On the Use of Differential Calculus in the Formation of Series *

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§198 Until now we only considered one single application of differential calculus in the doctrine of series which was the formation of series and which we mentioned above already, when there was the question how to expand the fraction whose denominator is an arbitrary power of a certain function into a series. But this method is similar to that one we already used several times, where the fraction to be converted into a series is set equal to a certain series with coefficients to determined from the constituted equality. But this determination is often simplified tremendously, if, before it is actually done, the equation is differentiated once and sometimes even twice. Since this method has very broad applications in integral calculus, let us explain it here more diligently.

§199 Therefore, at first let us repeat what we discussed above on the expansion of fractions into series without the application of differential calculus. Let an arbitrary fraction be propounded, i.e.

$$\frac{A + Bx + Cx^2 + Dx^3 + \text{etc.}}{\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{etc.}} = s,$$

which is to be converted into a powers series in x. Assume an undetermined series for s

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$$s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \mathfrak{F}x^5 + \mathfrak{G}x^6 + \text{etc.}$$

Therefore, since, having removed the fraction by multiplication,

$$\begin{aligned} A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6 + \text{etc.} \\ = s(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \zeta x^5 + \eta x^6 + \text{etc.}), \end{aligned}$$

if the assumed series is substituted for s, the following equation results

$$\begin{array}{rllllll} & A + Bx & + Cx^2 & + Dx^3 & + Ex^4 & + Fx^5 & + \text{etc.} \\ \hline = & \mathfrak{A}\alpha + \mathfrak{B}\alpha x & + \mathfrak{C}\alpha x^2 & + \mathfrak{D}\alpha x^3 & + \mathfrak{E}\alpha x^4 & + \mathfrak{F}\alpha x^5 & + \text{etc.} \\ & \quad + \mathfrak{A}\beta & + \mathfrak{B}\beta & + \mathfrak{C}\beta & + \mathfrak{D}\beta & + \mathfrak{E}\beta & + \text{etc.} \\ & & + \mathfrak{A}\gamma & + \mathfrak{B}\gamma & + \mathfrak{C}\gamma & + \mathfrak{D}\gamma & + \text{etc.} \\ & & & + \mathfrak{A}\delta & + \mathfrak{B}\delta & + \mathfrak{C}\delta & + \text{etc.} \\ & & & & + \mathfrak{A}\varepsilon & + \mathfrak{B}\varepsilon & + \text{etc.} \\ & & & & & + \mathfrak{A}\zeta & + \text{etc.} \end{array}$$

Therefore, having equated each term containing the same powers of x, it will be

$$\begin{aligned} &\mathfrak{A}\alpha - A = 0 \\ &\mathfrak{B}\alpha + \mathfrak{A}\beta - B = 0 \\ &\mathfrak{C}\alpha + \mathfrak{B}\beta + \mathfrak{A}\gamma - C = 0 \\ &\mathfrak{D}\alpha + \mathfrak{C}\beta + \mathfrak{B}\gamma + \mathfrak{A}\delta - D = 0 \\ &\mathfrak{E}\alpha + \mathfrak{D}\beta + \mathfrak{C}\gamma + \mathfrak{B}\delta + \mathfrak{A}\varepsilon - E = 0 \\ &\qquad\qquad\qquad \text{etc.} \end{aligned}$$

from which equations the assumed coefficients $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. are determined, and so the infinite series

$$\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \text{etc.}$$

equal to the propounded fraction s is found. And this form, if both the numerator and the denominator of the fraction s consist of a finite number of terms, contains all recurring series, which were treated in a lot greater detail above.

§200 But if either the numerator or the denominator or both were raised to an arbitrary power, then the series is found rather difficultly this way, since the task, if not a binomial function was raised, becomes very laborious. But by means of differential calculus this work can be simplified. At first, let the fraction consist only of a numerator and let

$$s = (A + Bx + Cxx)^n,$$

whence the power series in x equal to this power of the trinomial is to be found; it is plain that the series will be finite, if the exponent n was an positive integer. Again, assume an indefinite series for s, i.e.

$$S = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \mathfrak{F}x^5 + \mathfrak{G}x^6 + \text{etc.},$$

whose first term $\mathfrak{A}$ is known to be $= A^n$; for, if one puts $x = 0$, from the first propounded form $s = A^n$, but from the assumed series $s = \mathfrak{A}$. But this determination of the first term is to be derived from the nature of the series itself, if we want to use differentials, since it is not possible to determine the first coefficient from the differential, as it will be seen soon.

§201 Since $S = (A + Bx + Cx^2)^n$, taking logarithms, it will be

$$\log s = n \log(A + Bx + Cx^2)$$

and hence, having taken the differentials, one will have

$$\frac{ds}{s} = \frac{nBdx + 2nCxdx}{A + Bx + Cx^2} \quad \text{or} \quad (A + Bx + Cx^2)\frac{ds}{dx} = ns(B + 2Cx).$$

But from the assumed series

$$\frac{ds}{dx} = \mathfrak{B} + 2\mathfrak{C}x + 3\mathfrak{D}x^2 + 4\mathfrak{E}x^3 + 5\mathfrak{F}x^4 + \text{etc.}$$

Therefore, if this series is substituted for $\frac{ds}{dx}$ and for s the assumed series is substituted, the following equation will result

$$\begin{array}{rrrrrr}
A\mathfrak{B} + & 2A\mathfrak{C}x + & 3A\mathfrak{D}x^2 + & 4A\mathfrak{E}x^3 + & 5A\mathfrak{F}x^4 & + \text{etc.} \\
+ & B\mathfrak{B} + & 2B\mathfrak{C} & + 3B\mathfrak{D} & + 4B\mathfrak{E} & + \text{etc.} \\
& + & C\mathfrak{B} & + 2C\mathfrak{C} & + 3C\mathfrak{D} & + \text{etc.} \\
\hline
= nB\mathfrak{A} + & nB\mathfrak{B} + & nB\mathfrak{C} & + nB\mathfrak{D} & + nB\mathfrak{E} & + \text{etc.} \\
+ 2nC\mathfrak{A} & + 2nC\mathfrak{B} & + 2nC\mathfrak{C} & + 2nC\mathfrak{D} & & + \text{etc.}
\end{array}$$

Therefore, having equated the terms of the same power of x, it will be

$$\begin{aligned}
\mathfrak{B} &= \frac{nB\mathfrak{A}}{A} \\
\mathfrak{C} &= \frac{(n-1)B\mathfrak{B} + 2nC\mathfrak{A}}{2A} \\
\mathfrak{D} &= \frac{(n-2)B\mathfrak{C} + (2n-1)C\mathfrak{B}}{3A} \\
\mathfrak{E} &= \frac{(n-3)B\mathfrak{D} + (2n-2)C\mathfrak{C}}{4A} \\
\mathfrak{F} &= \frac{(n-4)B\mathfrak{E} + (2n-3)C\mathfrak{D}}{5A} \\
&\text{etc.}
\end{aligned}$$

Therefore, since, as we saw before, $\mathfrak{A} = A^n$, it will be $\mathfrak{B} = nA^{n-1}B$ and hence the remaining coefficients will successively be defined. But the law they follow is obvious from these formulas which would have remained immensely obscure, if we would have wanted to actually expand the trinomial.

