mathfrak{A}dz}{1\cdot 2dx} - \frac{(2^4-1)\mathfrak{B}d^3z}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}\right) + C,$$

which is the same expression we just found.

§184 Take a power of x for z, namely x^n, so that one finds the sum of the series

$$1 - 2^n + 3^n - 4^n + \cdots - \pm x^n$$

Because of

$$\frac{dz}{1dx} = \frac{n}{1}x^{n-1}, \quad \frac{d^3z}{1\cdot 2\cdot 3dx^3} = \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3} \quad \text{etc.},$$

introducing the coefficients A, B, C, D, E etc., it will be

$$\mp\frac{1}{2}\left\{\begin{array}{c} x^n + \frac{A}{2}nx^{n-1} - \frac{B}{4}\cdot\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3} + \frac{C}{6}\cdot\frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^{n-5} \\ -\frac{D}{8}\cdot\frac{n(n-1)\cdots(n-6)}{1\cdot 2\cdots 7}x^{n-7} + \text{etc.} + \text{Const.} \end{array}\right\}$$

where the upper sign holds, if x is an even number, the lower on the other hand, if it is an odd number. But the constant has to defined in such a way that the sum vanishes, if $x = 0$, in which case the above sign holds. Successively substituting the numbers 0, 1, 2, 3 etc. for n, the following sums will result

$$\text{I.} \quad 1 - 1 + 1 - 1 + \cdots \mp 1 = \mp\frac{1}{2}(1) + \frac{1}{2};$$

if the number of terms was even, the sum will be $= 0$, if odd, it will be $= +1$.

$$\text{II.} \quad 1 - 2 + 3 - 4 + \cdots \mp x = \mp\frac{1}{2}\left(x + \frac{1}{2}\right) + \frac{1}{4};$$

if the number of terms is even, the sum will be $= -\frac{1}{2}x$ and for an odd number of terms $= +\frac{1}{2}x + \frac{1}{2}$.

$$\text{III.} \quad 1 - 2^2 + 3^2 - 4^2 + \cdots \mp x^2 = \mp\frac{1}{2}\left(x^2 + x\right)$$

for an even number $= -\frac{1}{2}xx - \frac{1}{2}x$ and for an odd number $= +\frac{1}{2}xx + \frac{1}{2}x$.

$$\text{IV.} \quad 1 - 2^3 + 3^3 - 4^3 + \cdots \mp x^3 = \mp\frac{1}{2}\left(x^3 + \frac{3}{2}xx - \frac{1}{4}\right) - \frac{1}{8};$$

for even number of terms $= -\frac{1}{2}x^3 - \frac{3}{4}x^2$ and for an odd number $= \frac{1}{2}x^3 + \frac{3}{4}x^2 - \frac{1}{4}$.

$$\text{V.} \quad 1 - 2^4 + 3^4 - 4^4 + \cdots \mp x^4 = \mp\frac{1}{2}(x^4 - 2x^3 - x);$$

for an even number of terms $= -\frac{1}{2}x^4 - x^3 + \frac{1}{2}x^2$ and for an odd number $= \frac{1}{2}x^4 + x^3 - \frac{1}{2}x$ etc.

§185 Therefore, it is clear that in the sums of even powers except for the case $n = 0$ the constant to be added vanishes and in these cases the sum of an even number or an odd number of terms only differs with regard to the sign. Therefore, if x was an infinite number, since it is neither even nor odd, this consideration is not valid and the ambiguous signs are to be rejected; therefore, it follows that the sum of series, if they are continued infinity, of this kind are expressed by means of the constant to be added alone.

