Observations on the Comparison of Arcs of irrectifiable curves *

Leonhard Euler

§0 Mathematical speculations, concerning their utility, seem to have to be reduced to two classes; those, which provide extraordinary advantages both for common life and other arts, are to be referred to the first class, whose value is therefore estimated from the magnitude of this advantage. But the other class contains those speculations, which, even though they do not lead to such advantages, are nevertheless of such a nature, that they provide an occasion to promote the science of Analysis and to sharpen the power of our mind. For, since we are forced to put many speculations, whence highest utility could be expected, aside one because of the missing Analysis, nevertheless a certain value seems to be ascribed to them, which promises huge progress in the science of Analysis. But for this purpose especially those speculations seem to be accommodated, which are quasi made accidentally and were detected a posteriori and hence the way to get to them a priori and in a direct way is less obvious or not understood at all. For, so, having already found the truth, it will be easier to investigate methods, which will lead to them directly; and there is no doubt that the investigation of these new methods lead to huge progress in Analysis.
But I detected several observations of this kind, which were not made by a certain method and whose nature seems to be rather obscure, in the work of Fagnano published recently; therefore, they are worth one's complete attention

*Original Title: „Observationes De Comparatione Arcuum Curvarum irrectificabilium", first published in „*Novo Commentarii academiae scientiarum Petropolitinae* 6, 1761, pp. 58-84", reprinted in „*Opera Omnia*: Series 1, Volume 20, pp. 80 - 107", Eneström-Number E252, translated by: Alexander Aycock, Figures by: Artur Diener for the project „Euler-Kreis Mainz" and the eagerness, invested in their further investigation, well certainly be well-spent. In this work certain extraordinary properties are given, which the *Ellipse*, *Hyperbola* and the *Lemniscate* enjoy, and different arcs of these curves are compared to each other; therefore, since the reason for these properties seems very mysterious, I think it will be appropriate, if I examine them more diligently, and communicate everything what I could discover in addition to these results.

Therefore, concerning these curves, it is known that their rectification transcends all power of Analysis so that their arcs can not only not expressed algebraically, but they can not even reduced to the quadrature of the circle or hyperbola. Therefore, Fagnano's findings seem even more remarkable; for, he found that in the case of the ellipse and the hyperbola one can exhibited two arcs of such a kind, whose difference can be assigned algebraically, in infinitely many ways, but on the lemniscate there are infinitely pairs of two arcs of such a kind, which are either equal or the one has a ratio of two to one to the other, whence he then derived a way to also assign an arc of such a kind on this curves, which has an arbitrary ratio to another one.

But in the case of the ellipse and the hyperbola it has not been possible to find new additional results; therefore, I will be content to have given an easier construction of the arcs, whose difference can be exhibited geometrically. But for the lemniscate, following the same ideas, I found many more, even infinitely many, formulas, by means of which I am not only able to define two arcs of such a kind infinitely many ways, which are either equal or have a ratio of two to one, but also arcs of such a kind, which have an arbitrary ratio to each other.

I. On the Ellipse

§1 Let the elliptic quadrant be ABC (Fig. 1). whose center is in C, and put its semiaxes $CA = 1$ and $CB = c$; therefore, having taken an arbitrary abscissa $CP = x$, the ordinate corresponding to it will be $PM = y = c\sqrt{1-xx}$; since its differential is $dy = -\frac{cxdx}{\sqrt{1-xx}}$, the elliptic arc corresponding to the abscissa $CP = x$ will be

$$BM = \int \frac{\mathrm{d}x\sqrt{1-(1-cc)xx}}{\sqrt{1-xx}}.$$

For the sake of brevity put $1 - cc = n$ that the arc is

$$BM = \int \mathrm{d}x\sqrt{\frac{1-nxx}{1-xx}},$$

and, having taken another abscissa $CQ = u$, in like manner the arc corresponding to it will be

$$BN = \int du \sqrt{\frac{1 - nuu}{1 - uu}}.$$

Having constituted these, it is in question, how these two abscissas x and u must be related to each other that the sum of their arcs

$$BM + BN = \int dx \sqrt{\frac{1 - nxx}{1 - xx}} + \int du \sqrt{\frac{1 - nuu}{1 - uu}}$$

becomes integrable or can be exhibited geometrically.

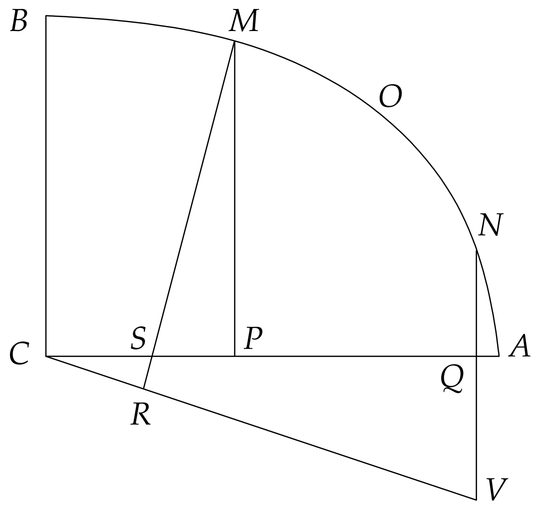

FIG. 1

§2 Therefore, the question reduces to this that it is determined, a function of x of which kind must be substituted for u that the differential formula

$$dx \sqrt{\frac{1 - nxx}{1 - xx}} + du \sqrt{\frac{1 - nuu}{1 - uu}}$$

admits an integration. But it is easily seen, if this question is considered in general, that its solution is based on the integration of each of the two formulas and hence transcends the limits of Analysis in the same way as the rectification of the ellipse. Therefore, since a general solution can not be expected at all, one will have to try to find particular solutions, not necessarily by a certain method but a least by trial and error; but hence their true origin, even though the solution has been found, can hardly be understood.

§3 First, the case $u = -x$ is immediately obvious, in which our differential formula becomes zero; but since hence two absolutely identical elliptic arcs result, as this case is too obvious, so also the question is not solved by this in a satisfactory manner. Therefore, since the problem must be solved by guessing and trying several solutions, assume

$$\sqrt{\frac{1-nxx}{1-xx}} = \alpha u$$

and assume α in such a way that vice versa

$$\sqrt{\frac{1-nuu}{1-uu}} = \alpha x;$$

for, this way one will have

$$BM + BN = \alpha \int u \mathrm{d}x + \alpha \int x \mathrm{d}u = \alpha x u + \text{Const.},$$

as it is required. But for the value of α we will have so

$$1 - nxx - \alpha\alpha uu + \alpha\alpha uuxx = 0 \quad \text{as} \quad 1 - nuu - \alpha\alpha xx + \alpha\alpha xxuu = 0;$$

whence it is plain that one has to put $\alpha\alpha = n$ and $\alpha = \sqrt{n}$ so that

$$u = \sqrt{\frac{1-nxx}{n-nxx}} \quad \text{and} \quad BM + BN = xu\sqrt{n} + \text{Const.}$$

§4 But even if this way the question seems to be solved, nevertheless these determinations can not hold in a ellipse. For, since $n < 1$, since $n = 1 - cc$, it will be $n - nxx < 1 - nxx$ and hence $u > 1$; therefore, the abscissa CQ exceeds the semiaxis CA and hence an imaginary arc would correspond to it so that hence no correct conclusion could be deduced.

