On the use of Differential Calculus in the resolution of Equations *

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§227 That an equation can be reduced to the nature of functions was demonstrated above already. For, let y denote any function of x; if one puts $y = 0$, this form contains completely all finite equations, may they be algebraic or transcendental. But the equation $y = 0$ is said to be solved, if that value x is found which substituted in the function y actually renders it equal to zero. But in most cases many of such values x exist, which are called the roots of the equation $y = 0$. Therefore, if we put that the numbers f, g, h, i etc. are roots of the equation $y = 0$, the function y will be of such a nature that, if in it either f or g or h or etc. is substituted for x, it indeed is $y = 0$.

§228 Therefore, since the function f vanishes, if in it one puts f or $x + (f - x)$ instead of x, where f is a root of the equation $y = 0$, by the results we demonstrated on functions above [§48] it will be

$$0 = y + \frac{(f-x)dy}{dx} + \frac{(f-x)^2 ddy}{2dx^2} + \frac{(f-x)^3 d^3y}{6dx^3} + \text{etc.},$$

from which equation the value of the root f is determined in such a way that, whatever was put for x and hence the value of the quantities y, $\frac{dy}{dx}$, $\frac{ddy}{2dx^2}$ etc. were substituted, always the equation expressing the true value of f results. That this is seen more clearly, let us put that it is

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$$y = x^3 - 2x^2 + 3x - 4;$$

it will be

$$\frac{dy}{dx} = 3xx - 4x + 3, \quad \frac{ddy}{2dx^2} = 3x - 2 \quad \text{and} \quad \frac{d^3y}{6dx^3} = 1.$$

Having substituted these values this equation results

$$0 = x^3 - 2x^2 + 3x - 4 + (f - x)(3xx - 4x + 3) + (f - x)^2(3x - 2) + (f - x)^3$$

or having actually done the multiplications

$$f^3 - 2ff + 3f - 4 = 0;$$

of course, the same equation as the propounded one results which therefore contains the same roots.

§229 But although this way one does not get to a new equation, from which the value of the root f can be determined in an easier way, nevertheless extraordinary auxiliary theorems for the invention of roots can be deduced from this. For, if a value already very close to a certain root was assumed for x such that $f - x$ is a very small quantity, then the terms of the equation

$$0 = y + \frac{(f - x)dy}{dx} + \frac{(f - x)^2 ddy}{2dx^2} + \frac{(f - x)^3 d^3y}{6dx^3} + \text{etc.}$$

will converge very rapidly and therefore this expression will not deviate much from the true value, if only the first two initial terms are considered. Therefore, if a value already close to a certain root of the equation $y = 0$ was assumed for x, it will approximately be

$$0 = y + \frac{(f - x)dy}{dx} \quad \text{or} \quad f = x - \frac{ydx}{dy},$$

from which formula a, even though not true, but nevertheless very good approximate value of the root f will be found, which, if it is substituted for x again, will yield a even better value for f and so one will continuously get closer to the true value of the root f.

§230 Therefore, at first the root of all powers of any number can be calculated this way. For, let the number $a^n + b$ be propounded and let its root of the power n to be extracted. Put $x^n = a^n + b$ or $x^n - a^n - b = 0$ that it is $y = x^n - a^n - b$; it will be

$$\frac{dy}{dx} = nx^{n-1}, \quad \frac{ddy}{2dx^2} = \frac{n(n-1)}{1 \cdot 2}x^{n-2}, \quad \frac{d^3y}{6dx^3} = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^{n-3} \quad \text{etc.}$$

Therefore, if the root in question is put $= f$ that it is $f = \sqrt[n]{a^n + b}$, it will be

$$0 = x^n - a^n - b + n(f-x)x^{n-1} + \frac{n(n-1)}{1 \cdot 2}(f-x)^2 x^{n-2} + \text{etc.}$$

Therefore, if one takes a number already coming close to the value of the root f in question for x, which will be achieved by putting $x = a$, if b was such a small number that $a^n + b < (a+1)^n$, it will approximately be $b = na^{n-1}(f-a)$ and hence

$$f = a + \frac{b}{na^{n-1}},$$

whence a much better approximation of the value of the root will be found. But if we want to take also the third term that it is

$$b = na^{n-1}(f-a) + \frac{n(n-1)}{1 \cdot 2}a^{n-2}(f-a)^2,$$

it will be

$$(f-a)^2 = -\frac{2a}{n-1}(f-a) + \frac{2b}{n(n-1)a^{n-2}}$$

and hence

$$f = a - \frac{a}{n-1} \pm \sqrt{\frac{aa}{(n-1)^2} + \frac{2b}{n(n-1)a^{n-2}}}$$

or

$$f = \frac{(n-2)a + \sqrt{aa + 2(n-1)b : na^{n-2}}}{n-1}.$$

Therefore, by means of the extraction of the square root an even closer value of the root f will be found.

EXAMPLE

Let us find the square root of any number c or let $xx - c = y$.

Therefore, put the number very close to the root $= a$ and $b = c - aa$; because of $aa + b = c$ and since it is $n = 2$, the first formula will become $f = a + \frac{c-aa}{2a} = \frac{c+aa}{2a}$; the other gives $f = \sqrt{c}$ which is the root in question itself. Therefore, because the root approximately is $= \frac{c+aa}{2a}$, write that value for a and $f = \frac{cc+6aac+a^4}{4a(c+aa)}$ will be an even better approximation. For the sake of an example let $c = 5$, from the first formula it will be $f = \frac{5}{2a} + \frac{a}{2}$. Therefore, put $a = 2$, it will be $f = 2.25$; now put $a = 2.25$, it will be $f = 2.236111$; further, set $a = 2.236111$, it will be $f = 2.2360679$ which value already hardly deviates from the value.