Therefore, it will be

$$\begin{aligned}
&1 - 1 \ + 1 \ - 1 \ + \text{etc. to infinity} &&= +\frac{1}{2} \\
&1 - 2 \ + 3 \ - 4 \ + \text{etc.} &&= +\frac{A}{4} = +\frac{(2^2-1)\mathfrak{A}}{2} \\
&1 - 2^2 + 3^2 - 4^2 + \text{etc.} &&= 0 \\
&1 - 2^3 + 3^3 - 4^3 + \text{etc.} &&= -\frac{B}{8} = -\frac{(2^4-1)\mathfrak{B}}{4} \\
&1 - 2^4 + 3^4 - 4^4 + \text{etc.} &&= 0 \\
&1 - 2^5 + 3^5 - 4^5 + \text{etc.} &&= +\frac{C}{12} = +\frac{(2^6-1)\mathfrak{C}}{6} \\
&1 - 2^6 + 3^6 - 4^6 + \text{etc.} &&= 0 \\
&1 - 2^7 + 3^7 - 4^7 + \text{etc.} &&= -\frac{D}{16} = -\frac{(2^8-1)\mathfrak{D}}{8} \\
&\qquad\qquad\text{etc.}
\end{aligned}$$

These same sums are found by means of the method to sum series in which the signs $+$ and $-$ alternate treated above [§ 8].

§186 If one takes negative numbers for n, the expression for the sums runs to infinity. Let $n = -1$; the sum of the series will be

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \mp \frac{1}{x}$$

$$= \mp\frac{1}{2}\left(\frac{1}{x} - \frac{A}{2x^2} + \frac{B}{4x^4} - \frac{C}{6x^6} + \frac{D}{8x^8} - \text{etc.}\right) + \text{Const.}$$

Because here the constant cannot be defined from the case $x = 0$, it is to be defined from another case. Put $x = 1$ and, because of the sum $= 1$ and the lower sign holds, it will be

$$\text{Const.} = 1 - \frac{1}{2}\left(\frac{1}{1} - \frac{A}{2} + \frac{B}{4} - \frac{C}{6} + \text{etc.}\right)$$

or

$$\text{Const.} = \frac{1}{2} + \frac{A}{4} - \frac{B}{8} + \frac{C}{12} - \frac{D}{16} + \text{etc.}$$

Or put $x = 2$; because of the sum $= \frac{1}{2}$, and since the upper sign holds, one will find

$$\text{Const.} = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} - \frac{A}{2 \cdot 2^2} + \frac{B}{4 \cdot 2^4} - \frac{C}{6 \cdot 2^6} + \text{etc.}\right)$$

or

$$\text{Const.} = \frac{3}{4} - \frac{A}{4 \cdot 2^2} + \frac{B}{8 \cdot 2^4} - \frac{C}{12 \cdot 2^6} + \frac{D}{16 \cdot 2^8} - \text{etc.}$$

But if one puts $x = 4$, it will be

$$\text{Const.} = \frac{17}{24} - \frac{A}{4 \cdot 4^2} + \frac{B}{8 \cdot 4^4} - \frac{C}{12 \cdot 4^6} + \frac{D}{16 \cdot 4^8} - \text{etc.}$$

But no matter how the constant is defined, the same value will result, which value at the same time will indicate the sum of the series continued to infinity which is $= \log 2$.

§187 Additionally, from these new numbers A, B, C, D, E etc. the sums of the series of the reciprocal powers in which only the odd numbers occur can be summed in a convenient way. For, if one puts

$$1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} + \frac{1}{6^{2n}} + \text{etc.} = s$$

it will be

$$+\frac{1}{2^{2n}} \qquad + \frac{1}{4^{2n}} \qquad + \frac{1}{6^{2n}} + \text{etc.} = \frac{s}{2^{2n}},$$

which subtracted from the other gives

$$1+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\text{etc.}=\frac{(2^{2n}-1)s}{2^{2n}},$$

Because we exhibited the values of s for each number n already above (§ 125), it will be

$$1++\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\text{etc.}=\frac{A}{1\cdot 2}\cdot\frac{\pi^2}{4}$$

$$1++\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\text{etc.}=\frac{B}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{\pi^4}{4}$$

$$1++\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\text{etc.}=\frac{C}{1\cdot 2\cdot 3\cdots 6}\cdot\frac{\pi^6}{4}$$

$$1++\frac{1}{3^8}+\frac{1}{5^8}+\frac{1}{7^8}+\text{etc.}=\frac{D}{1\cdot 2\cdot 3\cdots 8}\cdot\frac{\pi^8}{4}$$

$$1++\frac{1}{3^{10}}+\frac{1}{5^{10}}+\frac{1}{7^{10}}+\text{etc.}=\frac{E}{1\cdot 2\cdot 3\cdots 10}\cdot\frac{\pi^2}{4}$$

etc.