§5 Therefore, let us try other formulas and let so

$$\sqrt{\frac{1-nxx}{1-xx}} = \frac{\alpha}{u} \quad \text{as} \quad \sqrt{\frac{1-nuu}{1-uu}} = \frac{\alpha}{x},$$

whence, because of

$$\alpha\alpha - \alpha\alpha xx - uu + nxxuu = 0 \quad \text{and} \quad \alpha\alpha - \alpha\alpha uu - xx + nxxuu = 0$$

we conclude $\alpha = 1$ so that

$$1 - uu - xx + nxxuu = 0 \quad \text{and hence} \quad u = \sqrt{\frac{1 - xx}{1 - nxx}}.$$

But hence it result

$$BM + BN = \int \frac{\mathrm{d}x}{u} + \int \frac{\mathrm{d}u}{x} = \int \frac{x\mathrm{d}x + u\mathrm{d}u}{xu}.$$

But the equation $uu + xx = 1 + nxxuu$, having differentiated it, gives

$$x\mathrm{d}x + u\mathrm{d}u = nxu(x\mathrm{d}u + u\mathrm{d}x) \quad \text{or} \quad \frac{x\mathrm{d}x + u\mathrm{d}u}{xu} = n(x\mathrm{d}u + u\mathrm{d}x),$$

whence we conclude

$$BM + BN = n\int (x\mathrm{d}u + u\mathrm{d}x) = nxu + \text{Const.}$$

§6 This solution is not obstructed by any inconvenience; for, since $n < 1$, it will be $1 - nxx > 1 - xx$ and hence $u < 1$, as the nature of the subject demands it. Therefore, having taken an arbitrary abscissa $CP = x$, take the other

$$CQ = u = \sqrt{\frac{1 - xx}{1 - nxx}}$$

and the sum of the arcs will be $BM + BN = nxu + \text{Const.}$ To define this constant, let $x = 0$ that $BM = 0$; and it will be $u = 1$ and the arc BN goes over into the quadrant $BMNA$; hence $0 + BMNA = 0 + \text{Const.}$ and so this constant will be $= BMNA$. Having substituted this value, we have

$$BM + BN = nxu + BMNA$$

and hence

$$BM - AN = nxu = (1 - cc)xu = BN - AM.$$

§7 Therefore, given an arbitrary point M on the elliptic quadrant ACB, we can assign another point N so that the difference of the arcs $BM - AN$, or which is equal to $BN - AM$, can be expressed geometrically. To achieve this more easily, let us drop the perpendicular MS to the ellipse in the point M; the subnormal will be $PS = ccx$ and, because of $PM = c\sqrt{1 - xx}$, the normal itself will be

$$MS = c\sqrt{1 - xx + ccxx} = c\sqrt{1 - nxx}$$

and hence for the other point N the abscissa will be $CQ = u = \frac{PM}{MS}CA$. Or, drop the perpendicular to the elongated normal MS from C and elongate it to V that $CV = CA = 1$, and because of $\frac{CR}{CS} = \frac{PM}{MS}$ it will be $CQ = \frac{CR}{CS}CV$. Hence drop the perpendicular VQ to the axis CA from the point V, and it will denote the point Q and, if elongated, even the point N.

§8 Since $PS = ccx$, it will be $CS = x - ccx = nx$ and hence

$$CR = \frac{CQ \cdot CS}{CV} = \frac{u \cdot nx}{1} = nux.$$

Therefore, this perpendicular CR will exhibit the difference of the arcs $BM - AN$ or $BN - AM$. Hence this way the difference of the designated arcs will be $= nx\sqrt{\frac{1-xx}{1-nxx}}$, which therefore vanishes so in the case $x = 0$ as $x = 1$, in which the points M and N fall on the points B and A. But this difference becomes maximal, if $nx^4 - 2xx + 1 = 0$, this means, if $x = \frac{1}{\sqrt{1+c}}$, in which case it is $x = u$ and both points M and N merge into one point O; and in this case the difference of the arcs will be $BO - AO = nxx = 1 - c$ and hence it will become equal to the difference of the semiaxes $CA - CB$ so that $CA + AO = CB + BO$.

§9 If the point M is taken in this point O that

$$CP = x = \frac{1}{\sqrt{1+c}},$$

it will be

$$PM = \frac{c\sqrt{c}}{\sqrt{1+c}} \quad \text{and} \quad PS = \frac{cc}{\sqrt{1+c}}$$

and hence $MS = c\sqrt{c}$, whence the location of the point O can be defined conveniently in various ways. But since

$$CM = CO = \frac{\sqrt{1+c^3}}{\sqrt{1+c}} = \sqrt{1-c+cc} = \sqrt{1+cc-2c\cos 60^\circ},$$

whence a simple construction is deduced, it seems advisable to add the following theorems, whose proof is clear from the preceding paragraphs.

THEOREM 1

§10 *If in the elliptic quadrant ACB (Fig. 2) one draws the tangent HKM in the point M, which meats the other axis CB in H, and it is taken equal to the other semiaxis that $HK = CA$, but then one draws the line KN parallel to the axis CB through the point K and this parallel intersects the ellipse in N, the difference $BM - AN$ of the arcs BM and AN can be assigned geometrically; for, having dropped the perpendicular to the tangent from the center C, this difference of the arcs will be $BM - AN = MT$.*

The proof is immediately clear from the figure, since the tangent HMK is parallel and equal to the that line CRV (Fig. 1); but then it is perspicuous that $MT = CR$.

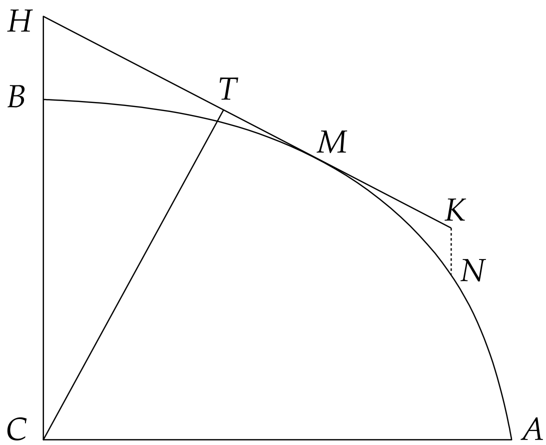

FIG. 2

THEOREM 2

§11 *If one constructs the equilateral triangle CAE over the one semiaxis CA of the elliptic quadrant ACB (Fig. 3) and in one of its sides AE one takes the portion $AF = CB$ and on the ellipse the line CO is equal to the line CF, the point O will have the property that*

$$CA + \text{arc } AO = CB + \text{arc } BO.$$

The proof is evident from § 9. For, since

$$CA = 1, \quad AF = c, \quad \text{and} \quad \text{angle } CAF = 60^\circ,$$

it will be

$$CF = \sqrt{1 + cc - 2c\cos 60^\circ}$$

and hence $= CO$.

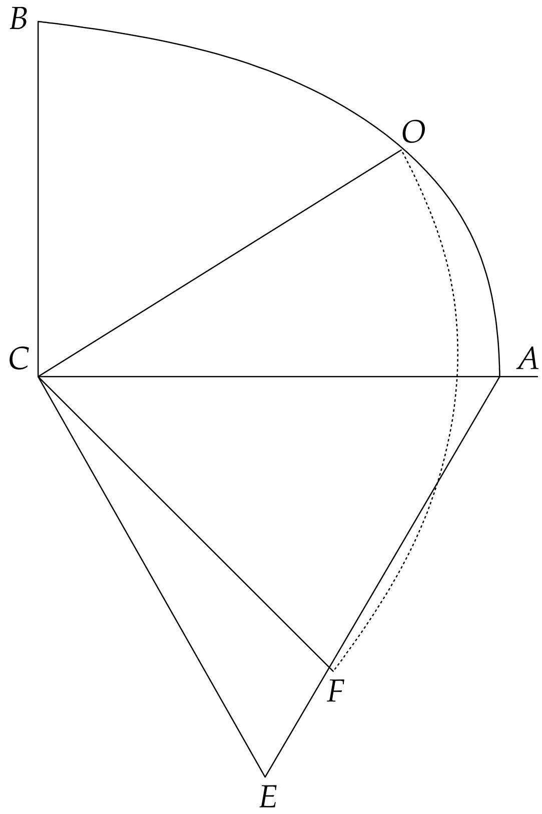

FIG. 3

II. On the Hyperbola

§12 Let C (Fig. 4) be the center of the hyperbola AMN and it transverse semiaxis $CA = 1$, the conjugated semiaxis $= c$; having taken an arbitrary abscissa $CP = x$ the ordinate will be $PM = c\sqrt{xx-1}$ and its differential $= \frac{cxdx}{\sqrt{xx-1}}$; hence the arc is

$$AM = \int \frac{\mathrm{d}x\sqrt{(1+cc)xx-1}}{\sqrt{xx-1}}.$$

For the sake of brevity put $1 + cc = n$; it will be

$$AM = \int \mathrm{d}x \sqrt{\frac{nxx - 1}{xx - 1}}.$$

Therefore, if in the same way another abscissa $CQ = u$ is taken, the arc corresponding to it will be

$$AN = \int \mathrm{d}u \sqrt{\frac{nuu - 1}{uu - 1}}.$$

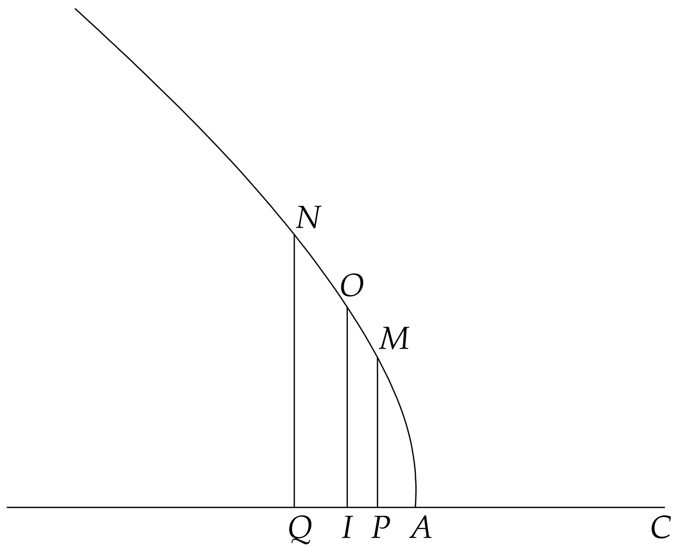

Fig. 4

§13 Having put these, let this new question be propounded to us that, given the point M, another one N is defined in such a way that the sum of the arcs $AM + AN$ or the expression

$$\int \mathrm{d}x \sqrt{\frac{nxx - 1}{xx - 1}} + \int \mathrm{d}u \sqrt{\frac{nuu1}{uu - 1}}$$

admits an integration absolutely; that this happens in the case $u = -x$ is immediately clear; but hence nothing can be concluded for our investigation.