§231 But in like manner the root of any equation can be found approximately by means of the equation $f = x - \frac{ydx}{dy}$, after having assumed a value differing hardly from a certain root of the equation for x, of course. To find a value of this kind for x, successively substitute various values for x and from them chose the one which minimizes the function y, which means, which indicates the value closest to zero. So, if it is

$$y = x^3 - 2xx + 3x - 4$$

$$\begin{array}{llll} \text{having put} & x = 0 & \text{it is} & y = -4 \\ & x = 1 & & y = -2 \\ & x = 2 & & y = +2, \end{array}$$

whence we see that the root is contained within the values 1 and 2 of x. Therefore, because it is $\frac{dy}{dx} = 3xx - 4x + 3$, in order to find the root f of the equation $x^3 - 2xx + 3x - 4 = 0$ one has to use this equation

$$f = x - \frac{ydx}{dy} = x - \frac{x^3 - 2xx + 3x - 4}{3xx - 4x + 3}.$$

Therefore, let $x = 1$; it will be $f = 1 + \frac{2}{2} = 2$. Now put $x = 2$; it will be $f = 2 - \frac{2}{7} = \frac{12}{7}$. Therefore, let $x = \frac{12}{7}$; it will be $f = \frac{12}{7} - \frac{104}{1701} = \frac{2812}{1701} = 1.653$. If we want to proceed further, we will use logarithms more conveniently.
Therefore, put $x = 1.653$ and it will be

$$\begin{array}{rcl|rcl}
\log x^1 & = & 0.2182729 & x^1 & = & 1.653000 \\
\log x^2 & = & 0.4365458 & x^2 & = & 2.732409 \\
\log x^3 & = & 0.6548187 & x^3 & = & 4.516673
\end{array}$$

$$\begin{array}{rcl}
x^3 & = & 4.516673 \\
3x & = & \underline{4.959000}
\end{array}$$

And hence

$$\begin{array}{rcrrcr}
x^3+3x & = & 9.475673 & 3xx+3 & = & 11.197227 \\
2xx+4 & = & \underline{9.464818} & 4x & = & \underline{6.612000} \\
\text{num.} & = & 0.010855 & \text{den.} & = & 4.585227 \\
\log\text{num.} & = & 8.0356298 & & & \\
\log\text{den.} & = & 0.6613608 & x & = & 1.6553000 \\
\log\text{fract.} & = & 7.3742690 & \text{fraction} & = & \underline{0.002367} \\
 & & & & & \\
 & & & f & = & \underline{1.650633,}
\end{array}$$

which value already comes very close to the true one.

§232 But we will be able to deduce faster approximations from the general expression. For, because having put any function $y = 0$, if the root of this equation was $x = f$, we will find that it is

$$0 = y + \frac{(f-x)dy}{dx} + \frac{(f-x)^2 ddy}{2dx^2} + \frac{(f-x)^3 d^3y}{6dx^3} + \text{etc.},$$

let $f - x = z$, such that the root is $f = x + z$, and put

$$\frac{dy}{dx} = p, \quad \frac{dp}{dx} = q, \quad \frac{dq}{dx} = r, \quad \frac{dr}{dx} = s \quad \text{etc.};$$

it will be

$$0 = y + zp + \frac{z^2q}{2} + \frac{z^3r}{6} + \frac{z^4s}{24} + \frac{z^5t}{120} + \text{etc.};$$

in this equation having taken any value for x, from which at the same time y, p, q, r, s etc. are determined, the quantity z must be found, having found which one will have the root $f = x + z$ of the propounded equation $y = 0$. Therefore, will have to focus on the task to find the value of the unknown z from this equation in the most convenient way possible.

§233 Assume this convergent series for z

$$z = A + B + C + D + E + \text{etc.}$$

and after the substitution it will be

$$\begin{aligned}
y &= \; y \\
pz &= Ap + Bp + Cp + Dp + Ep + \text{etc.} \\
\frac{1}{2}qz^2 &= \quad + \frac{1}{2}A^2q + ABq + ACq + ADq + \text{etc.} \\
&\qquad\qquad + \frac{1}{2}BBq + BCq + \text{etc.} \\
\frac{1}{6}rz^3 &= \qquad \frac{1}{6}A^3r + \frac{1}{2}A^2Br + \frac{1}{2}A^2Cr + \text{etc.} \\
&\qquad\qquad + \frac{1}{2}AB^2r + \text{etc.} \\
\frac{1}{24}sz^4 &= \qquad \frac{1}{24}A^4s + \frac{1}{6}A^3Bs + \text{etc.} \\
\frac{1}{120}tz^5 &= \qquad \frac{1}{120}A^5t + \text{etc.}
\end{aligned}$$

Therefore, one will obtain the following equations

$$
\begin{aligned}
A &= -\frac{y}{p} \\
B &= -\frac{yyq}{2p^3} \\
C &= -\frac{y^3qq}{2p^5} + \frac{y^3r}{6p^4} \\
D &= -\frac{5y^4q^3}{8p^7} + \frac{5y^4qr}{12p^6} - \frac{y^4s}{24p^5} \\
&\text{etc.}
\end{aligned}
$$

and hence it will be

$$z = -\frac{y}{p} - \frac{y^2q}{2p^3} - \frac{y^3qq}{2p^5} + \frac{y^3r}{6p^4} - \frac{5y^4q^3}{8p^7} + \frac{5y^4qr}{12p^6} - \frac{y^4s}{24p^5} - \text{etc.}$$

EXAMPLE

Let this equation be propounded $x^5 + 2x - 2 = 0$.

Therefore, it will be

$$y = x^5 + 2x - 2, \quad \frac{dy}{dx} = p = 5x^4 + 2, \quad \frac{dp}{dx} = q = 20x^3,$$
$$\frac{dq}{dx} = r = 60x^3, \quad \frac{dr}{dx} = s = 120x \quad \text{etc.}$$

But now put $x = 1$, since this value hardly deviates from the true root, it will be

$$y = 1, \quad p = 7, \quad q = 20, \quad r = 60, \quad s = 120,$$

whence it will be

$$z = -\frac{1}{7} - \frac{10}{7^3} - \frac{200}{7^5} + \frac{10}{7^4} - \frac{5 \cdot 1000}{7^7} + \frac{500}{7^6} - \frac{5}{7^5} + \text{etc.}$$

or

$$z = -\frac{1}{7} - \frac{10}{7^3} - \frac{130}{7^5} - \frac{1745}{7^7} - \text{etc.},$$

and therefore it will be $z = 0.18$ and the root $f = 0.82$; if this value is again substituted for x, a root very close to the true one will result.

§234 Therefore, we found an infinite series, which expresses the root of any equation; but it is inconvenient that the law of progression is not obvious, and it hence is too complex and not sufficiently useful. Therefore, let us consider the same problem in another way and try to find a more regular series expressing any root of the propounded equation.