But if all numbers enter the expression and the signs alternate, since it will be

$$1-\frac{1}{2^{2n}}+\frac{1}{3^{2n}}-\frac{1}{4^{2n}}+\text{etc.}=\frac{(2^{2n}-1)s-s}{2^{2n}},$$

one will have

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\text{etc.}=\frac{A-2\mathfrak{A}}{1\cdot 2}\cdot\frac{\pi^2}{4}=\frac{(2-1)\mathfrak{A}}{1\cdot 2}\cdot\pi^2$$

$$1-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\frac{1}{5^4}-\text{etc.}=\frac{B-2\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{\pi^4}{4}=\frac{(2^3-1)\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4}\cdot\pi^4$$

$$1-\frac{1}{2^6}+\frac{1}{3^6}-\frac{1}{4^6}+\frac{1}{5^6}-\text{etc.}=\frac{C-2\mathfrak{C}}{1\cdot 2\cdots 6}\cdot\frac{\pi^6}{4}=\frac{(2^5-1)\mathfrak{C}}{1\cdot 2\cdots 6}\cdot\pi^6$$

$$1-\frac{1}{2^8}+\frac{1}{3^8}-\frac{1}{4^8}+\frac{1}{5^8}-\text{etc.}=\frac{D-2\mathfrak{D}}{1\cdot 2\cdots 8}\cdot\frac{\pi^8}{4}=\frac{(2^7-1)\mathfrak{D}}{1\cdot 2\cdots 8}\cdot\pi^8$$

$$1-\frac{1}{2^{10}}+\frac{1}{3^{10}}-\frac{1}{4^{10}}+\frac{1}{5^{10}}-\text{etc.}=\frac{E-2\mathfrak{E}}{1\cdot 2\cdots 10}\cdot\frac{\pi^{10}}{4}=\frac{(2^9-1)\mathfrak{E}}{1\cdot 2\cdots 10}\cdot\pi^{10}$$

etc.

§188 As up to this point we contemplated series whose terms are the products of terms from the geometric progression p, p^2, p^3 etc. and terms of the arbitrary series a, b, c etc., so we will be able to investigate the series whose general terms are the products of terms of two arbitrary series one of which known. Let this series be known

$$\begin{matrix} 1 & & 2 & & 3 & & & & z \\ A & + & B & + & C & + & \cdots & + & Z, \end{matrix}$$

the other on the other hand unknown, which reads

$$a+b+c+\cdots+z,$$

and let the sum of this series be in question

$$Aa+Bb+Cc+\cdots+Zz,$$

which we want to put $=Zs$. Let the penultimate term in the known series be $=Y$ and having written $x-1$ instead of x the expression of the sum $S.Zz$ will go over into

$$Y\left(s-\frac{ds}{dx}+\frac{dds}{2dx^2}-\frac{d^3s}{6dx^3}+\frac{d^4s}{24dx^4}-\text{etc.}\right).$$

Since this sum expresses the series Zs decreased by the last term Zz, it will be

$$Zs - Zz = Ys - \frac{Yds}{dx} + \frac{Ydds}{2dx^2} - \frac{Yd^3s}{6dx^3} + \text{etc.},$$

which equation contains the relation how the sum Zs depends on Y, Z and z.