§14 Therefore, let us put

$$\sqrt{\frac{nxx - 1}{xx - 1}} = u\sqrt{n},$$

since it vice versa is

$$\sqrt{\frac{nuu-1}{uu-1}} = x\sqrt{n};$$

therefore, this equation $nuuxx - n(uu + xx) + 1 = 0$ results. But by this assumption the sum of the arcs results as

$$AM + AN = \int u\mathrm{d}x\sqrt{n} + \int x\mathrm{d}u\sqrt{n} = ux\sqrt{n} + \text{Const.}$$

Therefore, for this integration to hold, it has to be $u = \sqrt{\frac{nxx-1}{nxx-n}}$, whence, since because of $n > 1$ also $u > 1$ results, from the given point M one will always be able to assign the point N.

§15 To define the constant, it is clear that the case $x = 1$, in which the point M falls on the vertex A, is not helpful, since hence $u = \infty$ and the point N is removed to infinity. Therefore, to determine this constant correctly, another case must be considered; but there is no better one than that, where the points M and N coalesce into one or in which $u = x$ and $nx^4 - 2nxx + 1 = 0$. But hence it results

$$xx = 1 + \frac{c}{\sqrt{1+cc}} \quad \text{and} \quad x = \sqrt{1 + \frac{c}{\sqrt{1+cc}}}.$$

§16 Therefore, let O be this point, in which both points M and N coalesce, and having drawn the ordinate OI, it will be

$$CI = \sqrt{1 + \frac{c}{\sqrt{1+cc}}} \quad \text{and} \quad 2AO = c + \sqrt{1+cc} + \text{Const.}$$

Therefore, we hence obtain the constant in question

$$2AO - c - \sqrt{1+cc}$$

because $\sqrt{n} = \sqrt{1+cc}$. Having substituted this value, for different points M and N taken in such a way that $u = \frac{nxx-1}{nxx-n}$ the sum of the arcs will be

$$AM + AN = ux\sqrt{n} + 2AO - c - \sqrt{1+cc}$$

or

$$ON - OM = ux\sqrt{n} - c - \sqrt{1+cc}.$$

Therefore, this way we obtained two arcs ON and OM, whose difference $ON - OM$ can be assigned geometrically.

§17 But that it is understood more easily, how so the the point O as the point N can be defined from M, draw the perpendicular $AD = c$ starting from the point A and the line CD will be the aysmptote of the hyperbola CD; then, having put $CP = x$, $PM = y$ draw the tangent MT; because of

$$y = c\sqrt{xx-1} \quad \text{and} \quad \mathrm{d}y = \frac{cx\mathrm{d}x}{\sqrt{xx-1}}$$

the subtangent will be

$$PT = \frac{y\sqrt{xx-1}}{cx} = x - \frac{1}{x} \quad \text{and} \quad CT = \frac{1}{x}$$

and the tangent will be

$$MT = \frac{y\sqrt{nxx-1}}{cx}.$$

Hence it results

$$\sqrt{\frac{xx-1}{nxx-1}} = \frac{PT}{MT} \quad \text{and hence} \quad u = \frac{MT}{PT\sqrt{1+cc}} = \frac{CA^2 \cdot MT}{CD \cdot PT} = CQ.$$

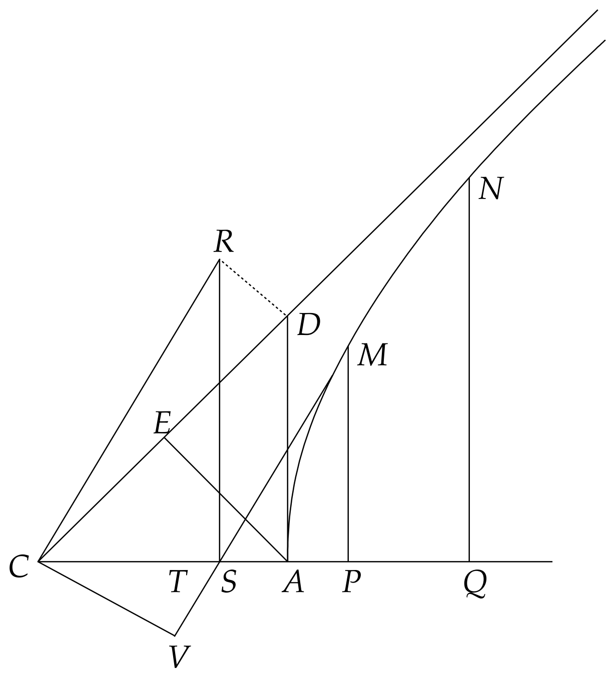

FIG. 5

§18 From the center C draw the line $CR = CD$ parallel to the tangent TM and, having dropped the perpendicular RS from R, it will be $CS = \frac{CD \cdot PT}{MT}$ and hence $CQ = \frac{CA^2}{CS}$. Hence CQ is to be taken as the third proportional to CS and CA. But the task will solved more conveniently without using the tangents; for, since

$$QN = \frac{cc}{\sqrt{n(xx-1)}} = \frac{c^3}{y\sqrt{n}},$$

it will be

$$PM \cdot QN = \frac{c^3}{\sqrt{1+cc}} = \frac{AD^3}{CD}$$

or, having dropped the perpendicular AE to the asymptote from A, it will be

$$PM \cdot QN = AD \cdot DE,$$

since $DE = \frac{AD^2}{CD}$, whence the following theorem results.

Theorem 3

§19 *While AOZ (Fig. 6) denotes the hyperbola, C its center, A its vertex and CDZ its asymptote, to which from A the perpendicular AD was dropped, and likewise the perpendicular AE was dropped to the asymptote, if the ordinate IO is constituted as the mean proportional of AD and DE and the ordinates PM and QN to the left and the right are chosen in such a way that IO is the mean proportional of them, then the difference of the arcs = N and OM can be assigned geometrically.*

$$ON - OM = \frac{CP \cdot CQ - CI \cdot CI}{CE}.$$

The proof is obvious from the preceding paragraph. For, since connecting the points M and N to O, let $IO \cdot IO = AD \cdot DE$, IO will be the mean proportional of AD and DE; and having found this it has to be $PM \cdot QN = OI \cdot OI$. But then it is understood from § 16 to be $ON - OM = (CP \cdot CQ - CI \cdot CI)\sqrt{n}$ and because of $\sqrt{n} = CD$ by satisfying the homogeneity it will be $ON - OM = (CP \cdot CQ - CI \cdot CI)\frac{CD}{CA^2}$. But it is $\frac{CA^2}{CD} = CE$ and so the truth of the theorem is established.