Let as before the equation $y = 0$ be propounded while y is any function of x and the question reduces to that the value of x is defined which substituted for x renders the function y equal to zero. But because y is a function of x, vice versa x can be considered as function of y and considering it like this the value of the function x is to be found which it obtains, if the quantity y vanishes. Therefore, if it is propounded to find the value of x which will be the root of the equation $y = 0$, since x goes over into f, if one sets $y = 0$, by the results demonstrated above [§ 67] it will be

$$f = x - \frac{ydx}{dy} + \frac{y^2ddx}{2dy^2} - \frac{y^3d^3}{6dy^3} + \frac{y^4d^4x}{24dy^4} - \text{etc.},$$

in which equation the differential dy is assumed to be constant. Therefore, if one puts

$$\frac{dx}{dy} = p, \quad \frac{dp}{dy} = q, \quad \frac{dq}{dy} = r, \quad \frac{dr}{dy} = s \quad \text{etc.},$$

having introduced these values, so that it is not necessary to consider a certain differential to be constant, it will be

$$f = x - py + \frac{1}{2}qy^2 - \frac{1}{6}ry^3 + \frac{1}{24}sy^4 - \frac{1}{120}ty^5 + \text{etc.}$$

§235 Therefore, having attributed any value to x at the same time the values of y and the quantities p, q, r, s etc. will be determined and having found these values one will have an infinite series expressing the value of the root f. But if the equation $y = 0$ has several roots, then these values result, if different values are assumed for x; for, because y can have the same value, even though different values are attributed to x, it is not surprising that the same series can often yield several values. To avoid this ambiguity in these cases and to render the series convergent, a value already close to the value of its root, which is in question, must be assumed for x. For, this way the value of y will become very small and the terms of the series will decrease immensely, such that by taking only a few terms one will already find a sufficiently correct value for f.

If this value is then substituted for x, the quantity y will become a lot smaller and the series will converge a lot more and this way the root f immediately becomes known so accurately that the error will be very small. And hence the advantages of this expression over the one we found before is clearly seen.

§236 Let us assume that the root of the power n of any number N is to be extracted. Therefore, having taken an approximate power of the exponent n the propounded number will easily be resolved into this form $N = a^n + b$. Therefore, it will be

$$x^n = a^n + b \quad \text{and} \quad y = x^n - a^n - b,$$

whence it is

$$\begin{aligned}
dy &= nx^{n-1}dx && \text{and} \quad \frac{dx}{dy} = p = \frac{1}{nx^{n-1}} \\
dp &= -\frac{(n-1)dx}{nx^n} && \text{and} \quad \frac{dp}{dy} = q = -\frac{n-1}{nnx^{2n-1}} \\
dq &= \frac{(n-1)(2n-1)dx}{nnx^{2n}} && \text{and} \quad \frac{dq}{dy} = r = \frac{(n-1)(2n-1)}{n^3x^{3n-1}} \\
dr &= -\frac{(n-1)(2n-1)(3n-1)dx}{n^3x^{3n}} && \text{and} \quad \frac{dr}{dy} = s = -\frac{(n-1)(2n-1)(3n-1)}{n^4x^{4n-1}} \\
& && \text{etc.}
\end{aligned}$$

Now, put $x = a$ and it will be $y = -b$ and the root in question $f = \sqrt[n]{a^n + b}$ will be expressed this way

$$f = a + \frac{b}{na^{n-1}} - \frac{(n-1)bb}{n \cdot 2na^{2n-1}} + \frac{(n-1)(2n-1)b^3}{n \cdot 2n \cdot 3na^{3n-1}} - \frac{(n-1)(2n-1)(3n-1)b^4}{n^4 \cdot 2n \cdot 3n \cdot 4na^{4n-1}} + \text{etc.}$$

and so the same series results which is usually found by expansion of the binomial $(a^n + b)^{\frac{1}{n}}$.

§237 Therefore, after the approximate root a was found in the actual extraction and at the same time the residue b was found then the value $\frac{b}{an^{n-1}}$ is to be added to the root, such that a root closer to the true one is obtained. But because of $N = a^n + b$ it will be

$$a^{n-1} = \frac{N-b}{a}.$$

But this way a root larger than the correct one will be found, since the third term must be subtracted. Therefore, to find a root a lot closer to the true one by means of division of the residue b a suitable divisor must be investigated, which we want to assume to be

$$na^{n-1} + \alpha b + \beta bb + \gamma b^3 + \text{etc.}$$

Therefore, because it must be

$$\frac{b}{na^{n-1} + \alpha b + \beta b^2 + \gamma b^3 + \text{etc.}}$$
$$= \frac{b}{na^{n-1}} - \frac{(n-1)bb}{2n^2a^{2n-1}} + \frac{(n-1)(2n-1)b^3}{6n^3a^{3n-1}} - \frac{(n-1)(2n-1)(3n-1)b^4}{24n^4a^{4n-1}} + \text{etc.},$$

after the multiplication by $na^{n-1} + \alpha b + \beta b^2 + \gamma b^3 + \text{etc.}$ it will be

$$\begin{aligned} b = b &- \frac{(n-1)bb}{2na^n} + \frac{(n-1)(2n-1)b^3}{6n^2a^{2n}} - \frac{(n-1)(2n-1)(3n-1)b^4}{24n^3a^{3n}} + \text{etc.} \\ &+ \frac{\alpha b^2}{na^{n-1}} \quad - \frac{(n-1)\alpha b^3}{2n^2a^{2n-1}} \quad + \frac{(n-1)(2n-1)\alpha b^4}{6n^3a^{3n-1}} \\ &\qquad\qquad + \frac{\beta b^3}{na^{n-1}} \quad - \frac{(n-1)\beta b^4}{2n^2a^{2n-1}} \\ &\qquad\qquad\qquad\qquad + \frac{\gamma b^4}{na^{n-1}} \end{aligned}$$

Therefore, the following determinations are deduced

$$\begin{aligned} \alpha &= \frac{n-1}{2a} \\ \beta &= \frac{(n-1)\alpha}{2na^n} - \frac{(n-1)(2n-1)}{6na^{n+1}} = -\frac{(n-1)(n+1)}{12na^{n+1}} \\ \gamma &= \frac{(n-1)\beta}{2na^n} - \frac{(n-1)(2n-1)\alpha}{6nna^{2n}} + \frac{(n-1)(2n-1)(3n-1)}{24n^2a^{2n+1}} = \frac{(n-1)(n+1)}{24na^{2n+1}}. \end{aligned}$$

Therefore, the fraction to be added to the root a already found will be

$$\frac{b}{na^{n-1} + \frac{(n-1)b}{2a} - \frac{(nn-1)bb}{12na^{n+1}} + \frac{(nn-1)b^3}{24na^{2n+1}} - \text{etc.}}$$

§238 Therefore, if the square root of the number N is to be extracted and the approximate root was already found to be $= a$ together with the residue $= b$, to the found root one additionally has to add the quotient, which results, if the residue b is divided by

$$2a + \frac{b}{2a} - \frac{bb}{8a^3} + \frac{b^3}{16a^5} - \text{etc.}$$

But if the cube root must be extracted, then the residue must be divided by

$$3a^2 + \frac{b}{a} - \frac{2bb}{9a^4} + \frac{b^3}{9a^7} - \text{etc.},$$

the use of which formulas we will seen in these examples.

