General Solution of certain Diophantine problems which usually seem to admit only special solutions *

Leonhard Euler

§1 Diophantine analysis, which is about solving undetermined problems by rational or even integer numbers, usually treats two kinds of problems, the difference of which is mostly in the nature of the solution. For, the one problems are of such a nature that general solutions can be exhibited which contain completely all satisfactory solutions; the other problems on the other hand only admit particular solutions, or at least by known methods it is only possible to find such solutions, such that, except the numbers which might be found, infinitely many others also solving the problem exist which are not contained in the found solution. Here it is convenient to note in general that problems of the first class a resolved a lot more easily than those which are referred to the other class, which in most cases require exceptional ingenuity together with extraordinary artifices, in which the very great power of this analysis is seen. For this reason Diophantine problems seem to be have to be divided into these two classes.

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§**2** Diophantus himself only gave the most simple solutions of all questions he considered and in most cases was contented to have indicated the numbers in which the one solution is contained. But on the other hand his method is not to be considered to be restricted to these very special solutions; for, since at that time the use of letters by which the indefinite numbers are denoted had not been received yet, further-extending solutions of this kind, as they are usually exhibited now, could not be expected from him; nevertheless, the methods which he used to solve a given problem extend as far as those which are used today; yes, we are even forced to confess that hardly anything in this branch of analysis has been found, the clear traces of which are not already detected in Diophantus' work. Therefore, since the apparent particularity of Diophantus' solutions is no obstruction, the disparity mentioned above is already manifestly seen in Diophantus' work, if we consider his methods; the ones of them are of such a nature that they can give completely all solutions which can solve the problem, the others on the other hand only yield some solutions or, even though they can be increased to infinity, nevertheless innumerable others which are also satisfactory are not contained.

§**3** An example of a problem the general solution of which can be exhibited is provided by the ordinary question in which one asks for two square numbers the sum of which is a square, or having taken x and y for the roots of these squares that $xx+yy$ is a square number. For, having taken three numbers a, p and q arbitrarily one will have this general solution

$$y = 2apq \quad \text{and} \quad x = a(pp-qq);$$

for, from these values it results

$$\sqrt{xx+yy} = a(pp+qq).$$

On this solution it is to be noted that there are no numbers to be substituted for x and y the sum of which becomes a square which are not at the same time contained in this formula. And this generality is not only seen from this that for the three letters a, p and q any numbers can be taken, whence already an infinite multitude of solutions is obtained, but the investigation of these formulas also reveals that there is no solution which is not comprehended by them. But on the other hand the last criterion is a lot more certain than the first, since often many indefinite letters can enter the solution and nevertheless the solution is not rendered general.

§**4** But the nature of the investigation in this example shows us the universality clearly. For, since $\sqrt{xx+yy}$ must be a rational number, it will certainly be greater than x; therefore, set it $= x+z$. But then, whatever the ratio of y to z is, one will be able to put $z = \frac{p}{q}y$ and this way the generality is not lost. But having put $\sqrt{xx+yy} = x + \frac{p}{q}y$ and having taken the squares we will have

$$xx + yy = xx + \frac{2q}{p}xy + \frac{pp}{qq}yy.$$

Having cancelled the term xx on both sides and having divided the remainer by y it will result

$$y = \frac{2q}{p}x + \frac{qq}{pp}y \quad \text{or} \quad (pp-qq)y = 2pqx.$$

Therefore, it will be

$$\frac{x}{y} = \frac{pp-qq}{2pq}$$

and hence x and y are either both multiples or both submultiples of the numbers $pp-qq$ and $2pq$. Therefore, having taken a for the general index of the multiple or submultiples we will obtain

$$y = 2apq \quad \text{and} \quad x = a(pp-qq),$$

and because of $z = \frac{q}{p}y = 2aqq$ it will be

$$x + z = \sqrt{xx+yy} = a(pp+qq).$$

§**5** But an example of a problem the general solution of which can not by exhibited by known methods is the question on finding three cubes the sum of which is a cube: or three numbers x, y and z are to be found such that

$$x^3 + y^3 + z^3 = \text{cube}.$$

This problem was solved both by Diophant and by more recent authors in many ways and certainly in such a way that an infinite multitude of solutions has been exhibited; and nevertheless no solution extends so far that it contains completely all cases solving this question. In this problem even either one cube x^3 or two $x^3 + y^3$ can be considered as given, whence either the two remaining cubes or just one must be found that the sum becomes a cube; but however the solution is found, it is nevertheless very particular.

§**6** To see this more clearly, let us briefly mention the usual solutions here. Therefore, first let the two cubes a^3 and b^3 be given and the third x^3 must be found that the sum of all three

$$a^3 + b^3 + x^3$$

becomes a cube again. But it is manifest that the cube root of this cube will be greater than x; but even though one sets it $= x + v$, nevertheless a quadratic equation for x results and so the difficulty is not reduced. Therefore, one usually sets $x = p - b$ that the sum of three cubes becomes

$$a^3 + 3bbp - 3bpp + p^3 = \text{cube} = v^3,$$

and by this the generality of the solution is certainly not restricted. But further a cube of such a kind must be assumed that the unknown p can be exhibited by a simple equation and hence rationally. But it is obvious that this can be done in two ways. For, first having taken $v = a + p$ it will become

$$a^3 + 3bbp - 3bpp + p^3 = a^3 + 3aap + 3app + p^3;$$

since here the terms a^3 and p^3 cancel each other, the remainder divided by $3p$ gives

$$bb - bp = aa + ap \quad \text{and hence} \quad p = \frac{bb - aa}{a + b} = b - a,$$