§189 To resolve this equation, first neglect the differential terms and it will be

$$s = \frac{Zz}{Z-Y};$$

put this value $\frac{Zz}{Z-Y} = P^{\mathrm{I}}$ and let the true sum be $s = P^{\mathrm{I}} + p$; having substituted this value in the equation, it will be

$$\begin{aligned}(Z-Y)p \;=\; & -\frac{YdP^{\mathrm{I}}}{dx} + \frac{YddP^{\mathrm{I}}}{2dx^2} - \text{etc.}\\ & -\frac{Ydp}{dx} + \frac{Yddp}{2dx^2} - \text{etc.};\end{aligned}$$

add YP^{I} on both sides, and since $P^{\mathrm{I}} - \frac{dP^{\mathrm{I}}}{dx} + \frac{YddP^{\mathrm{I}}}{2dx^2} - $etc. is the value of P^{I} which results, if one writes $x-1$ instead of x, let this value be P and it will be

$$(Z-Y)p + YP^{\mathrm{I}} = YP - \frac{Ydp}{dx} + \frac{Yddp}{2dx^2} - \text{etc.},$$

whence neglecting the differentials, it will be

$$p = \frac{Y(P-P^{\mathrm{I}})}{Z-Y}.$$

Put $\frac{Y(P-P^{\mathrm{I}})}{Z-Y} = Q^{\mathrm{I}}$ and let $p = Q^{\mathrm{I}} + q$; it will be

$$(Z-Y)q = -\frac{Y(dQ^{\mathrm{I}} + dq)}{dx} + \frac{Y(ddQ^{\mathrm{I}} + ddq)}{2dx^2} - \text{etc.}$$

and having put Q for the value of Q^{I} it has, if one writes $x-1$ instead of x, it will be

$$(Z-Y)q + YQ^{\mathrm{I}} = YQ - \frac{Ydq}{dx} + \frac{Yddq}{2dx^2} - \text{etc.},$$

whence, neglecting the differentials,

$$q = \frac{Y(Q - Q^{\mathrm{I}})}{Z - Y}.$$

Put $\frac{Y(Q-Q^{\mathrm{I}})}{Z-Y} = R^{\mathrm{I}}$ and let the true value be $q = R^{\mathrm{I}} + r$ and similarly one will find

$$r = \frac{Y(R - R^{\mathrm{I}})}{Z - Y};$$

and, proceeding this way, the sum in question will be

$$Zs = Z(P^{\mathrm{I}} + Q^{\mathrm{I}} + R^{\mathrm{I}} + \text{etc.}).$$

§190 Therefore, having propounded any series

$$Aa + Bb + Cc + \cdots + Yy + Zz,$$

its sum will be defined the following way

$$\begin{array}{llllll}
\text{Put} & & \text{having put} & x-1 & \text{instead of} & x \\
\dfrac{Zz}{Z-Y} & = P^{\mathrm{I}} & \text{and let} & P^{\mathrm{I}} & \text{go over into} & P \\
\dfrac{Y(P-P^{\mathrm{I}})}{Z-Y} & = Q^{\mathrm{I}} & \text{and let} & Q^{\mathrm{I}} & \text{go over into} & Q \\
\dfrac{Y(Q-Q^{\mathrm{I}})}{Z-Y} & = R^{\mathrm{I}} & \text{and let} & R^{\mathrm{I}} & \text{go over into} & R \\
\dfrac{Y(R-R^{\mathrm{I}})}{Z-Y} & = S^{\mathrm{I}} & \text{and let} & S^{\mathrm{I}} & \text{go over into} & S \\
 & \text{etc.} & & & &
\end{array}$$

Having found these values, the sum of the series will be

$$= ZP^{\mathrm{I}} + ZQ^{\mathrm{I}} + ZR^{\mathrm{I}} + ZS^{\mathrm{I}} + \text{etc.}$$

$+$ a constant rendering the sum $= 0$ for $x = 0$, or, what is the same, which forces the sum to be correct in a certain case.