EXAMPLE 1

Extract the square root of the number 200.

Put $N = 200$ and because the closest square is 196, it will be $a = 14$ and the residue $b = 4$, which therefore must be divided by

$$28 + \frac{1}{7} - \frac{1}{7 \cdot 196} + \frac{1}{7 \cdot 196 \cdot 98},$$

and therefore the divisor will be $= 28.142135$; if 4 is divided by it, one will obtain a decimal fraction to be added to 14, which will be correct up to 10 figures and more.

EXAMPLE 2

To extract the cube root of the number $N = 10$.

The closest cube is 8 and the residue is $= 2$, whence $a = 2$ and $b = 2$ and the divisor $= 12 + 1 - \frac{1}{18} = 12.9444$. Therefore, the cube root in question will approximately be $= 2\frac{2}{12.9444} = 2\frac{10000}{64722}$.

§239 The series found for the root can also be considered as a recurring series resulting from a certain fraction. For, this way many terms of the series will be reduced to a lot less, namely those which constitute the numerator and the denominator of the fraction. So, having paid a little attention, one will see that it will approximately be

$$(a+b)^n = a^n \cdot \frac{a + \frac{n+1}{2}b}{a - \frac{n-1}{2}b}$$

and even closer

$$(a+b)^n = a^n \cdot \frac{aa + \frac{n+2}{2}ab + \frac{(n+1)(n+2)}{12}bb}{aa - \frac{n-2}{2}ab + \frac{(n-1)(n-2)}{12}bb}.$$

In like manner by introducing several terms even more accurate fractions can be obtained:

$$(a+b)^n = a^n \cdot \frac{a^3 + \frac{n+3}{2}a^2b + \frac{(n+3)(n+2)}{10}ab^2 + \frac{(n+3)(n+2)(n+1)}{120}b^3}{a^3 - \frac{n-3}{2}a^2b + \frac{(n-3)(n-2)}{10}ab^2 - \frac{(n-3)(n-2)(n-1)}{120}b^3}.$$

An even more general form of this kind be exhibited, to express which conveniently let

$$\begin{array}{ll}
A = \dfrac{m(n+m)}{1\cdot 2m} & \mathfrak{A} = \dfrac{m(n-m)}{1\cdot 2m} \\
B = \dfrac{(m-1)(n+m-1)}{2(2m-1)}A & \mathfrak{B} = \dfrac{(m-1)(n-m+1)}{2(2m-1)}\mathfrak{A} \\
C = \dfrac{(m-2)(n+m-2)}{3(2m-2)}B & \mathfrak{C} = \dfrac{(m-2)(n-m+2)}{3(2m-2)}\mathfrak{B} \\
D = \dfrac{(m-3)(n+m-3)}{4(2m-3)}C & \mathfrak{D} = \dfrac{(m-3)(n-m+3)}{4(2m-3)}\mathfrak{C} \\
\quad \text{etc.} & \quad \text{etc.}
\end{array}$$

But having determined these values it will be

$$(a+b)^n = a^n \cdot \frac{a^m + Aa^{m-1}b + \mathfrak{B}a^{m-2}b^2 + Ca^{m-3}b^3 + \text{etc.}}{a^m - \mathfrak{A}a^{m-1}b + \mathfrak{B}a^{m-2}b^2 - \mathfrak{C}a^{m-3}b^3 + \text{etc.}}$$

§240 Therefore, if a fractional number is substituted for n here, these formulas will be very useful to extract the roots. So if any root of power n of the expression $a^n + b$ has to be extracted, the following formulas can be used

$$(a^n + b)^{\frac{1}{n}} = a \cdot \frac{2na^n + (n+1)b}{2na^n + (n-1)}$$

$$(a^n+b)^{\frac{1}{n}} = a\cdot\frac{12n^2a^{2n}+6n(2n+1)a^nb+(2n+1)(n+1)bb}{12n^2a^{2n}+6n(2n-1)a^nb+(2n-1)(n-1)bb}.$$

But if one puts $a^n+b=N$ that it is $a^n=N-b$, it will be

$$(a^n+b)^{\frac{1}{n}} = a\cdot\frac{2nN-(n-1)b}{2nN-(n+1)b}$$

$$(a^n+b)^{\frac{1}{n}} a\cdot\frac{12n^2N^2-6n(2n-1)Nb+(2n-1)(n-1)bb}{12n^2N^2-6n(2n+1)Nb+(2n+1)(n+1)bb}.$$

§241 Therefore, the general formula for finding the root of any equation, which consists of several terms, has the same use as the usual rule of a binomial has for the resolution of the pure equations $x^n=c$, and therefore our formula goes over into that rule in this case. But if the equation was affected or even transcendental, our general expression is always applied with the same success and yields an infinite series exhibiting the value of the root. Therefore, since the resolution of such equations is its most important application, let us demonstrate its use a little more diligently. Therefore, let this affected equation consisting of three terms be propounded

$$x^n+cx=N$$

while c and N denote any given quantities. Put $x^n+cx-N=y$; it will be $dy=(nx^{n-1}+c)dx$ and hence it will be $p=\frac{1}{nx^{n-1}+c}$; then it is

$$dp=-\frac{n(n-1)x^{n-2}dx}{(nx^{n-1}+c)^2}\quad\text{and}\quad q=-\frac{n(n-1)x^{n-2}}{(nx^{n-1}+c)^3}.$$

In like manner because of $r=\frac{dq}{dy}$, $s=\frac{dr}{dy}$ etc. one will find

$$r=\frac{n^2(n-1)(2n-1)x^{2n-4}-n(n-1)(n-2)cx^{n-3}}{(nx^{n-1}+c)^5}$$

$$s=\frac{-n^3(n-1)(2n-1)(3n-1)x^{3n-6}+4n^2(n-1)(n-2)(2n-1)cx^{2n-5}-n(n-1)(n-2)(n-3)c^2x^{n-4}}{(nx^{n-1}+c)^7}$$

$$t=\frac{\begin{Bmatrix}n^4(n-1)(2n-1)(3n-1)(4n-1)x^{4n-8}-n^3(n-1)(n-2)(2n-1)(29n-11)cx^{3n-7}\\+n^2(n-1)(n-2)(2n-1)(11n-29)c^2x^{2n-6}-n(n-1)(n-2)(n-3)(n-4)c^3x^{n-5}\end{Bmatrix}}{(nx^{n-1}+c)^9}$$

etc.