§202 The same method succeeds, if any polynomial function has to raised to a certain power. Let

$$s = (A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.})^n$$

and assume

$$s + \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \text{etc.};$$

it will be $\mathfrak{A} = A^n$ which value is concluded, if one puts $x = 0$. Now, having taken the logarithms, their differentials as before will be found to be

$$\frac{ds}{s} = \frac{nBdx + 2nCxdx + 3nDx^2dx + 4nEx^3dx + \text{etc.}}{A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}}$$

or

$$(A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.})\frac{ds}{dx}$$
$$= s(nB + 2nCx + 3nDx^2 + 4nEx^3 + \text{etc.}).$$

Therefore, since

$$\frac{ds}{dx} = \mathfrak{B} + 2\mathfrak{C}x + 3\mathfrak{D}x^2 + 4\mathfrak{E}x^3 + 5\mathfrak{F}x^4 + \text{etc.},$$

having substituted these series for s and $\frac{ds}{dx}$, it will be

$$\begin{array}{rrrrrl}
A\mathfrak{B} + & 2A\mathfrak{C}x + & 3A\mathfrak{D}x^2 + & 4A\mathfrak{E}x^3 + & 5A\mathfrak{F}x^4 & + \text{etc.} \\
+ & B\mathfrak{B} \;+ & 2B\mathfrak{C} \;+ & 3B\mathfrak{D} \;+ & 4B\mathfrak{E} & + \text{etc.} \\
 & + & C\mathfrak{B} \;+ & 2C\mathfrak{C} \;+ & 3C\mathfrak{D} & + \text{etc.} \\
 & & + & D\mathfrak{B} \;+ & 2D\mathfrak{C} & + \text{etc.} \\
 & & & + & E\mathfrak{B} & + \text{etc.} \\
\hline
= nB\mathfrak{A} + & nB\mathfrak{B} \;+ & nB\mathfrak{C} \;+ & nB\mathfrak{D} \;+ & nB\mathfrak{E} & + \text{etc.} \\
+\, 2nC\mathfrak{A} & +\, 2nC\mathfrak{B} & +\, 2nC\mathfrak{C} & +\, 2nC\mathfrak{D} & & + \text{etc.} \\
 & +\, 3nD\mathfrak{A} & +\, 3nD\mathfrak{B} & +\, 3nD\mathfrak{C} & & + \text{etc.} \\
 & & +\, 4nE\mathfrak{A} & +\, 4nE\mathfrak{B} & & + \text{etc.} \\
 & & & +\, 5nF\mathfrak{A} & & + \text{etc.}
\end{array}$$

Therefore, the following determinations are derived

$$A\mathfrak{B} = nB\mathfrak{A}$$

$$2A\mathfrak{C} = (n-1)B\mathfrak{B} + 2nC\mathfrak{A}$$

$$3A\mathfrak{D} = (n-2)B\mathfrak{C} + (2n-1)C\mathbb{B} + 3nD\mathfrak{A}$$

$$4A\mathfrak{E} = (n-3)B\mathfrak{D} + (2n-2)C\mathfrak{C} + (3n-1)D\mathfrak{B} + 4nE\mathfrak{A}$$

$$5A\mathfrak{F} = (n-4)B\mathfrak{E} + (2n-3)C\mathfrak{D} + (3n-2)D\mathfrak{C} + (4n-1)E\mathfrak{B} + 5nF\mathfrak{A}$$

etc.,

whence it becomes clear, how these assumed coefficients $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. depend on each other and are hence determined, since $\mathfrak{A} = A^n$.

§203 Since, if the quantity $A + Bx + Cx^2 + Dx^3 + \text{etc.}$ consists of a finite number of terms and the number n was a positive integer, any power also has to consist of a finite number of terms, it is obvious that in this case the formulas just found must finally vanish and, since all finite terms must occur until the first vanishes, at the same time all following ones must vanish. Let us put that the propounded formula $A + Bx + Cx^2$ is a trinomial and its cube is in question, i.e. that $n = 3$; it will be

$$\begin{array}{rcll}
\mathfrak{A} & = & 2A^3 & \text{and hence} \quad \mathfrak{A} = A^3 \\
A\mathfrak{B} & = & 3B\mathfrak{A} & \mathfrak{B} = 3A^2B \\
2A\mathfrak{C} & = & 2B\mathfrak{B} + 6C\mathfrak{A} & \mathfrak{C} = 3AB^2 + 3A^2C \\
3A\mathfrak{D} & = & 1B\mathfrak{C} + 5C\mathfrak{B} & \mathfrak{D} = B^3 + 6ABC \\
4A\mathfrak{E} & = & 0 \quad + 4C\mathfrak{C} & \mathfrak{E} = 3B^2C + 3AC^2 \\
5A\mathfrak{F} & = & -\ B\mathfrak{E} + 3C\mathfrak{D} & \mathfrak{F} = 3BC^2 \\
6A\mathfrak{G} & = & -\ 2B\mathfrak{F} + 2C\mathfrak{E} & \mathfrak{G} = C^3 \\
7A\mathfrak{H} & = & -\ 3B\mathfrak{G} + 2C\mathfrak{F} & \mathfrak{H} = 0 \\
8A\mathfrak{I} & = & -\ 4B\mathfrak{H} + 0 & \mathfrak{I} = 0
\end{array}$$

Therefore, since already two letters are $= 0$ and any arbitrary of the following letters depends on the two preceding ones, it is plain that all following ones must also vanish. And for this reason the law according to which these coefficients were found to depend on each other is even more noteworthy.

§204 If n was a negative number such that s becomes equal to a real fraction, the series will continue to infinity. Therefore, let

$$s = \frac{1}{(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{etc.})^n};$$

for its value assumes this series

$$s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \mathfrak{F}x^5 + \text{etc.}$$

And if in the above formulas one puts α, β, γ, δ etc. for the letters A, B, C, D etc. and at the same time n becomes negative, the following determinations of the coefficients $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. will result

$$\mathfrak{A} = \alpha^{-n} = \frac{1}{\alpha^n}$$

$$\alpha\mathfrak{B} + n\beta\mathfrak{A} = 0$$

$$2\alpha\mathfrak{C} + (n+1)\beta\mathfrak{B} + 2n\gamma\mathfrak{A} = 0$$

$$3\alpha\mathfrak{D} + (n+2)\beta\mathfrak{C} + (2n+1)\gamma\mathfrak{B} + 3n\delta\mathfrak{A} = 0$$

$$4\alpha\mathfrak{E} + (n+3)\beta\mathfrak{D} + (2n+2)\gamma\mathfrak{C} + (3n+1)\delta\mathfrak{B} + 4n\varepsilon\mathfrak{A} = 0$$

$$5\alpha\mathfrak{F} + (n+4)\beta\mathfrak{E} + (2n+3)\gamma\mathfrak{D} + (3n+2)\delta\mathfrak{C} + (4n+1)\varepsilon\mathfrak{B} + 5n\zeta\mathfrak{A} = 0$$

etc.

These formulas contain the same law of these coefficients of numbers we already observed above in the *Introductio* and whose validity has therefore been demonstrated rigorously now.

§205 Their nature is the same, if the numerator of the fraction was 1 or even any power of x, say x^m; for, in the second case it will only be necessary to multiply the series found first $\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \text{etc.}$ by x^m. But if the denominator consists of two or more terms, then we do not observe the law of progression found above: therefore, let us investigate it here by means of differentiation. Hence let

$$s = \frac{A + Bx + Cx^2 + Dx^3 + \text{etc.}}{(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{etc.})^n}$$

and assume the following series for the value of this fraction

$$s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \mathfrak{F}x^5 + \text{etc.};$$

to define its first term $\mathfrak{A}$ put $x = 0$ and from the first expression it will be $s = \frac{A}{\alpha^n}$, from the assumed series on the other hand $s = \mathfrak{A}$, whence it is necessary that $\mathfrak{A} = \frac{A}{\alpha^n}$. Having determined this term, the remaining ones will be found by means of differentiation.