§191 This formula, since it does not contain any differentials, is applied most easily in the most cases and will often even exhibit the true sum. So if this series is propounded

$$p + 4p^2 + 9p^3 + 16p^4 + \cdots + x^2p^x,$$

let $Z = p^x$ and $z = x^2$; it will be $Y = p^{x-1}$ and $\frac{Z}{Z-Y} = \frac{p}{p-1}$ and $\frac{Y}{Z-Y} = \frac{1}{p-1}$. Therefore, it will be

$$\begin{aligned} P^{\mathrm{I}} &= \frac{px^2}{p-1} & P &= \frac{pxx - 2px + p}{p-1} \\ Q^{\mathrm{I}} &= \frac{-2px + p}{(p-1)^2} & Q &= \frac{-2px + 3p}{(p-1)^2} \\ R^{\mathrm{I}} &= \frac{2p}{(p-1)^3} & R &= \frac{2p}{(p-1)^3} \\ S^{\mathrm{I}} &= 0 \end{aligned}$$

and all remaining vanish; therefore, the sum will be

$$\begin{aligned} &= p^x\left(\frac{px^2}{p-1} - \frac{2px-p}{(p-1)^2} + \frac{2p}{(p-1)^3}\right) - \frac{p}{(p-1)^2} - \frac{2p}{(p-1)^3} \\ &= p^{x+1}\left(\frac{x^2}{p-1} - \frac{2x}{(p-1)^2} + \frac{p+1}{(p-1)^3}\right) - \frac{p(p+1)}{(p-1)^3}. \end{aligned}$$

as we already found above [§ 176].

§192 In the same way we got to this expression of the sum we will be able to find another expression, if the propounded series is not composed of two others; this can mainly be used in those cases, in which one gets to vanishing denominators in the preceding expression. Therefore, let this series be propounded

$$\begin{array}{cccccccccccc} & & 1 & & 2 & & 3 & & 4 & & & x \\ s & = & a & + & b & + & c & + & d & + & \cdots & + z; \end{array}$$

since having written $x-1$ instead of x the sum is truncated by the last term, it will be

$$s - z = s - \frac{ds}{dx} + \frac{dds}{2dx^2} - \frac{d^3s}{6dx^3} + \frac{d^4s}{24dx^4} - \text{etc.}$$

or

$$z = \frac{ds}{dx} - \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} - \frac{d^4s}{24dx^4} + \text{etc.}$$

Since here the sum s does not occur, neglect the higher differentials and it will be $s = \int zdx$; put $\int zdx = P^{\mathrm{I}}$ whose value goes over into P, if one writes $x-1$ instead of x, and let the true value be P^{I}; it will be

$$z = \frac{dP^{\mathrm{I}}}{dx} - \frac{ddP^{\mathrm{I}}}{2dx^2} + \text{etc.} + \frac{dp}{dx} - \frac{ddp}{2dx^2} + \text{etc.};$$

since

$$P = P^{\mathrm{I}} - \frac{dP^{\mathrm{I}}}{dx} + \frac{ddP^{\mathrm{I}}}{2dx^2} - \text{etc.},$$

it will be

$$z - P^{\mathrm{I}} + P = \frac{dp}{dx} - \frac{ddp}{2dx^2} + \text{etc.},$$

whence

$$p = \int (z - P^{\mathrm{I}} - P)dx.$$

If one further puts $\int (z - P^{\mathrm{I}} + P)dx = Q^{\mathrm{I}}$ and this value goes over into Q, having written $x-1$ instead of x, let

$$\int (z - P^{\mathrm{I}} + P - Q^{\mathrm{I}} + Q)dx = R^{\mathrm{I}} = Q^{\mathrm{I}} - \int (Q^{\mathrm{I}} - Q)dx,$$

furthermore,

$$R^{\mathrm{I}} - \int (R^{\mathrm{I}} - R)dx = S^{\mathrm{I}}$$

etc.; the sum in question will be

$$s = P^{\mathrm{I}} + Q^{\mathrm{I}} + R^{\mathrm{I}} + S^{\mathrm{I}} + \text{etc.} + \text{Const.},$$

by means of which constant one single case is to be forced to be correct.