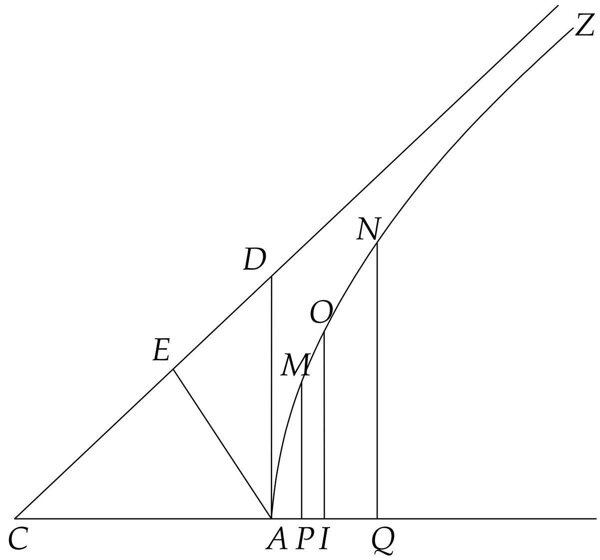

FIG. 6

III. ON THE LEMNISCATE

§20 This curve is famous for the many extraordinary properties it enjoys, but especially, since its arcs are equal to the arcs of the curva elastica. But this curve is of such a nature that, having put the orthogonal coordinates $CP = x$, $PM = y$ (Fig. 7), it is expressed by this equation

$$(xx + yy)^2 = xx - yy.$$

Hence it is plain that this curve is a line of fourth order, which has an angle of 45° to the semiaxis CA in the point C, which also is its center, but in A it passes through the axis $CA = 1$ perpendicularly. But the figure $CMNA$ exhibits the fourth part of the whole lemniscate, to which the three remaining parts around the center are equal; this follows from the fact that the equation remains the same, even if either the abscissa x or the ordinate y or even both is changed into is negative.

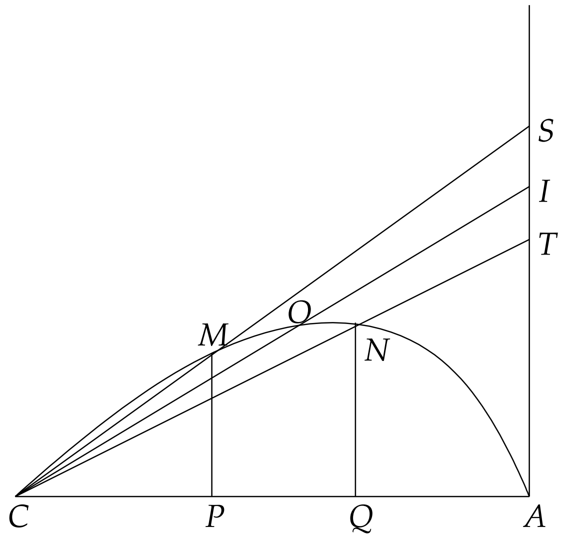

FIG. 7

§21 Therefore, concerning the expression of a certain arc CM of this curve, it is most conveniently defined from the cord CM. For, if we put this cord $CM = z$, because of $xx + yy = zz$ we will have $z^4 = xx - yy = 2xx - zz = zz - 2yy$, whence we find

$$x = z\sqrt{\frac{1+zz}{2}} \quad \text{and} \quad y = z\sqrt{\frac{1-zz}{2}}$$

and by differentiation

$$\mathrm{d}x = \frac{\mathrm{d}z(1+2zz)}{\sqrt{2(1+zz)}} \quad \text{and} \quad \mathrm{d}y = \frac{\mathrm{d}z(1-2zz)}{\sqrt{2(1-zz)}}.$$

Therefore, hence the element of the CM is calculated to be

$$\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \mathrm{d}z\sqrt{\frac{(1-zz)(1+2zz)^2 + (1+zz)(1-2zz)^2}{2(1+zz)(1-zz)}}$$

or

$$\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \frac{\mathrm{d}z}{\sqrt{1-z^4}}.$$

§22 Therefore, if an arbitrary cord, drawn from the center C, is put $CM = z$, the arc corresponding to it will be $CM = \int \frac{dz}{\sqrt{1-z^4}}$. Therefore, if in the same way another cord CN is called $= u$, the arc corresponding to it will be $CN = \int \frac{du}{\sqrt{1-u^4}}$, whose complement to the whole quadrant is the arc AN. Now Fagnano told, a function of z of which kind must be taken for u so that either the arc AN is equal to the arc CM or the CN is the double of the arc CM, or even that the arc AN is equal to the double arc CM. Therefore, I will first explain those cases, but then I will discuss everything else I was able to find on the proportions of arcs of this kind.

THEOREM 4

§23 *If on the lemniscate described just before one considers an arbitrary cord $CM = z$ and additionally another one, which we want to put*

$$CN = u = \sqrt{\frac{1-zz}{1+zz}},$$

the arc CM will be equal to the arc AN or also the arc CN will be equal to the arc AM.

PROOF

Since the cord is $CM = z$, the arc will be $CM = \int \frac{dz}{\sqrt{1-z^4}}$ and because of the cord $CN = u$ the arc CN will be $= \int \frac{du}{\sqrt{1-u^4}}$. But it is $u = \sqrt{\frac{1-zz}{1+zz}}$; hence

$$\mathrm{d}u = \frac{-2z\mathrm{d}z}{(1+zz)\sqrt{1-z^4}}.$$

But furthermore

$$u^4 = \frac{1-2zz+z^4}{1+2zz+z^4} \quad \text{and} \quad 1-u^4 = \frac{4zz}{(1+zz)^2} \quad \text{and} \quad \sqrt{1-u^4} = \frac{zz}{1+zz}.$$

Having substituted these values, one will have

$$\operatorname{arc} CN = -\int \frac{\mathrm{d}z}{\sqrt{1-z^4}} = -\operatorname{arc} CM + \text{Const.},$$

so that arc $CN +$ arc $CM =$ Const.. To define this constant, consider the case, in which $z = 0$ and hence the arc $CM = 0$; but in this case the cord $CN = u =$

$1 = CA$ and hence the arc CN goes over into the quadrant $CMNA$, from which one has $CMNA + 0 =$ Const. for this case. Therefore, having substituted these values, in general it will result arc $CN +$ arc $CM =$ arc $CMNA$ and hence

$$\text{arc } CM = \text{arc } AN,$$

and, adding the arc MN to both sides,

$$\text{arc } CMN = \text{arc } ANM.$$

Q.E.D.

COROLLARY 1

§24 Therefore, given an arbitrary arc CM terminated at the center C, whose cord is $CM = z$, another equal arc AN, originating from A, is separated from it by taking the cord

$$CN = u = \sqrt{\frac{1-zz}{1+zz}} \quad \text{or} \quad CN = CA\sqrt{\frac{CA^2 - CM^2}{CA^2 + CM^2}}$$

or completing the homogeneity by means of the axis $CA = 1$.

COROLLARY 2

§25 Since $u = \sqrt{\frac{1-zz}{1+zz}}$, it will vice versa be $z = \sqrt{\frac{1-uu}{1+uu}}$; hence the cords CM and CN can be permuted so that, if both cords $CM = z$ and $CN = u$ were of such a nature that

$$uuzz + uu + zz = 1,$$

also the points M and N can be permuted and hence so arc $CM =$ arc AN as arc $CN =$ arc AM.

COROLLARY 3

§26 Since $CN = u = \sqrt{\frac{1-zz}{1+zz}}$, it will be

$$\sqrt{\frac{1+uu}{2}} = \frac{1}{\sqrt{1+zz}} \quad \text{and} \quad \sqrt{\frac{1-uu}{2}} = \frac{z}{\sqrt{1+zz}}.$$

Hence, since from the nature of the lemniscate the coordinates for the point N are

$$CQ = u\sqrt{\frac{1+uu}{2}} \quad \text{and} \quad QN = u\sqrt{\frac{1-uu}{2}},$$

it will be

$$CQ = \frac{u}{\sqrt{1+zz}} \quad \text{and} \quad QN = \frac{uz}{\sqrt{1+zz}} \quad \text{and hence} \quad \frac{QN}{CQ} = z.$$

Hence, if the normal AT to the axis CA is drawn from A, until it intersects the cord CN in T, it will be $AT = z = CM$.

Corollary 4

§27 Therefore, from a given point M the other point N is most easily deduced this way: Take the tangent AT equal to the cord CM and having drawn the line CT it will intersect the curve in the point N in question. But for the same reason it is plain: If the cord CM is elongated until it meets the tangent in the point A in S, it will also be $AS = CN$.

Corollary 5

§28 Moreover, it is obvious that the points M and N can coalesce into one point O, by which hence the whole quadrant COA in divided into two equal parts. Therefore, one will find this point O, if one puts $u = z$, whence

$$z^4 + 2zz = 1 \quad \text{and hence} \quad zz + 1 = \sqrt{2};$$

therefore, the cord CO results as $= \sqrt{\sqrt{2}-1}$, to which at the same time the tangent AI will be equal, whence the position of this point O is easily assigned.

Corollary 6

§29 Therefore, having called this point O, in which the quadrant COA is split into two equal parts CMO and ANO, having defined the points M and N by the explained rule, it will also be arc. $MO =$ arc. ON so that this same point O splits the arc MN into two equal parts.