§206 Having taken logarithms, it will be

$$\begin{aligned} \log s &= \log(A + Bx + Cx^2 + Dx^3 + \text{etc.}) \\ &-n \log(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{etc.}) \end{aligned}$$

and hence by differentiation this equation will result

$$\begin{aligned} \frac{ds}{s} &= \frac{Bdx + 2Cdx + 3Dx^2dx + \text{etc.}}{A + Bx + Cx^2 + Dx^3 + \text{etc.}} \\ &- \frac{n\beta dx + 2n\gamma xdx + 3n\delta x^2dx + \text{etc.}}{\alpha + \beta x + \gamma x^2 + \delta x^3 + \text{etc.}} \end{aligned}$$

and, having got rid of the fractions by multiplication, it will be

$$\left\{ \begin{array}{lllll} A\alpha + A\beta x + & A\gamma x^2 + & A\delta x^3 & + \text{etc.} \\ \quad + B\alpha & + B\beta & + B\gamma & + \text{etc.} \\ & + C\alpha & + C\beta & + \text{etc.} \\ & & + D\alpha & + \text{etc.} \end{array} \right\} \frac{ds}{dx}$$

$$= \left\{ \begin{array}{lllll} B\alpha + & B\beta x + & B\gamma x^2 + & B\delta x^3 & + \text{etc.} \\ & + 2C\alpha & + 2C\beta & + 2C\gamma & + \text{etc.} \\ & & + 3D\alpha & + 3D\beta & + \text{etc.} \\ & & & + 4E\alpha & + \text{etc.} \end{array} \right\} s$$

$$-\left\{\begin{array}{rrrrr} A\beta + 2A\gamma x & + 3A\delta x^2 & + 4A\varepsilon x^3 & + \text{etc.} \\ + \quad B\beta & + 2B\gamma & + 3B\delta & + \text{etc.} \\ & + \quad C\beta & + 2C\gamma & + \text{etc.} \\ & & + \quad D\beta & + \text{etc.} \end{array}\right\} ns.$$

Since now $\frac{ds}{dx} = \mathfrak{B} + 2\mathfrak{C}x + 3\mathfrak{D}x^2 + 4\mathfrak{E}x^3 + \text{etc.}$, after the substitutions it will be

$$\left.\begin{array}{r} A\alpha\mathfrak{B} + nA\beta\mathfrak{A} \\ - \quad B\alpha\mathfrak{A} \end{array}\right\} = 0$$

$$\left.\begin{array}{rrr} 2A\alpha\mathfrak{C} + (n+1)A\beta\mathfrak{B} + & 2nA\gamma\mathfrak{A} \\ + \quad 0B\alpha\mathfrak{B} + & (n-1)B\beta\mathfrak{A} \\ - & 2C\alpha\mathfrak{A} \end{array}\right\} = 0$$

$$\left.\begin{array}{rrrr} 3A\alpha\mathfrak{D} + (n+2)A\beta\mathfrak{C} + & (2n+1)A\gamma\mathfrak{B} + & 3nA\delta\mathfrak{A} \\ + \quad B\alpha\mathfrak{C} + & nB\beta\mathfrak{B} + & (2n-1)B\gamma\mathfrak{A} \\ - & C\alpha\mathfrak{B} + & (n-2)C\beta\mathfrak{A} \\ & - & 3D\alpha\mathfrak{A} \end{array}\right\} = 0$$

$$\left.\begin{array}{rrrrr} 4A\alpha\mathfrak{E} + (n+3)A\beta\mathfrak{D} + & (2n+2)A\gamma\mathfrak{C} + & (3n+1)A\delta\mathfrak{B} + & 4nA\varepsilon\mathfrak{A} \\ + \quad B\alpha\mathfrak{D} + & (n+1)B\beta\mathfrak{C} + & 2nB\gamma\mathfrak{B} + & (3n-1)B\delta\mathfrak{A} \\ + & 0C\alpha\mathfrak{C} + & (n-1)C\beta\mathfrak{B} + & (3n-2)C\gamma\mathfrak{A} \\ & - & 2D\alpha\mathfrak{B} + & (n-3)D\beta\mathfrak{A} \\ & & - & 4E\alpha\mathfrak{A} \end{array}\right\} = 0.$$

etc.

Therefore, the law according to which these formulas proceed is easily seen; for, the first line of each equation follows the same law we had in § 204. But then the coefficients of the second line result, if $n+1$ is subtracted from the above coefficients, and in like manner the third line is formed from the second line and the following from the upper ones - always by subtracting $n+1$; but the letters constituting each term are immediately obvious considering the structure of the formulas.

§207 But if also the numerator of a fraction was a certain power, i.e.

$$s = \frac{(A + Bx + Cx^2 + Dx^3 + \text{etc.})^m}{(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{etc.})^n},$$

and one assumes this series

$$s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \text{etc.},$$

it will be $\mathfrak{A} = \frac{A^m}{\alpha^n}$; but the remaining coefficients will be determined from the following formulas

$$\left.\begin{array}{r} A\alpha\mathfrak{B} + nA\beta\mathfrak{A} \\ -\ mB\alpha\mathfrak{A} \end{array}\right\} = 0$$

$$\left.\begin{array}{rrr} 2A\alpha\mathfrak{C} + & (n+1)A\beta\mathfrak{B} + & 2nA\gamma\mathfrak{A} \\ - & (m-1)B\alpha\mathfrak{B} + & (n-m)B\beta\mathfrak{A} \\ & - & 2mC\alpha\mathfrak{A} \end{array}\right\} = 0$$

$$\left.\begin{array}{rrrr} 3A\alpha\mathfrak{D} + & (n+2)A\beta\mathfrak{C} + & (2n+1)A\gamma\mathfrak{B} + & 3nA\delta\mathfrak{A} \\ - & (m-2)B\alpha\mathfrak{C} + & (n-m+1)B\beta\mathfrak{B} + & (2n-m)B\gamma\mathfrak{A} \\ & - & (2m-1)C\alpha\mathfrak{B} + & (n-2m)C\beta\mathfrak{A} \\ & & - & 3mD\alpha\mathfrak{A} \end{array}\right\} = 0$$

$$\left.\begin{array}{rrrrr} 4A\alpha\mathfrak{E} + & (n+3)A\beta\mathfrak{D} + & (2n+2)A\gamma\mathfrak{C} + & (3n+1)A\delta\mathfrak{B} + & 4nA\varepsilon\mathfrak{A} \\ - & (m-3)B\alpha\mathfrak{D} + & (n+m-2)B\beta\mathfrak{C} + & (2n-m+1)B\gamma\mathfrak{B} + & (3n-m)B\delta\mathfrak{A} \\ & - & (2m-2)C\alpha\mathfrak{C} + & (n-2m+1)C\beta\mathfrak{B} + & (n-3m)C\gamma\mathfrak{A} \\ & & - & (3m-1)D\alpha\mathfrak{B} + & (n-3m)D\beta\mathfrak{A} \\ & & & - & 4mE\alpha\mathfrak{A} \end{array}\right\} = 0.$$

etc.

The rule, how these formulas are continued, becomes clear from the inspection of the above equation more quickly than it can be described by words. While descending down the column the coefficients are decreased by the difference $m + n$; but while proceeding horizontally the differences will continuously be increased by the difference $n - 1$.

§208 Therefore, this way the theory of recurring series is extended, since we discovered the previously unknown equations for the coefficients even for the cases, in which not only the denominator of the fraction was any power, but also the numerator consists of any arbitrary number of terms, to detect which equations induction alone did not suffice. But except for the many applications of recurring series we already explained, they are very useful to find the sums of certain series approximately; we exhibited a specimen of this already in the first chapter of this book, where we transformed the series into another one which often consists of a finite number of terms by means of the substitution $x = \frac{y}{1+ny}$. And the method could have been extended further, if other functions were substituted for x. Since then the law of progression of series, which had to be substituted for the power of x, was not sufficiently clear, it seemed advisable to mention this generalization just here, after the mentioned law had already been completely discovered. Nevertheless, considering this with more attention, we learn that the same task can be done without this law of progression only by using the method we used here to investigate the law.

§209 Therefore, let an arbitrary series be propounded, i.e.

$$s = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.}$$

which we want to transform into another one, each term of which are fractions whose denominators proceed according to powers of a formula of this kind

$$\alpha + \beta x + \gamma x^2 + \delta x^3 + \text{etc.}$$

To start from simpler cases, let us put that

$$s = \frac{\mathfrak{A}}{\alpha + \beta x} + \frac{\mathfrak{B}x}{(\alpha + \beta x)^2} + \frac{\mathfrak{C}x^2}{(\alpha + \beta x)^3} + \frac{\mathfrak{D}x^3}{(\alpha + \beta x)^4} + \text{etc.};$$

Having equated the series to this expression, multiply by $\alpha + \beta x$ everywhere and it will be

$$\left.\begin{aligned} A\alpha + B\alpha\, x + C\alpha x^2 + D\alpha x^3 + +\text{etc.} \\ A\beta x + b\beta \quad + C\beta \quad + +\text{etc.} \end{aligned}\right\} = \mathfrak{A} + \frac{\mathfrak{B}x}{\alpha + \beta x} + \frac{\mathfrak{C}x^2}{(\alpha + \beta x)^2} + \text{etc.}$$