§193 Having changed the letters a little bit, this summation reduces to this. Having propounded this series to be summed

$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & & & x \\ s & = a + & b + & c + & d & + & \cdots & + z \end{array}$$

$$\begin{array}{lll} \text{put} & & \text{having put } \; x-1 \; \text{ instead of } \; x \\ \int z dx & = P & \text{and let } \; P \; \text{ go over into } \; p \\ P - \int (P-p) dx & = Q & \text{and let } \; Q \; \text{ go over into } \; q \\ Q - \int (Q-q) dx & = R & \text{and let } \; R \; \text{ go over into } \; r \\ & \text{etc.} & \end{array}$$

having found which values the sum in question will be

$$s = P + Q + R + S + \text{etc.}$$

this expression will immediately show the sum, if these integral formulas can be exhibited. To give an example of its application, let $z = xx + x$ and it will be

$$P = \frac{1}{3}x^3 + \frac{1}{2}xx, \quad p = \frac{1}{3}x^3 - \frac{1}{2}xx + \frac{1}{6}, \quad P - p = xx - \frac{1}{6}$$

and

$$\int (P-p)dx = \frac{1}{3}x^3 - \frac{1}{6}x;$$

$$Q = \frac{1}{2}xx + \frac{1}{6}x, \quad q = \frac{1}{2}xx - \frac{5}{6}x + \frac{1}{3}, \quad Q - q = x - \frac{1}{3}$$

and

$$\int (Q-q)dx = \frac{1}{2}xx - \frac{1}{3}x;$$

$$R = \frac{1}{2}x, \quad r = \frac{1}{2}x - \frac{1}{2}, \quad R - r = \frac{1}{2}$$

and

$$\int (R-r)dx = \frac{1}{2}x;$$

$S = 0$ and the remaining values vanish. Therefore, the sum in question will be

$$\left.\begin{array}{l} \frac{1}{3}x^3 + \frac{1}{2}xx \\ \qquad + \frac{1}{2}xx + \frac{1}{6}x \\ \qquad\qquad + \frac{1}{2}x \end{array}\right\} = \frac{1}{3}x^3 + xx + \frac{2}{3}x = \frac{1}{3}x(x+1)(x+2).$$

And this way the sum of all series whose general terms are polynomial functions of x can be found by means of iterated integration. From these considerations it is easily seen what a broad field the doctrine of the summation of series actually is and several volumes will not suffice to capture all methods which already exist and which can still be thought of.

§194 Up to now we investigated sums of series from the first term up to the one whose index is x and if we know this sum, by putting $x = \infty$, the sum of the series continued to infinity is also found. But in most cases this is achieved more easily, if not the sum of terms from the first to that one whose index is x but the sum of the terms from the one whose index is x up to infinity is found; for, in this case especially the last expressions we found become more tractable. Therefore, let a series be propounded whose general term or the one corresponding to the index x we want to be $= z$, the following corresponding to the index $x+1$ then is $= z^{\mathrm{I}}$ and the ones following after this one are z^{II}, z^{III} etc. and let the sum of this infinite series be in question

$$\begin{array}{ccccccccc} & x & & x+1 & & x+2 & & x+3 & \text{etc.} \\ s = & z & + & z^{\mathrm{I}} & + & z^{\mathrm{II}} & + & z^{\mathrm{III}} & + \text{ etc. to infinity} \end{array}$$

Therefore, the sum s will be a function of x; if in this one writes $x+1$ instead of x, the first sum truncated by the term z will result. Since by this change s goes over into

$$s + \frac{ds}{dx} + \frac{dds}{2dx^2} + \text{etc.},$$

it will be

$$s - z = s + \frac{ds}{dx} + \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} + \frac{d^4s}{24dx^4} + \frac{d^5s}{120dx^5} + \text{etc.}$$

or

$$0 = z + \frac{ds}{dx} + \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} + \frac{d^4s}{24dx^4} + \frac{d^5s}{120dx^5} + \text{etc.}$$