Having found these values the root of the propounded equation will be

$$f = x - py + \frac{1}{2}qyy - \frac{1}{6}ry^3 + \frac{1}{24}sy^4 - \frac{1}{120}ty^5 + \text{etc.};$$

for, whatever is substituted for x, whence at the same time the letters y, p, q, r etc. obtain determined values, the sum of the series will become equal to the value of one single root.

EXAMPLE 1

Let this equation be propounded $x^3 + 2x = 2$.

It will be $c = 2$, $N = 2$ and $n = 3$ and $y = x^3 + 2x - 2$. Put $x = 1$; it will be $y = 1$ and

$$p = \frac{1}{5}, \quad q = -\frac{6}{5^3}, \quad r = \frac{78}{5^5}, \quad s = -\frac{16 \cdot 90}{5^7} \quad \text{etc.}$$

and the root of the equation will be

$$f = -\frac{1}{5} - \frac{3}{5^3} - \frac{13}{5^5} - \frac{60}{5^7} - \text{etc.} = 0.771072.$$

Now put $x = 0.77$, and since it is $y = x^3 + 2x - 2$,

$$p = \frac{1}{3xx + 2}, \quad q = -6p^3x, \quad r = 90xxp^5 - 12p^5$$

and

$$s = -2160p^7x^3 + 720p^7x,$$

by using logarithms one will have

$$\begin{array}{rcl|rcl}
\log x^1 & = & 9.8864907 & x^1 & = & 0.77 \\
\log x^2 & = & 9.7729814 & x^2 & = & 0.5929 \\
\log x^3 & = & 9.6594721 & x^3 & = & 0.456533 \\
 & & & 2x & = & 1.54 \\ \cline{6-6}
 & & & x^3 + 2x & = & 1.996533 \\
\text{Therefore} & & & y & = & -0.003467
\end{array}$$

And furthermore

$$\begin{array}{rcl|rcl}
\log(-y) & = & 7.5399538 & 3xx+2 & = & 3.7787 \\
\log p & = & 9.4226575 & \log(3xx+2) & = & 0.5773424 \\
\log(-py) & = & 6.9626113 & -py & = & 0.000917511 \\
\hline
\log p^3 & = & 8.2679725 & & & \\
\log x & = & 9.8864907 & & & \\
\log 3 & = & 0.4771213 & & & \\
\log y^2 & = & 5.0799076 & & & \\
\hline
\log\left(-\frac{1}{2}qyy\right) & = & 3.7114921 & -\frac{1}{2}qyy & = & 0.000000514
\end{array}$$

Therefore, the root is $f = 0.770916997$, which hardly deviates from the true value, just the last figure is not correct.

Example 2

Let the equation $x^4 - 2xx + 4x = 8$ *be propounded.*

Put $y = x^4 - 2xx + 4x - 8$; it will be $dy = 4dx(x^3 - x + 1)$,

$$p = \frac{1}{4(x^3 - x + 1)}, \quad \frac{dp}{dx} = \frac{-3xx + 1}{4(x^3 - x + 1)^2}.$$

Therefore,

$$q = \frac{-3xx+1}{16(x^3-x+1)^3}, \quad \frac{dq}{dx} = \frac{21x^4 - 12xx - 6x + 3}{16(x^3-x+1)^4} \quad \text{and} \quad r = \frac{21x^4 - 12xx - 6x + 3}{64(x^3-x+1)^5} \quad \text{etc.,}$$

from which the root of the propounded equation will be

$$f = x - \frac{y}{4(x^3-x+1)} - \frac{(3xx-1)yy}{32(x^3-x+1)^3} - \frac{(7x^4 - 4xx - 2x + 1)y^3}{128(x^3-x+1)^5} - \text{etc.}$$

Therefore, it is necessary to attribute an appropriate value to x, so that series becomes convergent. But at first it is perspicuous, if a value rendering $x^3 - x + 1 = 0$ would be attributed to x, that then all terms of there series except for the first would become infinite and nothing can be concluded from this.

Therefore, it is convenient to assign such a value to x that both y becomes small and $x^3 - x + 1$ not very large. Let $x = 1$; it will be $y = -5$ and

$$f = 1 + \frac{5}{4} - \frac{25}{16} + \frac{125}{64} - \text{etc.};$$

because the three terms $\frac{5}{4} - \frac{25}{16} + \frac{125}{64}$ agree with the terms of the geometric progression, whose sum is $\frac{5}{9}$, it will approximately be $f = \frac{14}{9}$. Therefore, let us put $x = \frac{3}{2}$; it will be

$$y = -\frac{23}{16} \quad \text{and} \quad x^3 - x + 1 = \frac{23}{8},$$

whence it is

$$f = \frac{3}{2} + \frac{1}{8} - \frac{1}{64} + \frac{391}{256 \cdot 529} - \text{etc.} = 1.61.$$

Now put $x = 1.61$; it will be

$\log x$	$=$	0.2068259		x	$=$	1.61	let	$x^3 - x + 1 = z$
$\log x^2$	$=$	0.4136518		x^2	$=$	2.5921		
$\log x^3$	$=$	0.6204777		x^3	$=$	4.173281		
$\log x^4$	$=$	0.8273036		x^4	$=$	6.718983		
			hence					
$\log(-y)$	$=$	8.4016934		y	$=$	$-$ 0.025217		
$\log z$	$=$	0.5518502		z	$=$	3.563281		
$\log \frac{-y}{z}$	$=$	7.8498432						
$\log 4$	$=$	0.6020600						
$\log \frac{-y}{4z}$	$=$	7.2477832		$\frac{-y}{4z}$	$=$	0.0017692		
$\log(3xx - 1)$	$=$	0.8309926		$3xx - 1$	$=$	6.7763		
$\log y^2$	$=$	6.8033868						
		7.6343794						
$\log z^3$	$=$	1.6555506						
		5.9788288						
$\log 32$	$=$	1.5051500		$\frac{(3xx-1)^2 y^2}{32z^3}$	$=$	0.00002976		
	$=$	4.4736788						

Therefore $f = 1.6117662$.