Put $\mathfrak{A} = A\alpha$ and let

$$\begin{aligned} A\beta + B\alpha &= A' \\ B\beta + C\alpha &= B' \\ C\beta + D\alpha &= C' \\ D\beta + E\alpha &= D' \\ &\text{etc.}; \end{aligned}$$

having divided by x, it will be

$$A' + B'x + C'x^2 + D'x^3 + \text{etc.} = \frac{\mathfrak{B}}{\alpha + \beta x} + \frac{\mathfrak{C}x}{(\alpha + \beta x)^2} + \frac{\mathfrak{D}x^2}{(\alpha + \beta x)^3} + \text{etc.}$$

Multiply by $\alpha + \beta x$ again and having put

$$\begin{aligned} A'\beta + B'\alpha &= A'' \\ B'\beta + C'\alpha &= B'' \\ C'\beta + D'\alpha &= C'' \\ &\text{etc.} \end{aligned}$$

it will be

$$A'\alpha + A''x + B''x^2 + C''x^3 + \text{etc.} = \mathfrak{B} + \frac{\mathfrak{C}x}{\alpha + \beta x} + \frac{\mathfrak{D}x^2}{(\alpha + \beta x)^2} + \text{etc.}$$

Therefore, let $\mathfrak{B} = A'\alpha$; and arguing exactly as before, if

$$\begin{aligned} A''\beta + B''\alpha &= A''' \qquad & A'''\beta + B'''\alpha &= A'''' \\ B''\beta + C''\alpha &= B''' \qquad & B'''\beta + C'''\alpha &= B'''' \\ C''\beta + D''\alpha &= C''' \qquad & C'''\beta + D'''\alpha &= C'''' \\ &\text{etc.} & &\text{etc.}, \end{aligned}$$

it will be $\mathfrak{C} = A''\alpha$, $\mathfrak{D} = A'''\alpha$, $\mathfrak{E} = A''''\alpha$; therefore, the sum of the propounded series will be expressed as follows

$$s = \frac{A\alpha}{\alpha+\beta x} + \frac{A'\alpha x}{(\alpha+\beta x)^2} + \frac{A'\alpha x^2}{(\alpha+\beta x)^3} + \frac{A'''\alpha x^3}{(\alpha+\beta x)^4} + \text{etc.}$$

This same series would have resulted from the substitution

$$\frac{x}{\alpha+\beta x} = y \quad \text{or} \quad x = \frac{\alpha y}{1-\beta y}.$$

§210 This transformation is applied with the greatest success, if the propounded series $A + Bx + Cx^2 + \text{etc.}$ was of such a nature that it is finally confounded with a recurring series or, even better, a geometric series resulting from the fraction $\frac{p}{\alpha+\beta x}$. For, then the values A', B', C', D' etc. will finally vanish and hence the letters A'', A''', A'''' etc. will even more constitute a highly converging series.

In like manner we will be able to use trinomial and any polynomial denominators which will have an extraordinary use, if the propounded series is finally confounded with a recurring series. Therefore, having propounded the series

$$s = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.},$$

set

$$s = \frac{\mathfrak{A} + \mathfrak{B}x}{\alpha+\beta x+\gamma x^2} + \frac{\mathfrak{A}'x^2 + \mathfrak{B}'x^3}{(\alpha+\beta x+\gamma x^2)^2} + \frac{\mathfrak{A}''x^4 + \mathfrak{B}''x^5}{(\alpha+\beta x+\gamma x^2)^3} + \frac{\mathfrak{A}'''x^6 + \mathfrak{B}'''x^7}{(\alpha+\beta x+\gamma x^2)^4} + \text{etc.}$$

Multiply by $\alpha + \beta x + \gamma x^2$ everywhere and, having put

$$\begin{aligned} A\gamma + B\beta + C\alpha &= A' \quad \text{and} \quad \mathfrak{A} = A\alpha \\ B\gamma + C\beta + D\alpha &= B' \quad \text{and} \quad \mathfrak{B} = A\beta + B\alpha \\ C\gamma + D\beta + E\alpha &= C', \end{aligned}$$

having divided by xx, an equation similar to the first will result, i.e.

$$\begin{aligned} &A' + B'x + C'x^2 + D'x^3 + E'x^4 + \text{etc.} \\ &= \frac{\mathfrak{A}' + \mathfrak{B}'x}{\alpha+\beta x+\gamma xx} + \frac{\mathfrak{A}'' + \mathfrak{B}''x}{(\alpha+\beta x+\gamma xx)^2} + \frac{\mathfrak{A}''' + \mathfrak{B}'''x}{(\alpha+\beta x+\gamma xx)^3} + \text{etc.} \end{aligned}$$

Therefore, if the operation is done as before by putting

$$\begin{array}{lll} A'\gamma + B'\beta + C'\alpha = A'' & \text{and} & \mathfrak{A}' = A'\alpha \\ B'\gamma + C'\beta + D'\alpha = B'' & \text{and} & \mathfrak{B} = A'\beta + B'\alpha \\ C'\gamma + D'\beta + E'\alpha = C'' & & \\ \text{etc.} & & \end{array}$$

and further

$$\begin{array}{lll} A''\gamma + B''\beta + C''\alpha = A''' & \text{and} & \mathfrak{A}'' = A''\alpha \\ B''\gamma + C''\beta + D''\alpha = B''' & \text{and} & \mathfrak{B} = A''\beta + B''\alpha \\ C''\gamma + D''\beta + E''\alpha = C''' & & \\ \text{etc.} & & \end{array}$$

and by investigating further values in this manner, it will be

$$s = \frac{A\alpha + (A\beta + b\alpha)x}{\alpha + \beta x + \gamma xx} + \frac{(A'\alpha + (A'\beta + B'\alpha))x^2}{(\alpha + \beta x + \gamma xx)^2} + \frac{(A''\alpha + (A''\beta + B''\alpha))x^4}{(\alpha + \beta x + \gamma xx)^3} + \text{etc.}$$

§211 If one puts $x = 1$, what can be done without loss of generality, because α, β, γ can be taken arbitrarily, and it was

$$s = A + B + C + D + E + F + G + \text{etc.},$$

having successively put the following values

$$\begin{array}{lll} A\gamma + B\beta + C\alpha = A' & A'\gamma + B'\beta + C'\alpha = A'' & \\ B\gamma + C\beta + D\alpha = B' & B'\gamma + C'\beta + D'\alpha = B'' & \text{and so forth} \\ C\gamma + D\beta + E\alpha = C' & C'\gamma + D'\beta + E'\alpha = C'' & \\ \text{etc.} & \text{etc.} & \end{array}$$

but for the sake of brevity one puts

$$\alpha + \beta + \gamma = m,$$

one will obtain the sum of the propounded series expressed this way

$$s = \left\{ \begin{array}{l} (\alpha+\beta)\left(\dfrac{A}{m} + \dfrac{A'}{m^2} + \dfrac{A''}{m^3} + \dfrac{A'''}{m^4} + \text{etc.}\right) \\ \qquad +\alpha\left(\dfrac{B}{m} + \dfrac{B'}{m^2} + \dfrac{B''}{m^3} + \dfrac{B'''}{m^4} + \text{etc.}\right) \end{array} \right\}$$

§212 The same denominators consisting of more terms can be taken, and since the operation is easily understood from the preceding, let us only expand the case for the polynomial of degree three. Therefore, let

$$s = A + B + C + D + E + F + G + \text{etc.}$$

Find the following values

$$\begin{array}{l} A\delta + B\gamma + C\beta + D\alpha = A' \\ B\delta + C\gamma + D\beta + E\alpha = B' \\ C\delta + D\gamma + E\beta + F\alpha = C' \\ \qquad\qquad \text{etc.} \end{array}$$

$$\begin{array}{l} A'\delta + B'\gamma + C'\beta + D'\alpha = A'' \\ B'\delta + C'\gamma + D'\beta + E'\alpha = B'' \\ C'\delta + D'\gamma + E'\beta + F'\alpha = C'' \\ \qquad\qquad \text{etc.} \end{array}$$

$$\begin{array}{l} A''\delta + B''\gamma + C''\beta + D''\alpha = A''' \\ B''\delta + C''\gamma + D''\beta + E''\alpha = B''' \\ C''\delta + D''\gamma + E''\beta + F''\alpha = C''' \\ \qquad\qquad \text{etc.} \end{array}$$

But then let $\alpha + \beta + \gamma + \delta = m$ and it will be

$$s = \begin{cases} (\alpha+\beta+\gamma)\left(\dfrac{A}{m} + \dfrac{A'}{m^2} + \dfrac{A''}{m^3} + \dfrac{A'''}{m^4} + \text{etc.}\right) \\ (\alpha+\beta)\left(\dfrac{B}{m} + \dfrac{B'}{m^2} + \dfrac{B''}{m^3} + \dfrac{B'''}{m^4} + \text{etc.}\right) \\ +\alpha\left(\dfrac{C}{m} + \dfrac{C'}{m^2} + \dfrac{C''}{m^3} + \dfrac{C'''}{m^4} + \text{etc.}\right) \end{cases}$$

whence at the same time the progression, if even more parts are attributed to the denominator m, is most clearly seen.