THEOREM 5

§30 *On the lemnicate, whose axis is $CA = 1$ (Fig. 8), if the ordinate is an arbitrary cord $CM = z$ and additionally we have another cord or ordinate*

$$CM^2 = u = \frac{2z\sqrt{1-z^4}}{1+z^4},$$

the arc corresponding this cord u, or CM^2 will twice as long as the arc corresponding that cord CM.

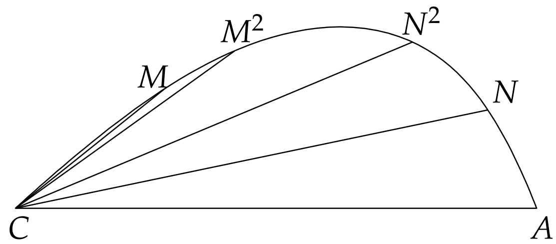

FIG. 8

PROOF

Since the cord is $CM = z$, the arc CM will be $= \int \frac{dz}{\sqrt{1-z^4}}$ and similarly because of the cord $CM^2 = u$ the arc CM^2 will be $\int \frac{du}{\sqrt{1-u^4}}$. But since $u = \frac{2z\sqrt{1-z^4}}{1+z^4}$, it will be

$$uu = \frac{4zz - 4z^6}{1 + 2z^4 + z^8}$$

and hence

$$\sqrt{1-uu} = \frac{1-2zz-z^4}{1+z^4} \quad \text{and} \quad \sqrt{1+uu} = \frac{1+2zz-z^4}{1+z^4},$$

whence

$$\sqrt{1-u^4} = \frac{1-6z^4+z^8}{(1+z^4)^2}.$$

Hence by differentiating one concludes

$$\mathrm{d}u = \frac{2\mathrm{d}z(1-z^8) - 4z^4\mathrm{d}z(1+z^4) - 8z^4\mathrm{d}z(1-z^4)}{(1+z^4)^2\sqrt{1-z^4}}$$

or

$$\mathrm{d}u = \frac{2\mathrm{d}z - 12z^4\mathrm{d}z + 2z^8\mathrm{d}z}{(1+z^4)^2\sqrt{1-z^4}} = \frac{2\mathrm{d}z(1-6z^4+z^8)}{(1+z^4)^2\sqrt{1-z^4}}.$$

Therefore, we obtain

$$\frac{\mathrm{d}u}{\sqrt{1-u^4}} = \frac{2\mathrm{d}z}{\sqrt{1-z^4}}$$

and by integrating $\operatorname{arc}.\,CM^2 = 2\operatorname{arc}.\,CM + \text{Const.}$. But since for $z=0$ also $u=0$ and hence both arcs CM and CM^2 vanish, also the constant vanishes. And so, having taken the cord $CM^2 = u = \frac{2z\sqrt{1-z^4}}{1+z^4}$ it will be

$$\operatorname{arc}\ CM^2 = 2\operatorname{arc}\ CM.$$

Q.E.D.

COROLLARY 1

§31 If one takes the cord $CN = \sqrt{\frac{1-zz}{1+zz}}$, the arc AN will be $\operatorname{arc}.\,CM$ and hence also the arc CM^2 will be $2\operatorname{arc}.AN$. In like manner, if one takes the cord $CN^2 = \frac{1-uu}{1+uu}$, the arc AN^2 will be $= \operatorname{arc}.\,CM^2$ and so also, starting from the vertex A, it will be $\operatorname{arc}.\,AN^2 = 2\operatorname{arc}.\,AN$. Therefore, this way one will obtains four equal arcs, namely $\operatorname{arc}.\,CM$, $\operatorname{arc}.\,MM^2$, $\operatorname{arc}.\,AN$ and $\operatorname{arc}.\,NN^2$.

COROLLARY 2

§32 But since

$$u = \frac{2z\sqrt{1-z^4}}{1+z^4}, \quad \sqrt{1-uu} = \frac{1-2zz-z^4}{1+z^4} \quad \text{and} \quad \sqrt{1+uu} = \frac{1+2zz-z^4}{1+z^4},$$

one will have these four cords expressed in such a way

$$CM = z, \quad CN = \sqrt{\frac{1-zz}{1+zz}}, \quad CM^2 = \frac{2z\sqrt{1-z^4}}{1+z^4}, \quad CN^2 = \frac{1-2zz-z^4}{1+2zz-z^4}.$$

COROLLARY 3

§33 Let both points M^2 and N^2 coalesce in the middle point O of the curve, for which we saw above that the cord CO is $= \sqrt{\sqrt{2}-1}$, and in this case the whole curve COA will be split into four parts in the points M, O and N.

Therefore, this happens, if $CM^2 = CN^2 = \sqrt{\sqrt{2}-1}$ so that, having for the sake of brevity set $\sqrt{\sqrt{2}-1} = \alpha$, we have

$$1 - 2zz - z^4 = \alpha + 2\alpha zz - \alpha z^4 \quad \text{or} \quad z^4 = \frac{-2(1+\alpha)zz + 1 - \alpha}{1-\alpha}$$

and

$$zz = \frac{-(1+\alpha) + \sqrt{2(1+\alpha\alpha)}}{1-\alpha} \quad \text{or} \quad zz = \frac{-1 - \sqrt{\sqrt{2}-1} + \sqrt{2\sqrt{2}}}{1 - \sqrt{\sqrt{2}-1}}.$$

Hence we conclude

$$CM = z = \sqrt{\frac{-1-\alpha+\sqrt{2(1+\alpha\alpha)}}{1-\alpha}} \quad \text{and} \quad CN = \sqrt{\frac{-1+\alpha+\sqrt{2(1+\alpha\alpha)}}{1+\alpha}}$$

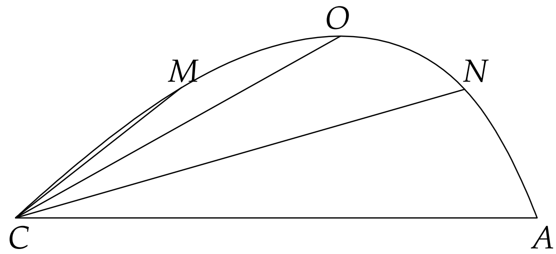

Fig. 9

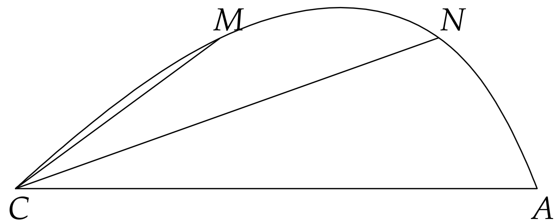

Fig. 10

Corollary 4

§34 Let both points M^2 and N (Fig. 10) coalesce and the points M and N^2 will coalesce in the same way and so the whole curve $CMNA$ will be split into three parts in the points M and N. Therefore, for this case we will have

$$\frac{2z\sqrt{1-z^4}}{1+z^4} = \sqrt{\frac{1-zz}{1+zz}} \quad \text{or} \quad z = \frac{1-2zz-z^4}{1+2zz-z^4},$$

the second of which gives $1 - z - 2zz - 2z^3 - z^4 + z^5 = 0$ and this divided by $1 + z$ then $1 - 2z - 2z^3 + z^4 = 0$; assume its factors to be

$$(1 - \mu z + zz)(1 - \nu z + zz) = 0$$

and it will be $\mu + \nu = 2$ and $\mu\nu = -2$, whence $\mu - \nu = 2\sqrt{3}$ and hence

$$\mu = 1 + \sqrt{3} \quad \text{and} \quad \nu = 1 - \sqrt{3}.$$

Therefore, it will be

$$z = \frac{1 + \sqrt{3} \pm \sqrt{2\sqrt{3}}}{2} = CM$$

and because of $zz = \frac{4+4\sqrt{3}\pm 2(1+\sqrt{3})\sqrt{2\sqrt{3}}}{4}$ it will result

$$CN = \sqrt{\frac{1-zz}{1+zz}} = \sqrt{\frac{-2\sqrt{3} \mp (1+\sqrt{3})\sqrt{2\sqrt{3}}}{4 + 2\sqrt{3} \pm (1+\sqrt{3})\sqrt{2\sqrt{3}}}} = \sqrt{\frac{\mp\sqrt{2\sqrt{3}}}{1+\sqrt{3}}}.$$

Therefore,

$$CM = \frac{1 + \sqrt{3} - \sqrt{2\sqrt{3}}}{2} \quad \text{and} \quad CN = \sqrt{\frac{\sqrt{2\sqrt{3}}}{1+\sqrt{3}}}.$$