§242 This method to find the roots of equations approximately extends to transcendental quantities in like manner. Let us find the number x, whose logarithm has a given ratio of 1 to n to the number x itself, and one will have this equation $x - n\log x = 0$; but let k be the modulus of these logarithms, such that these logarithms are obtained, if the hyperbolic logarithms are multiplied by k; it will be $d.\ln x = \frac{kdx}{x}$. Therefore, put $x - n\log x = y$ and let f be the value of f in question which renders $x = n\log x$. Therefore, because it is $y = x - n\log x$, it will be

$$dy = dx - \frac{kndx}{x} = \frac{dx(x-kn)}{x}$$

and

$$\frac{dx}{dy} = p = \frac{x}{x-kn}, \qquad \text{whence} \quad dp = -\frac{kndx}{(x-kn)^2},$$

therefore

$$\frac{dp}{dy} = q = \frac{-knx}{(x-kn)^3}, \qquad dq = \frac{2knxdx + k^2n^2dx}{(x-kn)^4}$$

$$\frac{dq}{dy} = r = \frac{knx(2x+kn)}{(x-kn)^5}.$$

Therefore, it will be

$$f = x - \frac{xy}{x-kn} - \frac{knxyy}{2(x-kn^3)} - \frac{knxy^3(2x+kn)}{6(x-kn)^5} - \text{etc.}$$

Below [§ 272] we will show that this problem only admits a solution, if $kn > e$ while e is the number whose hyperbolic logarithm is $= 1$, or it must be $kn > 2.7182818$.

Example

A number except for 10 *is to be found, whose tabulated logarithm becomes equal to the tenth part of the number itself.*

Since the question is about tabulated logarithms, it will be $k = 0.43429448190325$ and because of $n = 10$ one will have $kn = 4.3429448190325$. Now, having put $x = 1$ it will be $y = 1$ and it will be

$$f = 1 + \frac{1}{3.3429} + \frac{2.1714724}{(3.3429)^3} - \text{etc.}$$

and so it will approximately be $f = 1.37$. Therefore, set $x = 1.37$; it will be $\log x = 0.136720567156406$ and because of $y = x - 10\log x$ it will be

$$y = 0.00279432843594 \quad \text{and} \quad -x + kn = 2.9729448190325.$$

Therefore, let

$$\begin{array}{lcl} \log x & = & 0.1367205 \\ \log y & = & \underline{7.4462773} \\ & & 7.5829978 \\ \log(kn - x) & = & \underline{0.4731866} \\ & & 7.1098112 \qquad \dfrac{-xy}{x - kn} = 0.00128769 \end{array}$$

Further, because the third term is $-\frac{knxyy}{2(x-kn)^2} = \frac{kny}{2(x-kn)^2} \cdot \frac{-xy}{x-kn}$, it will be

$$\begin{array}{rcl} \log \dfrac{-xy}{x - kn} & = & 7.1098112 \\ \log y & = & 7.4462773 \\ \log kn & = & 0.6377842 \\ \hline & & 5.1938727 \\ \log(kn - x)^2 & = & \underline{0.9463732} \\ & & 4.2474995 \\ \log 2 & = & \underline{0.3010300} \\ \text{log third term} & = & 3.9464695 \\ \text{I. term } x & = & 1.37 \\ \text{II. term} & = & 0.00128769 \\ \text{III. term} & = & \underline{0.00000088} \\ f & = & 1.37128857 \\ \log f & = & 0.137128857 \end{array}$$

§243 If the equation was an exponential equation, it can be reduced to any logarithmic one; so, if the value of x is in question, that it is $x^x = a$, it will be $x \ln x = \ln a$. Therefore, having put $y = x \ln x - \ln a$ it will be

$$dy = dx \ln x + dx \quad \text{and} \quad \frac{dx}{dy} = p = \frac{1}{1 + \ln x}$$

and then

$$dp = \frac{-dx}{x(1+\ln x)^2} \quad \text{and} \quad \frac{dp}{dy} = q = \frac{-1}{x(1+\ln x)^3},$$

$$dq = \frac{dx}{xx(1+\ln x)^3} + \frac{3dx}{xx(1+\ln x)^4} \quad \text{and hence} \quad \frac{dq}{dy} = r = \frac{1}{xx(1+\ln x)^4} + \frac{3}{xx(1+\ln x)^5};$$

further, it will be

$$dr = \frac{-2dx}{x^3(1+\ln x)^4} - \frac{10dx}{x^3(1+\ln x)^5} - \frac{15dx}{x^3(1+\ln x)^6},$$

therefore

$$s = \frac{-2}{x^3(1+\ln x)^5} - \frac{10}{x^3(1+\ln x)^6} - \frac{15}{x^3(1+\ln x)^7}$$

and

$$t = \frac{6}{x^4(1+\ln x)^6} + \frac{40}{x^4(1+\ln x)^7} + \frac{105}{x^4(1+\ln x)^8} + \frac{105}{x^4(1+\ln x)^9},$$

$$u = \frac{-24}{x^5(1+\ln x)^7} - \frac{196}{x^5(1+\ln x)^6} - \frac{700}{x^5(1+\ln x)^9} - \frac{1260}{x^5(1+\ln x)^{10}} - \frac{945}{x^5(1+\ln x)^{11}}.$$

Therefore, if the true value of x is $= f$, such that $f^f = a$, it will be

$$\begin{aligned} f = x - \frac{y}{1+\ln x} - \frac{yy}{2x(1+\ln x)^3} &- \frac{y^3}{2xx(1+\ln x)^5} - \frac{5y^4}{8x^3(1+\ln x)^7} - \frac{7y^5}{8x^4(1+\ln x)^9} \\ &- \frac{y^3}{6x^2(1+\ln x)^4} - \frac{5y^4}{12x^3(1+\ln x)^6} - \frac{7y^5}{8x^4(1+\ln x)^8} \\ &\qquad\qquad - \frac{y^4}{12x^3(1+\ln x)^5} - \frac{y^5}{3x^4(1+\ln x)^7} \\ &\qquad\qquad\qquad\qquad - \frac{y^5}{20x^4(1+\ln x)^6} \end{aligned}$$

etc.

Therefore, this expression having continued it to infinity, whatever value is substituted for x, and having taken $y = x \ln x - \ln a$ will give the true value of f. So, if one puts $x = 1$, it will be $y = -\ln a$ and

$$f = 1 + \ln a - \frac{(\ln a)^2}{2} + \frac{2(\ln a)^3}{3} - \frac{9(\ln a)^4}{8} + \frac{32(\ln a)^5}{15} - \frac{625(\ln a)^6}{144} - \text{etc.},$$

where it is to be noted that $\ln a$ is the hyperbolic logarithm of a.

EXAMPLE

Find the number f that it is $f^f = 100$.