§213 And it is not necessary at all that the denominators of the fractions, to which we reduced the sum of the series, are powers of the same formula

$$\alpha + \beta x + \gamma x^2 + \text{etc.},$$

but this can be varied in each term. In order to clarify this, let us at first only take two terms and assume that the series

$$s = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.}$$

is converted into this series of fractions

$$s = \frac{\mathfrak{A}}{\alpha+\beta x} + \frac{\mathfrak{A}' x}{(\alpha+\beta x)(\alpha'+\beta' x)} + \frac{\mathfrak{A}'' x^2}{(\alpha+\beta x)(\alpha'+\beta' x)(\alpha''+\beta'' x)} + \text{etc.}$$

At first, multiply both sides by $\alpha + \beta x$ and put

$$\begin{aligned} A\beta + B\alpha &= A' \\ B\beta + C\alpha &= B' \quad \text{and} \quad \mathfrak{A} = A\alpha \\ C\beta + D\alpha &= C' \\ \text{etc.} \end{aligned}$$

and, having divided by x, it will be

$$A' + B' + C'x^2 + D'x^3 + \text{etc.} = \frac{\mathfrak{A}}{\alpha' + \beta' x} + \frac{\mathfrak{A}'' x}{(\alpha'+\beta' x)(\alpha''+\beta'' x)} + \text{etc.}$$

Further, in like manner by multiplying by $\alpha' + \beta' x$ and then by $\alpha'' + \beta'' x$ and so forth, if one sets

$$\begin{array}{lll} A'\beta' + B'\alpha' = A'' & A''\beta'' + B''\alpha'' = A''' & A'''\beta''' + B'''\alpha''' = A'''' \\ B'\beta' + C'\alpha' = B'' & B''\beta'' + C''\alpha'' = B''' & B'''\beta''' + C'''\alpha''' = B'''' \quad \text{etc.} \\ C'\beta' + D'\alpha' = C'' & C''\beta'' + D''\alpha'' = C''' & C'''\beta''' + D'''\alpha''' = C'''' \\ \text{etc.} & \text{etc.} & \text{etc.} \end{array}$$

it will be

$$\mathfrak{A}' = A'\alpha', \quad \mathfrak{A}'' = A''\alpha'', \quad \mathfrak{A}''' = A'''\alpha''' \quad \text{etc.}$$

and hence the propounded series will be converted into this one

$$s = \frac{A\alpha}{\alpha + \beta x} + \frac{A'\alpha' x}{(\alpha + \beta x)(\alpha' + \beta' x)} + \frac{A''\alpha'' x}{(\alpha + \beta x)(\alpha' + \beta' x)(\alpha'' + \beta'' x)} + \text{etc.},$$

where the values α, β, α', β', α'', β'' etc. are arbitrary, but can be taken in such a way for each case that this new series is highly convergent.

§214 Let us apply this also to trinomial factors and, having propounded an arbitrary series $s = A + B + C + D + E + F + G + \text{etc.}$, let

$$\begin{array}{ll} A\gamma + B\beta + C\alpha = A' & A'\gamma' + B'\beta' + C'\alpha' = A'' \\ B\gamma + C\beta + D\alpha = B' & B'\gamma' + C'\beta' + D'\alpha' = B'' \\ C\gamma + D\beta + E\alpha = C' & C'\gamma' + D'\beta' + E'\alpha' = C'' \\ \text{etc.} & \text{etc.} \end{array}$$

$$\begin{array}{ll} A''\gamma' + B''\beta'' + C''\alpha'' = A''' & A'''\gamma''' + B'''\beta''' + C'''\alpha''' = A'''' \\ B''\gamma'' + C''\beta'' + D''\alpha'' = B''' & B'''\gamma''' + C'''\beta''' + D'''\alpha''' = B'''' \\ C''\gamma'' + D''\beta'' + E''\alpha'' = C''' & C'''\gamma''' + D'''\beta''' + E'''\alpha''' = C'''' \\ \text{etc.} & \text{etc.} \end{array}$$

Further, for the sake of brevity put

$$\begin{aligned}\alpha + \beta + \gamma &= m \\ \alpha' + \beta' + \gamma' &= m' \\ \alpha'' + \beta'' + \gamma'' &= m'' \\ \alpha''' + \beta''' + \gamma''' &= m''' \\ &\text{etc.}\end{aligned}$$

and the sum of the propounded series will be

$$s = \frac{\alpha(A+B)}{m} + \frac{\alpha'(A'+B')}{mm'} + \frac{\alpha''(A''+B'')}{mm'm''} + \frac{\alpha'''(A'''+B''')}{mm'm''m'''} + \text{etc.}$$
$$+\frac{\beta A}{m} + \frac{\beta' A'}{mm'} + \frac{\beta'' A''}{mm'm''} + \frac{\beta''' A'''}{mm'm''m'''} + \text{etc.}$$

§215 Since these formulas extend so far that their use can be seen less clearly, let us restrict our considerations to the case of the transformation given in § 213 and let $x = -1$ that one has this series

$$s = A - B + C - D + E - F + G - \text{etc.}$$

and set

$$\begin{array}{llll} B - A = A' & B' - 2A' = A'' & B'' - 3A'' = A''' & B''' - 4A''' = A'''' \\ C - B = B' & C' - 2B' = B'' & C'' - 3B'' = B''' & C''' - 4B''' = B'''' \\ D - C = C' & D' - 2C' = C'' & D'' - 3C'' = C''' & D''' - 4C''' = C'''' \\ \text{etc.} & \text{etc.} & \text{etc.} & \text{etc.} \end{array}$$

Having found these values, the sum of the propounded series will be equal to the following series

$$s = \frac{A}{2} - \frac{A'}{2\cdot 3} + \frac{A''}{2\cdot 3\cdot 4} - \frac{A'''}{2\cdot 3\cdot 4\cdot 5} + \frac{A''''}{2\cdot 3\cdot 4\cdot 5\cdot 6} - \text{etc.}$$

Therefore, any propounded series can be transformed into innumerable others equal to it, among which without any doubt a most convergent series will be found, by means of which the propounded sum can be found approximately.

§216 But let us return to the invention of series whose law of progression is revealed by differential calculus. Therefore, because this was already achieved for algebraic quantities, let us proceed to transcendental functions and let the series equal to this logarithm be in question

$$s = \log(1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \text{etc.});$$

assume that the series in question reads

$$s = \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \mathfrak{E}x^5 + \mathfrak{F}x^6 + \text{etc.}$$

Therefore, because from the differentiation of the first equation it follows

$$\frac{ds}{dx} = \frac{\alpha + 2\beta x + 3\gamma x^2 + 4\delta x^3 + 5\varepsilon x^4 + \text{etc.}}{1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \text{etc.}},$$

it will be

$$(1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \text{etc.})\frac{ds}{dx} = \alpha + 2\beta x + 3\gamma x^2 + 4\delta x^3 + \text{etc.}$$