Corollary 5

§35 Given an arbitrary arc CM^2 (Fig. 8) one can also find its half CM; for, if the cord of that arc is set $CM^2 = u$ and the cord of the arc in question $CM = z$, it will be

$$u = \frac{2z\sqrt{1-z^4}}{1+z^4} \quad \text{and} \quad 1 - \frac{4zz}{uu} + 2z^4 + \frac{4z^6}{uu} + z^8 = 0,$$

whose factor we assume to be

$$(1 - \mu zz - z^4)(1 - \nu zz - z^4) = 0,$$

whence one obtains $\mu + \nu = \frac{4}{uu}$ and $\mu\nu = 4$; therefore, it will be

$$\mu - \nu = 4\sqrt{\frac{1}{u^4} - 1} = \frac{4}{uu}\sqrt{1 - u^4}$$

and hence

$$\mu = \frac{2+2\sqrt{1-u^4}}{uu} \quad \text{and} \quad \nu = \frac{2-2\sqrt{1-u^4}}{uu},$$

therefore,

$$zz = \frac{-1-\sqrt{1-u^4}+\sqrt{2(1+\sqrt{1-u^4})}}{uu},$$

whence two real values are found for z, the one

$$z = \frac{\sqrt{-1-\sqrt{1-u^4}+\sqrt{2(1+\sqrt{1-u^4})}}}{u} = \frac{\sqrt{(1-\sqrt{1-uu})(\sqrt{1+uu}-1)}}{u}$$

the other

$$z = \frac{\sqrt{(-1+\sqrt{1-u^4})+\sqrt{2(1-\sqrt{1-u^4})}}}{u} = \frac{\sqrt{(1+\sqrt{1-uu})(\sqrt{1+uu})-1}}{u}.$$

Corollary 6

§36 These two values are indeed true; for, since the same cord CM^2 (Fig. 11) and Cm^2 corresponds to two different arcs CM^2 and CM^2m^2, the one value of z will yield the cord of the arc CM, which is the half of the arc CM^2, but the other value of z gives the cord of the arc Cm, which is the half of the arc CM^2m^2; and the first value certainly holds for that case, the second on the other hand for this case.

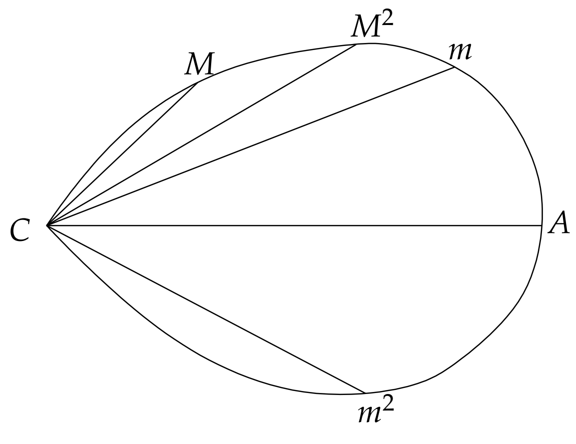

Fig. 11

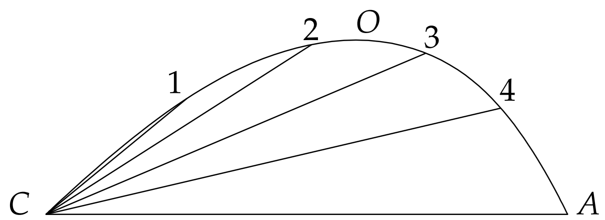

FIG. 12

COROLLARY 7

§37 This way the lemniscate CA can also be split into five equal parts. For, let the cord of the simple part be $C1 = z$ (Fig. 12), the cord of the doubled part is

$$C2 = \frac{2z\sqrt{1-z^4}}{1+z^4} = u;$$

the cord of the quadrupled part will be

$$C4 = \frac{2u\sqrt{1-u^4}}{1+u^4} = \sqrt{\frac{1-zz}{1+zz}},$$

since $A4 = C1$, whence the cord z is defined; having found it, since $C2 = A3$, the cord $C3$ will be $= \frac{1-uu}{1+uu}$.

COROLLARY 8

§38 Since hence, having put a certain cord $= z$, one can find the cords of the doubled, quadrubled, eigtht times as large, sixteen times as long etc. arcs, it is obvious that this way the lemniscate can be split in so many parts, whose number is $2^m(1+2^n)$. But this formula contains the following numbers

$$1, \ 2, \ 3, \ 4, \ 5, \ 6, \ 8, \ 9, \ 10, \ 12, \ 16, \ 17, \ 18, \ 20, \ 24, \ 32, \ 33, \ \text{etc}$$

But hence it is not always possible to assign all points of division.

SCHOLIUM

§39 Therefore, these are the results, which Fagnano observed on the lemniscate or which can be derived from his findings. For, even if, having propounded an arbitrary arc, he only taught to assign its double, nevertheless hence by doubling this arc again also the cords of the quadrupled, eight times as long, sixteen times as long etc. arcs will be found from this. And if the cord of the simple are is set $= z$, of the doubled arc $= u$, the quadrupled arc $= p$, the eight times as long arc $= q$, sixteen times as long arc $= r$ etc., it will be

$$u = \frac{2z\sqrt{1-z^4}}{1+z^4}$$

$$p = \frac{2u\sqrt{1-u^4}}{1+u^4} = \frac{4z(1+z^4)(1-6z^4+z^8)\sqrt{1-z^4}}{(1+z^4)^4+16z^4(1-z^4)^2}$$

$$q = \frac{2p\sqrt{1-p^4}}{1+p^4}$$

$$r = \frac{2q\sqrt{1-q^4}}{1+q^4}$$

But it is not possible to assign the cords of other multiple arcs from them. Therefore, I will investigate here, how the cords of certain multiple arcs are expressed, to exhaust this subject, as far as the limits of analysis certainly allow it, completely. First, certainly by trial and error, I found, if the cord of the simple arc is $= z$, that then the cord of the tripled arc will be $= \frac{z(3-6z^4-z^8)}{1+6z^4-3z^8}$; but after this I understood that the subject can be treated generally in the following way.

THEOREM 6

§40 *If the cord of the simple arc CM (Fig. 13) is $= z$ and the cord of the n-tupled arc CM^n is $= u$, the arc of the $(n+1)$-tupled arc will be*

$$CM^{n+1} = \frac{z\sqrt{\frac{1-uu}{1+uu}} + u\sqrt{\frac{1-zz}{1+zz}}}{1 - uz\sqrt{\frac{(1-uu)(1-zz)}{(1+uu)(1+zz)}}}.$$

PROOF

Therefore, the simple arc itself will be

$$CM = \int \frac{dz}{\sqrt{1-z^4}}$$

and the n-tupled arc

$$CM^n = \int \frac{du}{\sqrt{1-u^4}} = n\int \frac{dz}{\sqrt{1-z^4}}$$

and hence we have $\mathrm{d}u = \frac{n\mathrm{d}z\sqrt{1-u^4}}{\sqrt{1-z^4}}$. For the sake of brevity let us put

$$z\sqrt{\frac{1-uu}{1+uu}} = P \quad \text{and} \quad u\sqrt{\frac{1-zz}{1+zz}} = Q$$

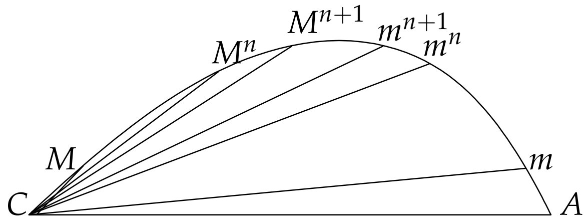

FIG. 13

that the cord exhibited for the $(n+1)$-tupled arc is $CM^{n+1} = \dfrac{P+Q}{1-PQ}$, which we want to call $= s$, and it has to be demonstrated that the arc corresponding to this cord is

$$\int \frac{\mathrm{d}s}{\sqrt{1-s^4}} = (n+1)\int \frac{\mathrm{d}z}{\sqrt{1-z^4}} \quad \text{oder} \quad \frac{\mathrm{d}s}{\sqrt{1-s^4}} = \frac{(n+1)\mathrm{d}z}{\sqrt{1-z^4}}.$$