§195 If we now argue as before, having neglected the higher differentials, it will be $s = C - \int zdx$. Therefore, put $\int zdx = P$ and let the true value be $s = C - P + p$; it will be

$$0 = z - \frac{dP}{dx} - \frac{ddP}{2dx^2} - \frac{d^3P}{6dx^3} - \text{etc.}$$

$$+ \frac{dp}{dx} + \frac{ddp}{2dx^2} + \frac{d^3p}{6dx^3} + \text{etc.}$$

Let P go over into P^{I}, if one writes $x + 1$ instead of x, and it will be

$$0 = z + P - P^{\mathrm{I}} + \frac{dp}{dx} + \frac{ddp}{2dx^2} + \frac{d^3p}{6dx^3} + \text{etc.}$$

Therefore, having neglected the higher differentials, it will be $p = \int (P^{\mathrm{I}} - P)dx - P$. Set $\int (P^{\mathrm{I}} - P)dx - P = -Q$ and let $p = -Q + q$; it will be

$$0 = z + P - P^{\mathrm{I}} - \frac{dQ}{dx} - \frac{ddQ}{2dx^2} - \text{etc.} + \frac{dq}{dx} + \frac{ddq}{2dx^2} + \text{etc.}$$

Let Q go over into Q^{I}, if one writes $x + 1$ instead of x, and it will be

$$0 = z + P - P^{\mathrm{I}} + Q - Q^{\mathrm{I}} + \frac{dq}{dx} + \frac{ddq}{2dx^2} + \text{etc.},$$

whence $q = \int (Q^{\mathrm{I}} - Q)dx - Q$ follows. Therefore, if the sign I attached to the quantity denotes its value which it takes having put $x + 1$ instead of x, and one puts

$$\begin{aligned} \int zdx &= P \\ P - \int (P^{\mathrm{I}} - P)dx &= Q \\ Q - \int (Q^{\mathrm{I}} - Q)dx &= R \\ R - \int (R^{\mathrm{I}} - R)dx &= S \\ \text{etc.}, \end{aligned}$$

the sum of the propounded series $z + z^{\mathrm{I}} + z^{\mathrm{II}} + z^{\mathrm{III}} + z^{\mathrm{IV}} + \text{etc.}$ will be

$$= C - P - Q - R - S - \text{etc.},$$

where the constant C has to defined in such a way that for $x = \infty$ the total sum vanishes. But since the application of this expression requires integrations, it is not possible to show its use here.

§196 But to avoid integral formulas, let us set the sum of the series $= ys$ while y is a known function of x whose values y^{I}, y^{II} etc., which result by writing $x+1$, $x+2$ etc. instead of x, will be known. If one now writes $x+1$ instead of x, the above series truncated by the first term will result, whose sum will therefore be

$$y^{\mathrm{I}}\left(s + \frac{ds}{dx} + \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} + \text{etc.}\right) = ys - z$$

or

$$z + \frac{y^{\mathrm{I}}ds}{dx} + \frac{y^{\mathrm{I}}dds}{2dx^2} + \frac{y^{\mathrm{I}}d^3s}{6dx^3} + \text{etc.} = (y - y^{\mathrm{I}})s,$$

whence, having neglected the differentials, $s = \frac{z}{y - y^{\mathrm{I}}}$ results. Set $\frac{z}{y^{\mathrm{I}} - y} = P$ and let the true value be $s = -P + p$; it will be

$$\left.\begin{aligned} -\frac{y^{\mathrm{I}}dP}{dx} - \frac{y^{\mathrm{I}}ddP}{2dx^2} - \frac{y^{\mathrm{I}}d^3P}{6dx^3} - \text{etc.} \\ +\frac{y^{\mathrm{I}}dp}{dx} + \frac{y^{\mathrm{I}}ddp}{2dx^2} + \frac{y^{\mathrm{I}}d^3p}{6dx^3} - \text{etc.} \end{aligned}\right\} = (y - y^{\mathrm{I}})p$$

and hence

$$\frac{y^{\mathrm{I}}dp}{dx}+\frac{y^{\mathrm{I}}ddp}{2dx^2}+\frac{y^{\mathrm{I}}d^3p}{6dx^3}+\text{etc.}=y^{\mathrm{I}}(P^{\mathrm{I}}-P)-(y^{\mathrm{I}}-y)p.$$