But since from the assumed equation

$$\frac{ds}{dx} = \mathfrak{A} + 2\mathfrak{B}x + 3\mathfrak{C}x^2 + 4\mathfrak{D}x^3 + 5\mathfrak{E}x^4 + \text{etc.},$$

having done the substitution, this equation results

$$\begin{array}{rrrrrr}
\mathfrak{A} & + 2\mathfrak{B}x + & 3\mathfrak{C}x^2 + & 4\mathfrak{D}x^3 + & 5\mathfrak{E}x^4 & + \text{etc.} \\
 & + \mathfrak{A}\alpha & + 2\mathfrak{B}\alpha & + 3\mathfrak{C}\alpha & + 4\mathfrak{D}\alpha & + \text{etc.} \\
 & & + \mathfrak{A}\beta & + 2\mathfrak{B}\beta & + 3\mathfrak{C}\beta & + \text{etc.} \\
 & & & + \mathfrak{A}\gamma & + 2\mathfrak{B}\gamma & + \text{etc.} \\
 & & & & + \mathfrak{A}\delta & + \text{etc.} \\
\hline
= \alpha + & 2\beta x + & 3\gamma x^2 + & 4\delta x^3 + & 5\varepsilon x^4 & + \text{etc.}
\end{array}$$

From it one obtains the following determinations

$$\mathfrak{A} = \alpha$$

$$\mathfrak{B} = -\frac{1}{2}\mathfrak{A}\alpha + \beta$$

$$\mathfrak{C} = -\frac{2}{3}\mathfrak{B}\alpha - \frac{1}{3}\mathfrak{A}\beta + \gamma$$

$$\mathfrak{D} = -\frac{3}{4}\mathfrak{C}\alpha - \frac{2}{4}\mathfrak{B}\beta - \frac{1}{4}\mathfrak{A}\gamma + \delta$$

$$\mathfrak{E} = -\frac{4}{5}\mathfrak{D}\alpha - \frac{3}{5}\mathfrak{C}\beta - \frac{2}{5}\mathfrak{B}\gamma - \frac{1}{5}\mathfrak{A}\delta + \text{etc.}$$

etc.

§217 Now, let this exponential quantity be propounded

$$s = e^{\alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \text{etc.}}$$

in which e denotes the number whose hyperbolic logarithm is $= 1$, and assume this series in question

$$s = 1 + \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \mathfrak{E}x^5 + \text{etc.}$$

For, from the case $x = 0$ it is plain that the first term must be 1. Therefore, since by taking logarithms,

$$\log s = \alpha + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \zeta x^6 + \text{etc.},$$

having taken the differentials, it will be

$$\frac{ds}{dx} = s(\alpha + 2\beta x + 3\gamma x^2 + 4\delta x^3 + 5\varepsilon x^4 + \text{etc.})$$

But from the assumed equation it will be

$$
\begin{array}{rl}
\dfrac{ds}{dx} & = \mathfrak{A} + 2\mathfrak{B}x + 3\mathfrak{C}x^2 + 4\mathfrak{D}x^3 + 5\mathfrak{E}x^4 + \text{etc.} \\
\hline
& = \alpha + \mathfrak{A}\alpha x + \mathfrak{B}\alpha x^2 + \mathfrak{C}\alpha x^3 + \mathfrak{D}\alpha x^4 + \text{etc.} \\
& \quad + 2\beta + 2\mathfrak{A}\beta + 2\mathfrak{C}\beta + 2\mathfrak{C}\beta + \text{etc.} \\
& \quad\quad + 3\gamma + 3\mathfrak{A}\gamma + 3\mathfrak{B}\gamma + \text{etc.} \\
& \quad\quad\quad + 4\delta + 4\mathfrak{A}\delta + \text{etc.} \\
& \quad\quad\quad\quad + 5\varepsilon + \text{etc.},
\end{array}
$$

from which the following determinations of the letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. result

$$
\begin{aligned}
\mathfrak{A} &= \alpha \\
\mathfrak{B} &= \beta + \frac{1}{2}\mathfrak{A}\beta \\
\mathfrak{C} &= \gamma + \frac{2}{3}\mathfrak{A}\beta + \frac{1}{3}\mathfrak{B}\alpha \\
\mathfrak{D} &= \delta + \frac{3}{4}\mathfrak{A}\gamma + \frac{2}{4}\mathfrak{B}\beta + \frac{1}{4}\mathfrak{C}\alpha \\
\mathfrak{E} &= \varepsilon + \frac{4}{5}\mathfrak{A}\delta + \frac{3}{5}\mathfrak{B}\gamma + \frac{2}{5}\mathfrak{C}\beta + \frac{1}{5}\mathfrak{D}\alpha \\
&\text{etc.}
\end{aligned}
$$

§218 If the arc, whose sine or cosine is in question, is expressed by a binomial or polynomial or even an infinite series, this way one can even express its sine and cosine by means of an infinite series. But to do this the most convenient way, it does not suffice to consider only the first differentials, but it is necessary to use the differentials of the second order. Therefore, let

$$
s = \sin(\alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \text{etc.})
$$

and assume the series in question to be

$$
s = \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \mathfrak{E}x^5 + \text{etc.}
$$

For, it is plain that the first term vanishes; but since one has to descend to the second differentials, the coefficient $\mathfrak{A}$ also has to be determined from elsewhere, which will happen, if we put x to be infinitely small. For then, because of the arc $= \alpha x$, the sine itself will become equal to it and it will therefore be $\mathfrak{A} = \alpha$. Now, for the sake of brevity let us put

$$z = \alpha x + \beta x^2 + \gamma x^3 + \text{etc.},$$

that $s = \sin z$; by differentiating it will be $ds = dz \cos z$ and by differentiating again it will be $dds = ddz \cos z - dz^2 \sin z$. Therefore, since $\sin z = s$ and $\cos z = \frac{ds}{dz}$, it will be

$$dds = \frac{dsddz}{dz} - sdz^2 \quad \text{and} \quad dzdds + sdz^3 = dsddz.$$

§219 Let us put that the arc z is only expressed by a binomial and

$$z = \alpha x + \beta x^2;$$

it will be

$$dz = (\alpha + 2\beta x)dx$$

and, having put dx to be constant,

$$ddz = 2\beta dx^2$$

and

$$dz^3 = (\alpha^3 + 6\alpha^2\beta x + 12\alpha\beta^2 x^2 + 8\beta^3 x^3)dx^3.$$

Further, because of $s = \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \text{etc.}$, it will be

$$\frac{ds}{dx} = \mathfrak{A} + 2\mathfrak{B}x + 3\mathfrak{C}x^2 + 4\mathfrak{D}x^3 + \text{etc.}$$

and

$$\frac{dds}{dx^2} = 2\mathfrak{B} + 6\mathfrak{C}x + 12\mathfrak{D}x^2 + \text{etc.}$$

Having substituted these values in the differential equation, it will be

$$
\begin{array}{rcccccc}
\dfrac{dzdds}{dx^3} = 1\cdot 2\mathfrak{B}\alpha & + \quad 2\cdot 3\mathfrak{C}\alpha x & + \quad 3\cdot 4\mathfrak{D}\alpha x^2 & + \quad 4\cdot 5\mathfrak{E}\alpha x^3 & + \quad 5\cdot 6\mathfrak{F}\alpha x^4 & + \text{etc.} \\
 & + 2\cdot 1\cdot 2\mathfrak{B}\beta & + 2\cdot 2\cdot 3\mathfrak{C}\beta & + 2\cdot 3\cdot 4\mathfrak{D}\beta & + 2\cdot 4\cdot 5\mathfrak{E}\beta & + \text{etc.} \\
\dfrac{sdz^3}{dx^3} = & + \quad \mathfrak{A}\alpha^3 & + \quad \mathfrak{B}\alpha^3 & + \quad \mathfrak{C}\alpha^3 & + \quad \mathfrak{D}\alpha^3 & + \text{etc.} \\
 & & + \quad 6\mathfrak{A}\alpha^2\beta & + \quad 6\mathfrak{B}\alpha^2\beta & + \quad 6\mathfrak{C}\alpha^2\beta & + \text{etc.} \\
 & & & + \quad 12\mathfrak{A}\alpha\beta^2 & + \quad 12\mathfrak{B}\beta^2 & + \text{etc.} \\
 & & & & + \quad 8\mathfrak{A}\beta^3 & + \text{etc.} \\
\hline
\dfrac{dsddz}{dx^3} = \quad 2\mathfrak{A}\beta & + \quad 4\mathfrak{B}\beta & + \quad 6\mathfrak{C}\beta & + \quad 8\mathfrak{D}\beta & + \quad 10\mathfrak{E}\beta & + \text{etc.}
\end{array}
$$