But since $s = \frac{P+Q}{1-PQ}$, it will be

$$\mathrm{d}s = \frac{\mathrm{d}P(1+QQ) + \mathrm{d}Q(1+PP)}{(1-PQ)^2}$$

but then one finds

$$\begin{aligned} 1 - s^4 &= \frac{(1-PQ)^4 - (P+Q)^4}{(1-PQ)^4} \\ &= \frac{(1+PP+QQ+PPQQ)(1-PP-QQ-4PQ+PPQQ)}{(1-PQ)^4} \end{aligned}$$

hence

$$\sqrt{1-s^4} = \frac{\sqrt{(1+PP)(1+QQ)(1-PP-QQ-4PQ+PPQQ)}}{(1-PQ)^2},$$

whence one finds

$$\frac{\mathrm{d}s}{\sqrt{1-s^4}} = \frac{\mathrm{d}P\sqrt{\frac{1+QQ}{1+PP}} + \mathrm{d}Q\sqrt{\frac{1+PP}{1+QQ}}}{\sqrt{1-PP-QQ-4PQ+PPQQ}},$$

the value of which expression we want to investigate.
And first certainly

$$1 + PP = \frac{1 + uu + zz - uuzz}{1 + uu} \quad \text{and} \quad 1 + QQ = \frac{1 + uu + zz - uuzz}{1 + zz},$$

so that $\frac{1+PP}{1+QQ} = \frac{1+zz}{1+uu}$ and hence

$$\frac{\mathrm{d}s}{\sqrt{1 - s^4}} = \frac{\mathrm{d}P\sqrt{\frac{1+uu}{1+zz}} + \mathrm{d}Q\sqrt{\frac{1+zz}{1+uu}}}{\sqrt{1 - PP - QQ + PPQQ - 4PQ}}.$$

Further, because of

$$1 - PP = \frac{1 + uu - zz + uuzz}{1 + uu} \quad \text{and} \quad 1 - QQ = \frac{1 + zz - uu + uuzz}{1 + zz}$$

it will be

$$(1 - PP)(1 - QQ) = 1 - P^2 - Q^2 + P^2Q^2 = \frac{1 - z^4 - u^4 + 4uuzz + u^4z^4}{(1 + zz)(1 + uu)}$$

and

$$4PQ = \frac{4uz\sqrt{(1 - z^4)(1 - u^4)}}{(1 + zz)(1 + uu)}$$

and hence one concludes the denominator

$$\begin{aligned}
&\sqrt{1 - PP - QQ + PPQQ - 4PQ} \\
=&\frac{\sqrt{(1 - z^4 - u^4) + 4uuzz + u^4z^4 - 4uz\sqrt{(1 - z^4)(1 - u^4)}}}{\sqrt{(1 + zz)(1 + uu)}} \\
=&\frac{\sqrt{(1 - z^4)(1 - u^4)} - 2uz}{\sqrt{(1 + zz)(1 + uu)}},
\end{aligned}$$

whence one will obtain

$$\frac{\mathrm{d}s}{\sqrt{1 - s^4}} = \frac{\mathrm{d}P(1 + uu) + \mathrm{d}Q(1 + zz)}{\sqrt{(1 - z^4)(1 - u^4)} - 2uz}.$$

But now by differentiating

$$\mathrm{d}P = \mathrm{d}z\sqrt{\frac{1 - uu}{1 + uu}} - \frac{2zu\mathrm{d}u}{(1 + uu)\sqrt{1 - u^4}}$$

$$\mathrm{d}Q = \mathrm{d}u\sqrt{\frac{1 - zz}{1 + zz}} - \frac{2zu\mathrm{d}u}{(1 + zz)\sqrt{1 - z^4}},$$

whence because of

$$du = \frac{n\mathrm{d}z\sqrt{1-u^4}}{\sqrt{1-z^4}}$$

it will be

$$\begin{aligned}
\mathrm{d}P &= \mathrm{d}z\sqrt{\frac{1-uu}{1+uu}} - \frac{2nuz\mathrm{d}z}{(1+uu)\sqrt{1-z^4}} \\
\mathrm{d}Q &= \frac{n\mathrm{d}z\sqrt{1-u^4}}{1+zz} - \frac{2uz\mathrm{d}z}{(1+zz)\sqrt{1-z^4}},
\end{aligned}$$

whence one gets the numerator

$$\mathrm{d}P(1+uu)+\mathrm{d}Q(1+zz) = \mathrm{d}z\sqrt{1-u^4} - \frac{2nuz\mathrm{d}z}{\sqrt{1-z^4}} + n\mathrm{d}z\sqrt{1-u^4} - \frac{2uz\mathrm{d}z}{\sqrt{1-z^4}}$$

or

$$\begin{aligned}
\mathrm{d}P(1+uu)+\mathrm{d}Q(1+zz) &= (n+1)\mathrm{d}z\sqrt{1-u^4} - \frac{2(n+1)uz\mathrm{d}z}{\sqrt{1-z^4}} \\
&= \frac{(n+1)\mathrm{d}z}{\sqrt{1-z^4}}\left(\sqrt{(1-z^4)(1-u^4)} - 2uz\right),
\end{aligned}$$

whence it is perspicuous that

$$\frac{\mathrm{d}s}{\sqrt{1-s^4}} = \frac{(n+1)\mathrm{d}z}{\sqrt{1-z^4}}$$

and

$$\text{arc } CM^{n+1} = (n+1)\cdot\text{arc } CM$$

Q.E.D.

COROLLARY 1

§41 If from the vertex A the arcs Am, Am^n, Am^{n+1}, which are equal to the arcs CM, CM^n, CM^{n+1}, Cm will be the cord of the complement of the arc CM, Cm the cord of the complement of the arc CM^n, Cm^{n+1} the cord of the arc CM^{n+1}. But because of the cords $CM = z$, $CM^n = u$, $CM^{n+1} = s$ the complement of the cords will be

$$Cm = \sqrt{\frac{1-zz}{1+zz}}, \quad Cm^n = \sqrt{\frac{1-uu}{1+uu}}, \quad Cm^{n+1} = \sqrt{\frac{1-ss}{1+ss}}.$$

But since

$$s = \frac{z\sqrt{\frac{1-uu}{1+uu}} + u\sqrt{\frac{1-zz}{1+zz}}}{1 - zu\sqrt{\frac{(1-uu)(1-zz)}{(1+uu)(1+zz)}}} = \frac{P+Q}{1-PQ},$$

it will be

$$\sqrt{\frac{1-ss}{1+ss}} = \sqrt{\frac{1-PP-QQ-4PQ+PPQQ}{(1+PP)(1+QQ)}} = \frac{\sqrt{(1-z^4)(1-u^4)}-2uz}{1+uu+zz-uuzz}$$

which is reduced to this form

$$\sqrt{\frac{1-ss}{1+ss}} = \frac{\sqrt{\frac{(1-zz)(1-uu)}{(1+zz)(1+uu)}} - uz}{1+uz\sqrt{\frac{(1-zz)(1-uu)}{(1+zz)(1+uu)}}}.$$

Corollary 2

§42 Therefore, if one puts

$$\text{the cords of the simple arc } = z, \quad \text{the cord of the complement } = Z,$$

$$\text{the cords of the n-tuple arc } = u, \quad \text{the cord of the complement } = U,$$

that

$$Z = \sqrt{\frac{1-zz}{1+zz}} \quad \text{and} \quad U = \sqrt{\frac{1-uu}{1+uu}},$$

the cord of the $(n+1)-$tupled arc will be

$$\frac{zU+uZ}{1-zuUZ},$$

the cord of the complement will be

$$\frac{ZU-zu}{1+zuZU}.$$

COROLLARY 3

§43 Therefore, the invention of the cords of certain multiple arcs together with the cords of the complements we look as follows:

Cord of the arc	Cord of the complement
single = a	single = A
double = b = $\frac{2aA}{1-aaAA}$	double = $\frac{AA-aa}{1+aaAA}$ = B
triple = c = $\frac{aB+bA}{1-abAB}$	triple = $\frac{AB-ab}{1+abAB}$ = C
quadruple = d = $\frac{aC+cA}{1-acAC}$	quadruple = $\frac{AC-ac}{1+acAC}$ = D
quintuple = e = $\frac{aD+dA}{1-adAD}$	quintuple = $\frac{AD-ad}{1+adAD}$ = E
etc.	etc.