Because it is

$$a = 100 \quad \text{and} \quad y = x \ln x - \ln a = x \ln x - \ln 100,$$

since it is clear that it is $f > 3$ and < 4, put $x = \frac{7}{2}$ and it will be

$$\begin{array}{rcl} \log x & = & 1.25276296849 \\ x \log x & = & 4.38467038972 \\ \log 100 & = & \underline{4.60517018599} \\ y & = & -0.22049979627 \\ 1 + \log x & = & 2.25276296849. \end{array}$$

Therefore, by using ordinary logarithms it will be

$$\begin{array}{lcl} \log(-y) & = & 9.3434083 \\ \log(1 + \log x) & = & \underline{0.3527156} \\ & & 8.9906927 \\ \log y^2 & = & 8.6868166 \\ 3 \log(1 + \log x) & = & \underline{1.0581468} \\ & & 7.6286698 \\ \log 2x = \log 7 & = & \underline{0.8450980} \\ & & 6.7835718 \end{array} \qquad \begin{array}{l} \dfrac{-y}{1 + \log x} = 0.0978797 \\ \\ \\ \\ \dfrac{y^2}{2x(1 + \log x)^3} = 0.0006075. \end{array}$$

Therefore, it will approximately be $f = 3.5972722;$

but having additionally taken the following terms, it will be $f = 3.5972852$.

§244 But moreover differential calculus has an extraordinary use in the resolution of equations, if a certain relation among the roots was known. Let the equation $y = 0$ be propounded, in which y is an arbitrary function of x. If now, for the sake of an example, it is known that the roots of this equation differ by the given quantity a, these two roots will easily be found the following way. Let x denote the smaller of these two roots; the larger will be $= x + a$; therefore, because the function y vanishes, if x denotes any of the roots of the equation $y = 0$, it will also vanish, if one puts $x + a$ instead of x. Therefore, it will be

$$0 = y + \frac{ady}{dx} + \frac{a^2 ddy}{2dx^2} + \frac{a^3 d^3 y}{6dx^3} + \text{etc.}$$

Therefore, because it is $y = 0$, it will also be

$$0 = \frac{ady}{dx} + \frac{a^2 ddy}{2dx^2} + \frac{a^3 d^3 y}{6dx^3} + \text{etc.}$$

which two equations taken at the same time, using the method of elimination, will give the value of the root x, which is smaller than the other root by the quantity a.

Example

Let this equation $x^5 - 24x^4 + 49xx - 36 = 0$ *be propounded, which is known from anywhere to have two roots differing* 1.

Having put $y = x^5 - 24x^3 + 49xx - 36$ it will be

$$\frac{dy}{dx} = 5x^4 \quad -72x^2 + 98x$$
$$\frac{ddy}{2dx^2} = 10x^3 - 72x \quad +49$$
$$\frac{d^3y}{6dx^3} = 10x^2 - 24$$
$$\frac{d^4y}{24dx^4} = 5x$$
$$\frac{d^5y}{120dx^5} = 1.$$

But because of $a = 1$ it will be

$$A \cdots 5x^4 + 10x^3 - 62x^2 + 31x + 26 = 0.$$

But it is

$$B \cdots x^5 - 24x^3 + 49xx - 36 = 0.$$

Multiply the upper equation by x and the lower equation by 5 and subtract the one from the other and it will remain

$$10x^4 + 58x^3 - 214x^2 + 26x + 180 = 0$$

or

$$C \cdots 5x^4 + 229x^3 - 107x^2 + 13x + 90 = 0,$$

having subtracted the first A from which it will be

$$\begin{array}{rrcrcrcrcrcrcr}
D\cdots & 19x^3 & - & 45x^2 & - & 18x & + & 64 & = & 0. & & & \\
D\cdot 5x\cdots & 95x^4 & - & 225x^3 & - & 90x^2 & + & 320x & = & 0. & & & \\
A\cdot 19\cdots & 95x^4 & + & 190x^3 & - & 1178x^2 & + & 589x & + & 494 & = & 0. \\
\hline
E\cdots & & & 415x^3 & - & 1088x^2 & + & 269x & + & 494 & = & 0. \\
D\cdot 415\cdots & & & 7885x^3 & - & 18675x^2 & - & 7470x & + & 26560 & = & 0. \\
E\cdot 19\cdots & & & 7885x^3 & - & 20672x^2 & + & 5111x & + & 8386 & = & 0. \\
\hline
F\cdots & & & & & 1997x^2 & - & 12581x & + & 17174 & = & 0.
\end{array}$$

And

$$\begin{array}{rrcrcrcrcr}
D \cdot 247 \cdots & 4693x^3 & - & 11115x^2 & - & 4446x & + & 15808 & = & 0 \\
D \cdot 32 \cdots & 13280x^3 & - & 34816x^2 & + & 8608x & + & 15808 & = & 0 \\
\hline
& 8587x^3 & - & 23701x^2 & + & 13054x & = & 0 & & \\
\hline
G \cdots & & & 8587x^2 & - & 23701x & + & 13054 & = & 0 \\
F \cdot 8587 \cdots & & & 17148239x^2 & - & 108033047x & + & 147473138 & = & 0 \\
G \cdot 1997 \cdots & & & 17148239x^2 & - & 47330897x & + & 26068838 & = & 0 \\
\hline
& & & & & 60702150x & - & 121404300 & = & 0.
\end{array}$$

From this equation it follows that $x = 2$ and therefore also $x = 3$ will be a root of the equation, both of which values indeed satisfy the equation.

§245 But this operation can be done without use of differential calculus, because the same equation the differential calculus yielded, results, if in the propounded equation one puts $x + 1$ instead of x. Furthermore, this method of elimination is too laborious, and if the equations would be of higher degree, the labor would be simply too much to handle; and this holds even more for transcendental equations. But if we put that two roots of the propounded equation $y = 0$ are equal to each other, then because of $x = a$ the differential equation goes over into this one $\frac{dy}{dx} = 0$. Therefore, if any equation $y = 0$ had two equal roots, it will be $\frac{dy}{dx} = 0$ and these two roots taken together will yield the value of x, to which these two roots are equal. Therefore, vice versa, if the two equations $y = 0$ and $\frac{dy}{dx} = 0$ have a common root, it will be a double root of the equation $y = 0$. But this happens, if, after the quantity x was completely eliminated by means of these two equations $y = 0$ and $\frac{dy}{dx} = 0$, one gets to an identical equation. So if the equation

$$x^3 - 2xx - 4x + 8 \tag{1}$$

was propounded, it will also be $3xx - 4x - 4 = 0$, whose double added to the first gives

$$x^3 + 4xx - 12x = 0 \quad \text{or} \quad xx + 4x - 12 = 0,$$

whose triple is

$$\begin{array}{rrrrrrrrr}
 & 3xx & + & 12x & - & 36 & = & 0 \\
\text{subtract} & 3xx & - & 4x & - & 4 & = & 0 \\
\hline
 & & & 16x & - & 32 & = & 0 \\
 & & & x & - & 2 & = & 0
\end{array}$$

Therefore, because $x = 2$ results, substitute this value in one of the preceding $3xx - 4 - 4 = 0$ and the identical equation $12 - 8 - 4 = 0$ will result, whence one concludes that the propounded equation $x^3 - 2xx - 4x + 8 = 0$ has to equal roots, namely $x = 2$.