Therefore, the coefficients will be defined the following way:

$$
\begin{aligned}
\mathfrak{B} &= \frac{2\mathfrak{A}\beta}{2\alpha} \\
\mathfrak{C} &= 0 - \frac{\mathfrak{A}\alpha^2}{2\cdot 3} \\
\mathfrak{D} &= -\frac{2\mathfrak{C}\beta}{4\alpha} - \frac{6\mathfrak{A}\alpha\beta}{3\cdot 4} - \frac{\mathfrak{B}\alpha^2}{3\cdot 4} \\
\mathfrak{E} &= -\frac{4\mathfrak{D}\beta}{5\alpha} - \frac{12\mathfrak{A}\beta^2}{4\cdot 5} - \frac{6\mathfrak{B}\alpha\beta}{4\cdot 5} - \frac{\mathfrak{C}\alpha^2}{4\cdot 5} \\
\mathfrak{F} &= -\frac{6\mathfrak{E}\beta}{6\alpha} - \frac{8\mathfrak{A}\beta^3}{5\cdot 6\alpha} - \frac{12\mathfrak{B}\beta\beta}{5\cdot 6} - \frac{6\mathfrak{C}\alpha\beta}{5\cdot 6} - \frac{\mathfrak{D}\alpha^2}{5\cdot 6} \\
\mathfrak{G} &= -\frac{8\mathfrak{F}\beta}{7\alpha} - \frac{8\mathfrak{B}\beta^3}{6\cdot 7\alpha} - \frac{12\mathfrak{C}\beta\beta}{6\cdot 7} - \frac{6\mathfrak{D}\alpha\beta}{6\cdot 7} - \frac{\mathfrak{E}\alpha^2}{6\cdot 7} \\
&\text{etc.}
\end{aligned}
$$

Having found these values, it will be

$$\sin(\alpha x + \beta x^2) = \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \text{etc.}$$

while $\mathfrak{A} = \alpha$.

§220 In like manner, the cosine of any angle is converted into a series; but since an arc is very rarely expressed by a polynomial, let us show the use of differential equations for the invention of the series for the cosine of the arc x. Therefore, let $s = \cos x$ and assume

$$s = 1 - \mathfrak{A}x^2 + \mathfrak{B}x^4 - \mathfrak{C}x^6 + \mathfrak{D}x^8 - \text{etc.}$$

Since $ds = -dx \sin x$ and $dds = -dx^2 \cos x = -sdx^2$, it will be

$$dds + sdx^2 = 0;$$

after the substitution it will be

$$\begin{aligned} \frac{dds}{dx^2} &= -1 \cdot 2\mathfrak{A} + 3 \cdot 4\mathfrak{B}x^2 - 5 \cdot 6\mathfrak{C}x^4 + 7 \cdot 8\mathfrak{D}x^6 - \text{etc.} \\ s &= \quad 1 - \quad \mathfrak{A}x^2 \quad - \mathfrak{B}x^4 - \quad \mathfrak{C}x^6 + \text{etc.} \end{aligned}$$

and by comparing the coefficients it follows

$$\begin{aligned} \mathfrak{A} &= \frac{1}{1 \cdot 2} \\ \mathfrak{B} &= \frac{\mathfrak{A}}{3 \cdot 4} = \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \\ \mathfrak{C} &= \frac{\mathfrak{B}}{5 \cdot 6} = \frac{1}{1 \cdot 2 \cdot 3 \cdots 6} \\ \mathfrak{D} &= \frac{\mathfrak{C}}{7 \cdot 8} = \frac{1}{1 \cdot 2 \cdot 3 \cdots 8} \\ &\text{etc.} \end{aligned}$$

Therefore, it is plain, what we demonstrated in more detail above already, that

$$\cos x = 1 - \frac{x^2}{1 \cdot 2} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{x^6}{1 \cdot 2 \cdot 3 \cdots 6} + \frac{x^8}{1 \cdot 2 \cdot 3 \cdots 8} - \text{etc.};$$

the first series for the sine, having put $\beta = 0$ and $\alpha = 1$, will give

$$\sin x = \frac{x}{1} - \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{x^7}{1 \cdot 2 \cdot 3 \cdots 7} + \frac{x^9}{1 \cdot 2 \cdot 3 \cdots 9} - \text{etc.}$$

§221 From the well-known series for the sine and cosine the series for the tangent, cotangent, secant, cosecant of a certain angle are deduced. For, the tangent results, if the sine is divided by the cosine, the cotangent, if the cosine is divided by the sine, the secant, if the radius 1 is divided by the cosine, and the cosecant, if the radius is divided by the sine. But the series result from these divisions seem to be most irregular; but, with the exception of the series exhibiting the secant, all remaining the others can be reduced to a simple law by means of the Bernoulli numbers $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. For, since we found above (§ 127) that

$$\frac{\mathfrak{A}u^2}{1 \cdot 2} + \frac{\mathfrak{B}u^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{\mathfrak{C}u^6}{1 \cdot 2 \cdot 3 \cdots 6} + \frac{\mathfrak{D}u^8}{1 \cdot 2 \cdot 3 \cdots 8} + \text{etc.} = 1 - \frac{u}{2} \cot \frac{1}{2}u,$$

having put $\frac{1}{2}u = x$, it will be

$$\cot x = \frac{1}{x} - \frac{2^2\mathfrak{A}x}{1 \cdot 2} - \frac{2^4\mathfrak{B}x^3}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{2^6\mathfrak{C}x^5}{1 \cdot 2 \cdot 3 \cdots 6} - \frac{2^8\mathfrak{D}x^7}{1 \cdot 2 \cdot 3 \cdots 8} - \text{etc.},$$

and if one puts $\frac{1}{2}x$ for x, it will be

$$\cot \frac{1}{2}x = \frac{2}{x} - \frac{2\mathfrak{A}x}{1 \cdot 2} - \frac{2\mathfrak{B}x^3}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{2\mathfrak{C}x^5}{1 \cdot 2 \cdot 3 \cdots 6} - \frac{2^{\mathfrak{D}}x^7}{1 \cdot 2 \cdot 3 \cdots 8} - \text{etc.},$$

§222 But hence the tangent of any arc will be expressed by means of an infinite series the following way. Since

$$\tan 2x = \frac{2 \tan x}{1 - \tan x},$$

it will be

$$\cot 2x = \frac{1}{2 \tan x} - \frac{\tan x}{2} = \frac{1}{2} \cot x - \frac{1}{2} \tan x$$

and hence

$$\tan x = \cot x - 2\cot 2x.$$

Therefore, since

$$\cot x = \frac{1}{x} - \frac{2^2\mathfrak{A}x}{1\cdot 2} - \frac{2^4\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4} - \frac{2^6\mathfrak{C}x^5}{1\cdot 2\cdots 6} - \frac{2^8\mathfrak{D}x^7}{1\cdot 2\cdots 8} - \text{etc.},$$

$$2\cot 2x = \frac{1}{x} - \frac{2^4\mathfrak{A}x}{1\cdot 2} - \frac{2^8\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4} - \frac{2^{12}\mathfrak{C}x^5}{1\cdot 2\cdots 6} - \frac{2^{16}\mathfrak{D}x^7}{1\cdot 2\cdots 8} - \text{etc.},$$

subtracting this series from the first, it will be

$$\tan x = \frac{2^2(2^2-1)\mathfrak{A}x}{1\cdot 2} + \frac{2^4(2^4-1)\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4} + \frac{2^6(2^6-1)\mathfrak{C}x^5}{1\cdot 2\cdots 6} + \frac{2^8(2^8-1)\mathfrak{D}x^7}{1\cdot 2\cdots 8} + \text{etc.}$$

Therefore, if the numbers A, B, C, D etc. found in § 182 are introduced here, it will be

$$\tan x = \frac{2Ax}{1\cdot 2} + \frac{2^3Bx^3}{1\cdot 2\cdot 3\cdot 4} + \frac{2^5Cx^5}{1\cdot 2\cdot 3\cdot 6} + \frac{2^7Dx^7}{1\cdot 2\cdots 8} + \text{etc.}$$