COROLLARY 4

§44 In like manner, if the cord of the $m-$tupled arc is $= r$, the cord of the complement $= R$ and the cord of the $n-$tupled arc $= s$ and the cord of its complement $= S$ that

$$R = \sqrt{\frac{1-rr}{1+rr}} \quad \text{and} \quad S = \sqrt{\frac{1-ss}{1+ss}},$$

the cord of the $(m+n)-$tuple arc $= \frac{rS+sR}{1-rsRS}$ and the cord of the complement $= \frac{RS-rs}{1+rs+RS}$. Yes, even by taking a negative number for n, since then the cord s goes over into its negative, the cord of the difference of those arcs can be exhibited; of course, the cord of the $(m-n)-$tupled arc $= \frac{rS-sR}{1+rsRS}$ and the cord of its complement $= \frac{RS+rs}{1-rsRS}$.

COROLLARY 5

§45 Therefore, having introduced the notations in corollary 3, it will also be

$$d = \frac{2bB}{1-bbBB} \quad \text{and} \quad D = \frac{BB-bb}{1+bbBB}$$

$$e = \frac{bC+cB}{1-bcBC} \quad \text{and} \quad E = \frac{BC-bc}{1+bcBC}.$$

Corollary 6

§46 From these one concludes, if the cord of the simple arc is set $= z$, that the values of the cords used in corollary 3 will be

$$\begin{aligned}
a &= z & A &= \sqrt{\frac{1-zz}{1+zz}} \\
b &= \frac{2z\sqrt{1-z^4}}{1+z^4} & B &= \frac{1-2zz-z^4}{1+2zz-z^4} \\
c &= \frac{z(3-6z^4-z^8)}{1+6z^4-3z^8} & C &= \frac{(1+z^4)^2-4zz(1-zz)^2}{(1+z^4)^2+4zz(1-zz)^2}\sqrt{\frac{1-zz}{1+zz}} \\
d &= \frac{4z(1+z^4)(1-6z^4+z^8)\sqrt{1-z^4}}{(1+z^4)^4+16z^4(1-z^4)^2)} & D &= \frac{(1-6z^4+z^8)^2-8zz(1-z^4)(1+z^4)^2)}{(1-6z^4+z^8)^2+8zz(1-z^4)(1+z^4)^2}.
\end{aligned}$$

Scholium 1

§47 The composition of the formulas $\frac{rS+sR}{1-rsRS}$ and $\frac{RS-rs}{1+rs+RS}$ deserves some more attention, since it is similar to the rule, by which the tangent of the sum or difference of two angles are usually defined. For, if $rS = \tan\alpha$ and $sR = \tan\beta$, it will be $\frac{rS+sT}{1-rsRS} = \tan(\alpha+\beta)$ and for the difference exhibited in corollary 4 $\frac{rS-sR}{1+rsRS} = \tan(\alpha-\beta)$. And in like manner, if one puts $RS = \tan\gamma$ and $rs = \tan\delta$, it will be

$$\frac{RS-rs}{1+rsRS} = \tan(\gamma-\delta) \quad \text{and} \quad \frac{RS+rs}{1-rsRS} = \tan(\gamma+\delta).$$

But the nature of this composition will be represented more conveniently, if the cord of the $m-$tupled arc is put $r = M\sin\mu$, the cord of the complement $R = M\cos\mu$, the cord of the $n-$tuple $s = N\sin\nu$, the cord of the complement $S = N\cos\nu$; for, then it will be

$$\begin{aligned}
&\text{cord of the } (m+n)\text{-tuple arc} &&= \frac{MN\sin(\mu+\nu)}{1-M^2N^2\sin\mu\sin\nu\cos\mu\cos\nu} \\
&\text{arc of its complement} &&= \frac{MN\cos(\mu+\nu)}{1+M^2N^2\sin\mu\cos\mu\cos\nu\sin\nu} \\
&\text{arc of the } (m-n)\text{-tuple arc} &&= \frac{MN\sin(\mu-\nu)}{1-M^2N^2\sin\mu\sin\nu\cos\mu\cos\nu} \\
&\text{the cord of its complement} &&= \frac{MN\cos(\mu-\nu)}{1+M^2N^2\sin\mu\sin\nu\cos\mu\cos\nu}
\end{aligned}$$

But since $1-rr-RR = rrRR$, it will be $1-MM = M^4(\sin^2\mu)(\cos^2\mu)$ and hence

$$M^2\sin\mu\cos\mu = \sqrt{1-MM} \quad \text{and} \quad N^2\sin\nu\cos\nu = \sqrt{1-NN},$$

whence the denominator of these formulas will go over into

$$1-\sqrt{(1-MM)(1-NN)} \quad \text{and} \quad 1+\sqrt{(1-MM)(1-NN)}.$$

Furthermore, from that equation $1-MM = M^4 \sin^2 \mu \cos^2 \mu$

$$\frac{1}{MM} = \frac{1}{2} + \frac{1}{2}\sqrt{1+\sin 2\mu \sin 2\mu},$$

because of $\sin 2\mu = 2\sin \mu \cos \mu$. But hence those formulas do not become shorter.

SCHOLIUM 2

§48 From these observations very important progresses for the integral calculus follow, since hence we are able to exhibit the particular integrals of many differential equations, whose integration can not be hoped for in general. So having propounded the differential equation

$$\frac{\mathrm{d}u}{\sqrt{1-u^4}} = \frac{\mathrm{d}z}{\sqrt{1-z^4}},$$

except the case of the integral $u = z$, which is obvious, we know that it is satisfied by $u = -\sqrt{\frac{1-zz}{1+zz}}$. Therefore, since in general the integration involves an arbitrary constant, say C, u will be equal to a certain function of the quantity z and C; it will nevertheless be of such a nature that for a certain value of C we have $u = z$ and for another value of C we find $u = -\sqrt{\frac{1-zz}{1+zz}}$. Therefore, two values exist, which, if attributed to this constant, convert that function into that so simple algebraic expression.
In like manner having propounded this equation

$$\frac{\mathrm{d}u}{\sqrt{1-u^4}} = \frac{2\mathrm{d}z}{\sqrt{1-z^4}},$$

we have two values, which we know to satisfy it,

$$u = \frac{2z\sqrt{1-z^4}}{1+z^4} \quad \text{and} \quad u = \frac{-1+2zz+z^4}{1+2zz-z^4}$$

and in like manner we taught to exhibit two values, which satisfy this equation in general

$$\frac{m\mathrm{d}u}{\sqrt{1-u^4}} = \frac{n\mathrm{d}z}{\sqrt{1-z^4}},$$

whence the way to find the general integrals of these formulas seems to be paved almost completely.

Further, what was mentioned about the ellipse and the hyperbola, yields the following special integrations of differential equations.
For, having propounded this equation of § 3

$$\mathrm{d}x\sqrt{\frac{1-nxx}{1-xx}}+\mathrm{d}u\sqrt{\frac{1-nuu}{1-uu}}=(x\mathrm{d}u+u\mathrm{d}x)\sqrt{n},$$

we know that it is satisfied by this integral equation

$$1-nxx-nuu+nuuxx=0.$$

This equation taken from § 5

$$\mathrm{d}x\sqrt{\frac{1-nxx}{1-xx}}+\mathrm{d}u\sqrt{\frac{1-nuu}{1-uu}}=n(x\mathrm{d}u+u\mathrm{d}x)$$

was found to be satisfied by this equation

$$1-xx-uu+nuuxx=0.$$

Further, the following equation derived from the hyperbola § 14

$$\mathrm{d}x\sqrt{\frac{nxx-1}{xx-1}}+\mathrm{d}u\sqrt{\frac{nuu-1}{uu-1}}=(x\mathrm{d}u+u\mathrm{d}x)\sqrt{n}$$

is also satisfied by

$$1-nxx-nuu+nuuxx=0,$$

which certainly agrees with the first derived from the ellipse, since

$$\sqrt{\frac{nxx-1}{xx-1}}=\sqrt{\frac{1-nxx}{1-xx}}.$$

But hence it is easy to conclude that this equation

$$\mathrm{d}x\sqrt{\frac{f-gxx}{h-kxx}}+\mathrm{d}u\sqrt{\frac{f-guu}{h-kuu}}=(x\mathrm{d}u+u\mathrm{d}x)\sqrt{\frac{g}{h}}$$

is satisfied by this special integral equation

$$fh-gh(xx+uu)+gkxxuu=0,$$

but this other equation

$$\mathrm{d}x\sqrt{\frac{f-gxx}{n-kxx}}+\mathrm{d}u\sqrt{\frac{f-guu}{n-kuu}}=(x\mathrm{d}u+u\mathrm{d}x)\frac{g}{\sqrt{fk}}$$

is satisfied by this special integral equation

$$fh-fk(xx+uu)+gkxxuu=0.$$

These are the results I considered had to be explained, since the seem to provide me some motivation to expand the limits of Analysis even further.