§246 Therefore, if one has an algebraic equation of no matter how many dimensions

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + \text{etc.} = 0,$$

which has two equal roots, it will also be

$$nx^{n-1} + (n-1)Ax^{n-2} + (n-2)Bx^{n-2} + (n-3)Cx^{n-4} + (n-4)Dx^{n-5} + \text{etc.} = 0.$$

This double root of that equation will at the same time be a root of this last equation, of course. Multiply the first equation by n and subtract the second multiplied by x from it and this new equation will result

$$Ax^{n-1} + 2Bx^{n-2} + 3Cx^{n-3} + 4Dx^{n-4} + \text{etc.} = 0.$$

Now add the first multiplied by a and the latter multiplied by b; it will be

$$ax^n + (a+b)Ax^{n-1} + (a+2b)Bx^{n-2} + (a+3b)Cx^{n-3} + \text{etc.} = 0,$$

which equation combined with the propounded itself will show equal roots, if the propounded one has some. Therefore, because the quantities a and b can be taken arbitrarily, the coefficients a, $a+b$, $a+2b$ etc. represent any arithmetic progression. Therefore, if any equation has two equal roots, they will be found, if the single terms of the propounded equation are multiplied by terms of a certain arithmetic progression, respectively; for, the new equation resulting this way will also contain the root, which is contained twice in the propounded one. So, if the the terms of the equation

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + \text{etc.}$$

are multiplied by this arithmetic progression

$$a, \quad a+b, \quad a+2b, \quad a+3b, \quad a+4b \quad \text{etc.};$$

this new equation will result

$$ax^n + (a+b)Ax^{n-1} + (a+2b)Bx^{n-2} + (a+3b)x^{n-2} + (a+3b)Cx^{n-3} + \text{etc.} = 0,$$

which combined with the latter will show the equal roots. And this is the well-known rule to find equal roots of any equation.

§247 If the equation $y = 0$ has three equal roots, it will not only be $\frac{dy}{dx} = 0$, but it will also be $\frac{ddy}{dx^2} = 0$, if one substitutes the value of the root for x, which is contained in the equation $y = 0$ trice. To show this let us put that the equation $y = 0$ has three roots x, $x + a$, $x + b$ etc., of which the first differs from the other ones by a and b, respectively; and since y vanishes, if one writes $x + a$ or $x + b$ instead of x, it will be

$$\begin{aligned}
&y && = 0\\
&y + \frac{ady}{dx} + \frac{a^2ddy}{2dx^2} + \frac{a^3d^3y}{6dx^3} + \frac{a^4d^4y}{24dx^4} + \text{etc.} && = 0\\
&y + \frac{bdy}{dx} + \frac{b^2ddy}{2dx^2} + \frac{b^3d^3y}{6dx^3} + \frac{b^4d^4y}{24dx^4} + \text{etc.} && = 0;
\end{aligned}$$

if the first is subtracted from the two last ones, it will be

$$\begin{aligned}
&\frac{dy}{dx} + \frac{addy}{2dx^2} + \frac{a^2d^3y}{6dx^3} + \frac{a^3d^4y}{24dx^4} + \text{etc.} = 0\\
&\frac{dy}{dx} + \frac{bddy}{2dx^2} + \frac{b^2d^3y}{6dx^3} + \frac{b^3d^4y}{24dx^4} + \text{etc.} = 0
\end{aligned}$$

Also subtract these from each other and having divided by $a - b$ it will be

$$\frac{ddy}{2dx^2} + \frac{(a+b)d^3y}{6dx^3} + \frac{(aa+ab+bb)d^4y}{24dx^4} + \text{etc.} = 0.$$

Now put $a = 0$ and $b = 0$ such that these three roots are equal to each other, and because of the vanishing terms it will be

$$y = 0, \quad \frac{dy}{dx} = 0 \quad \text{and} \quad \frac{ddy}{dx^2} = 0.$$

§248 Therefore, if the equation $y = 0$ has three equal roots, say f, f, f, then this quantity f will also be a root not only of this equation $\frac{dy}{dx} = 0$, but also of this one $\frac{ddy}{dx^2} = 0$. Therefore, recalling the results we demonstrated before on two equal roots of equations, it is obvious, because f is the common root of the equation $\frac{dy}{dx} = 0$ and its differential $\frac{ddy}{dx^2} = 0$, that it has to be contained twice in the equation $\frac{dy}{dx} = 0$. Therefore, if the equation

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + \text{etc.} = 0$$

contains three equal roots f, f, f, if its terms are multiplied by the terms of a certain arithmetic progression, then the resulting equation will have two equal roots f and f; therefore, it can be multiplied by an arithmetic progression again so that an equation containing the root f once results. Therefore, one will obtain three equations having the common root f, from whose combination this root will easily be found. For, if arithmetic progressions of such a kind are chosen, whose either first or last terms are $= 0$, then an equation of one degree lower will result and so the elimination will be even easier.

§249 In like manner it will be shown, if the equation $y = 0$ has four equal roots f, f, f, f, that than for $x = f$ it will not only be $y = 0$, $\frac{dy}{dx} = 0$ and $\frac{ddy}{dx^2} = 0$, but it will also be $\frac{d^3y}{dx^3} = 0$. As the equation $y = 0$ contains the root $x = f$ four times, so the equation $\frac{dy}{dx}$ will contain the same trice, the equation $\frac{ddy}{dx^2} = 0$ twice and the equation $\frac{d^3y}{dx^3} = 0$ once. This will also be seen more easily, if we consider that the function y has to have a form of this kind $(x - f)^4 X$ in this case, where X denotes any function of x. Having assumed this form it will be

$$\frac{dy}{dx} = (x - f)^3 \left(4X + \frac{(x - f)dX}{dx} \right)$$

and hence be divisible by $(x - f)^3$. Further, $\frac{ddy}{dx^2}$ will have the factor $(x - f)^2$ and $\frac{d^3y}{dx^3}$ the factor $x - f$; from this it is perspicuous, if the root f is contained in the equation $y = 0$, it has to be contained in the equation $\frac{dy}{dx} = 0$ trice, in the equation $\frac{ddy}{dx^2} = 0$ twice and in $\frac{d^3y}{dx^3} = 0$ still once.