§223 But the cosecant will be found the following way. Since

$$\cot x = \tan x + 2\cot 2x = \frac{1}{\cot x} + 2\cot 2x,$$

it will be

$$\cot^2 x = 2\cot x\cot 2x + 1$$

and having extracted the root

$$\cot x = \cot 2x + \csc 2x,$$

whence

$$\csc 2x = \cot x - \cot 2x$$

and having put x for $2x$

$$\csc x = \cot\frac{1}{2}x - \cot x.$$

Therefore, because we have the cotangent, i.e.

$$\cot\frac{1}{2}x = \frac{2}{x} - \frac{2\mathfrak{A}x}{1\cdot 2} - \frac{2\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4} - \frac{2\mathfrak{C}x^5}{1\cdot 2\cdots 6} - \text{etc.}$$

$$\cot x = \frac{1}{x} - \frac{2^2\mathfrak{A}x}{1\cdot 2} - \frac{2^4\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4} - \frac{2^5\mathfrak{C}x^5}{1\cdot 2\cdots 6} - \text{etc.},$$

having subtracted this series from the first, it will be

$$\csc x = \frac{1}{x} + \frac{2(2-1)\mathfrak{A}x}{1\cdot 2} + \frac{2(2^3-1)\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4} + \frac{2(2^5-1)\mathfrak{C}x^5}{1\cdot 2\cdots 6} + \text{etc.}$$

§224 But the secant cannot be expressed by means of these Bernoulli numbers, but it requires other numbers which enter the sums of the odd powers of the reciprocals. For, if one puts

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \text{etc.} = \alpha \cdot \frac{\pi}{2^2}$$

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \text{etc.} = \frac{\beta}{1\cdot 2} \cdot \frac{\pi^3}{2^4}$$

$$1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \text{etc.} = \frac{\gamma}{1\cdot 2\cdot 3\cdot 4} \cdot \frac{\pi^5}{2^6}$$

$$1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \frac{1}{9^7} - \text{etc.} = \frac{\delta}{1\cdot 2\cdots 6} \cdot \frac{\pi^7}{2^8}$$

$$1 - \frac{1}{3^9} + \frac{1}{5^9} - \frac{1}{7^9} + \frac{1}{9^9} - \text{etc.} = \frac{\varepsilon}{1\cdot 2\cdots 8} \cdot \frac{\pi^9}{2^{10}}$$

$$1 - \frac{1}{3^{11}} + \frac{1}{5^{11}} - \frac{1}{7^{11}} + \frac{1}{9^{11}} - \text{etc.} = \frac{\zeta}{1\cdot 2\cdots 10} \cdot \frac{\pi^{11}}{2^{12}}$$

etc.,

it will be

$$\begin{aligned}
\alpha &= 1\\
\beta &= 1\\
\gamma &= 5\\
\delta &= 61\\
\varepsilon &= 1385\\
\zeta &= 50521\\
\eta &= 2702765\\
\theta &= 199360981\\
\iota &= 19391512145\\
\varkappa &= 2040487967544\text{1etc.}
\end{aligned}$$

and from these values one will obtain

$$\sec x = \alpha + \frac{\beta}{1\cdot 2}xx + \frac{\gamma}{1\cdot 2\cdot 3\cdot 4}x^4 + \frac{\delta}{1\cdot 2\cdots 6}x^6 + \frac{\varepsilon}{1\cdot 2\cdots 8}x^8 + \text{etc.}$$

§225 To show the connection of this series to the numbers α, β, γ, δ etc., let us consider the series treated above [§ 33]

$$\frac{\pi}{n\sin\frac{m}{n}\pi} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{m+n} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \text{etc.}$$

Put $m = \frac{1}{2}n - k$ and it will be

$$\frac{\pi}{2n\cos\frac{k}{n}} = \frac{1}{n-2k} + \frac{1}{n+2k} - \frac{1}{3n-2k} - \frac{1}{3n+2k} + \frac{1}{5n-2k} + \text{etc.}$$

Let $\frac{k\pi}{n} = x$ or $k\pi = nx$; it will be

$$\frac{\pi}{2n}\sec x = \frac{\pi}{n\pi-2nx} + \frac{\pi}{n\pi+2nx} - \frac{\pi}{3n\pi-2nx} - \frac{\pi}{3n\pi+2nx} + \frac{\pi}{5n\pi-2nx} + \text{etc.}$$

or

$$\sec x = \frac{2}{\pi - 2x} + \frac{2}{\pi + 2x} - \frac{2}{3\pi - 2x} - \frac{2}{3\pi + 2x} + \frac{2}{5\pi - 2x} + \text{etc.}$$

or

$$\sec x = \frac{4\pi}{\pi^2 - 4x^2} - \frac{4 \cdot 3\pi}{9\pi^2 - 4xx} + \frac{4 \cdot 5\pi}{25\pi^2 - 4xx} - \frac{4 \cdot 7\pi}{49\pi^2 - x^2} + \text{etc.}$$

If now each terms is converted into series, it will be

$$\begin{aligned}\sec x = \ & \frac{4}{\pi} \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \text{etc.}\right)\\ & + \frac{2^4 x^2}{\pi^3}\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \text{etc.}\right)\\ & + \frac{2^6 x^4}{\pi^5}\left(1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \text{etc.}\right)\\ & \text{etc.;}\end{aligned}$$

if the values assigned above are substituted for these series, the same series we gave for the secant will result.

§226 Therefore, at the same time the law is plain, according to which the numbers α, β, γ, δ etc. appearing in the expressions of the sums of the odd powers, proceed. For, since

$$\sec x = \frac{1}{\cos x} = \alpha + \frac{\beta}{1 \cdot 2}x^2 + \frac{\gamma}{1 \cdot 2 \cdot 3 \cdot 4}x^4 + \frac{\delta}{1 \cdot 2 \cdots 6}x^6 + \text{etc.},$$

it is necessary that this series is equal to the fraction

$$\frac{1}{1 - \frac{xx}{1 \cdot 2} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{x^6}{1 \cdot 2 \cdots 6} + \frac{x^6}{1 \cdot 2 \cdots 8} - \text{etc.}};$$

therefore, having equated the two expressions, it will be

$$
\begin{aligned}
1 = \alpha + \frac{\beta}{1\cdot 2}x^2 &+ \frac{\gamma}{1\cdot 2\cdot 3\cdot 4}x^4 + \frac{\delta}{1\cdot 2\cdots 6}x^6 + \frac{\varepsilon}{1\cdot 2\cdots 8}x^8 + \text{etc.}\\
- \frac{\alpha}{1\cdot 2} &- \frac{\beta}{1\cdot 2\cdot 1\cdot 2} - \frac{\gamma}{1\cdot 2\cdot 1\cdots 4} - \frac{\delta}{1\cdot 2\cdot 1\cdots 6} - \text{etc.}\\
&+ \frac{\alpha}{1\cdot 2\cdot 3\cdot 4} + \frac{\beta}{1\cdots 4\cdot 1\cdot 2} + \frac{\gamma}{1\cdots 4\cdot 1\cdots 4} - \text{etc.}\\
&\qquad - \frac{\alpha}{1\cdot 2\cdots 6} + \frac{\beta}{1\cdots 6\cdot 1\cdots 2} - \text{etc.}\\
&\qquad\qquad + \frac{\alpha}{1\cdot 2\cdots 8} + \text{etc.},
\end{aligned}
$$

whence these equations follow

$$
\begin{aligned}
\alpha &= 1\\
\beta &= \frac{2\cdot 1}{1\cdot 2}\alpha\\
\gamma &= \frac{4\cdot 3}{1\cdot 2}\beta - \frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4}\alpha\\
\delta &= \frac{6\cdot 5}{1\cdot 2}\gamma - \frac{6\cdot 5\cdot 4\cdot 3}{1\cdot 2\cdot 3\cdot 4}\beta + \frac{6\cdots 1}{1\cdots 6}\alpha\\
\varepsilon &= \frac{8\cdot 7}{1\cdot 2}\delta - \frac{8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4}\gamma + \frac{8\cdots 3}{1\cdots 6}\beta - \frac{8\cdots 1}{1\cdots 8}\alpha\\
&\text{etc.}
\end{aligned}
$$

And from these formulas the values of these letters were found which we exhibited in § 224 and by means of which the sums of the series contained in this form

$$1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \text{etc.},$$

if n was an odd number, can be expressed.