Set $Q=\frac{y^{\mathrm{I}}(P^{\mathrm{I}}-P)}{y^{\mathrm{I}}-y}$ and let $p=Q+q$; it will be

$$y^{\mathrm{I}}(Q^{\mathrm{I}}-Q)+y^{\mathrm{I}}\left(\frac{dq}{dx}+\frac{ddq}{2dx^2}+\text{etc.}\right)=-(y^{\mathrm{I}}-y)q.$$

Put $R=\frac{y^{\mathrm{I}}(Q^{\mathrm{I}}-Q)}{y^{\mathrm{I}}-y}$ and let be $q=-R+r$.

And if we proceed this way, the sum of the propounded series

$$z+z^{\mathrm{I}}+z^{\mathrm{II}}+z^{\mathrm{III}}+z^{\mathrm{IV}}+\text{etc.}$$

will be found the following way. Having taken an arbitrary function of x and called this function $=y$, set

$$\begin{aligned}
P &= \frac{z}{y^{\mathrm{I}}-y} &&= \frac{z}{\Delta y}\\
Q &= \frac{y^{\mathrm{I}}(P^{\mathrm{I}}-P)}{y^{\mathrm{I}}-y} &&= \frac{y\Delta P}{\Delta y}+\Delta P\\
R &= \frac{y^{\mathrm{I}}(Q^{\mathrm{I}}-Q)}{y^{\mathrm{I}}-y} &&= \frac{y\Delta Q}{\Delta y}+\Delta Q\\
S &= \frac{y^{\mathrm{I}}(R^{\mathrm{I}}-R)}{y^{\mathrm{I}}-y} &&= \frac{y\Delta R}{\Delta y}+\Delta R\\
& &&\text{etc.}
\end{aligned}$$

And hence the sum in question will be

$$=C-Py+Qy-Ry+Sy-\text{etc.}$$

having taken a constant of such a kind for C that for $x=\infty$ the sum vanishes.

§197 Assume $y=a^x$; because of $y^{\mathrm{I}}=a^{x+1}$, it will be $y^{\mathrm{I}}-y=a^x(a-1)$, whence it will become

$$P = \frac{z}{a^x(a-1)} \qquad P^{\mathrm{I}} = \frac{z^{\mathrm{I}}}{a^{x+1}(a-1)}$$

$$Q = \frac{a(P^{\mathrm{I}} - P)}{a-1} = \frac{z^{\mathrm{I}} - az}{a^x(a-1)^2} \qquad Q^{\mathrm{I}} = \frac{z^{\mathrm{II}} - az^{\mathrm{I}}}{a^{x+1}(a-1)^2}$$

$$R = \frac{a(Q^{\mathrm{I}} - Q)}{a-1} = \frac{z^{\mathrm{II}} - 2az^{\mathrm{I}} + aaz}{a^x(a-1)^3} \qquad R^{\mathrm{I}} = \frac{z^{\mathrm{III}} - 2az^{\mathrm{II}} + aaz^{\mathrm{I}}}{a^{x+1}(a-1)^2}$$

$$S = \frac{a(R^{\mathrm{I}} - R)}{a-1} = \frac{z^{\mathrm{III}} - 3az^{\mathrm{II}} + 3a^2z^{\mathrm{I}} - a^3}{a^x(a-1)^4}$$

etc.

Therefore, the sum of the propounded series will be

$$C - \frac{z}{a-1} + \frac{z^{\mathrm{I}} - az}{(a-1)^2} - \frac{z^{\mathrm{II}} - 2az^{\mathrm{I}} + a^2z}{(a-1)^3} + \frac{z^{\mathrm{III}} - 3az^{\mathrm{II}} + 3a^2z^{\mathrm{I}} - a^3z}{(a-1)^4} - \text{etc.}$$

This same expression of the sum was indeed found already above in the first chapter. Hence, taking other values for y, infinitely many other expressions can be found, whence the one, which is the most convenient in each case, can be selected.