whence $x = p - b = -a$, in which case of course

$$a^3 + b^3 + x^3 = a^3 + b^3 - a^3 = b^3 = \text{a cube}.$$

§**7** But that this solution is a very particular one is clear from the assumption of the value $v = a + p$, since it can certainly happen that the quantity

$$a^3 + 3bbp - 3bpp + p^3$$

is a cube, the root of which is not $a + p$, such that by this the solution is highly restricted, whence it happens that it even exhibited a value for p and hence for x which is to be considered to have not even yielded a suitable solution, since we found $x = -a$, which case is so obvious immediately that it can certainly not even admitted as a solution. Therefore, usually another value is assumed for v, such a one nevertheless that the invention of p leads to a simple equation what happens by putting $v = a + \frac{bb}{aa}p$; for, it will be

$$a^3 + 3bbp - 3bpp + p^3 = a^3 + 3bbp + \frac{3b^4}{a^3}pp + \frac{b^6}{a^6}p^3,$$

which, having cancelled the terms $a^3 + 3bbp$ on both sides, divided by pp gives

$$-3b + p = \frac{3b^4}{a^3} + \frac{b^6}{a^6}p \quad \text{and} \quad p = \frac{3a^6b + 3a^3b^4}{a^6 - b^6}.$$

§**8** Therefore, since we hence found

$$p = \frac{3a^3b(a^3 + b^3)}{a^6 - b^6} = \frac{3a^3b}{a^3 - b^3},$$

it will be

$$x = p - b = \frac{2a^3b + b^4}{a^3 - b^3} = \frac{b(2a^3 + b^3)}{a^3 - b^3},$$

which is the root of the third cube to be added to the two given ones, i.e. $a^3 + b^3$, that the sum becomes a cube. But the cube root of the sum by assumption will be

$$= v = a + \frac{bb}{aa}p = a + \frac{3ab^3}{a^3 - b^3}$$

or

$$v = \frac{a^4 + 2ab^3}{a^3 - b^3} = \frac{a(a^3 + 2b^3)}{a^3 - b^3}.$$

Therefore, whatever number had been assumed for a and b, hence one will have three cubes the sum of which is a cube. Of course, these will be

$$a^3 + b^3 + \left(\frac{b(2a^3 + b^3)}{a^3 - b^3}\right)^3 = \left(\frac{a(a^3 + 2b^3)}{a^3 - b^3}\right)^3.$$

But that even this solution is a very special one is perspicuous from the investigation itself, since we completely arbitrarily assumed the root of the sum of the three cubes $v = a + \frac{bb}{aa}p$, although without a doubt it can obtain infinitely many other values.

§**9** But further, given two cubes the third cube is found which combined with them produces a cube; but it is obvious that infinitely many cubes of this kind exist. For, if $a = 4$ and $b = 3$, the root of the third cube hence results as

$$x = \frac{3(2 \cdot 64 + 27)}{64 - 27} = \frac{465}{37} \quad \text{and} \quad v = \frac{472}{37},$$

such that

$$4^3 + 3^3 + \left(\frac{465}{37}\right)^3 = \left(\frac{472}{37}\right)^3.$$

But we know the cube of five added to these cubes $4^3 + 5^3$ also produces a cube, of course the cube of six, or that

$$3^3 + 4^3 + 5^3 = 6^3,$$

which case is nevertheless not contained in this solution. Hence if, in order to solve this problem that $x^3 + y^3 + z^3 = v^3$, somebody says that one has to take

$$x = a, \quad y = b \quad \text{and} \quad z = \frac{b(2a^3 + b^3)}{a^3 - b^3}$$

and that it will then be $v = \frac{a(a^3+2b^3)}{a^3-b^3}$, these formulas are certainly satisfactory, but even though because of the two completely arbitrary numbers a and b hence infinitely many triples of cubes can be exhibited the sum of which is a square, nevertheless infinitely many other triples of cubes exist achieving the same which are not contained in these formulas, as this case $x = 3$, $y = 4$ and $z = 5$, for which $v = 6$.

§**10** A further-extending solution is found, if just one of the three cubes is assumed to be given such that it has to be

$$a^3 + x^3 + y^3 = v^3.$$

To this end, put $x = pu + r$ and $y = qu - r$, which position does not restrict the generality, and it will be

$$a^3 + 3rr(p+q)u + 3r(pp - qq)uu + (p^3 + q^3)u^3 = v^3.$$

But that hence the quantity u can be defined in rational terms, assume $v = a + \frac{rr}{aa}(p+q)u$, by which position the solution is restricted vehemently, of course; but from it one will obtain

$$v^3 = a^3 + 3rr(p+q)u + \frac{3r^4}{a^3}(p+q)^2uu + \frac{r^6}{a^6}(p+q)^3u^3.$$

Therefore, having cancelled the terms $a^3 + 3rr(p+q)u$ on both sides and having divided the remainer by $(p+q)uu$ this equation will emerge

$$3r(p-q) + (pp-pq+qq)u = \frac{3r^4}{a^3}(p+q) + \frac{r^6}{a^6}(p+q)^2u,$$

from which one finds

$$u = \frac{3a^3r^4(p+q) - 3a^6r(p-q)}{a^6(pp-pq+qq) - r^6(p+q)^2}.$$

§11 Therefore, having found this value for u, it will be

$$x = pu + r = \frac{3a^3pr^4(p+q) - a^6r(2pp-2pq-qq) - r^7(p+q)^2}{a^6(pp-pq+qq) - r^6(p+q)^2},$$

$$y = qu - r = \frac{3a^3qr^4(p+q) - a^6r(pp+2pq-2qq) + r^7(p+q)^2}{a^6(pp-pq+qq) - r^6(p+q)^2}$$

and

$$v = a + \frac{rr}{aa}(p+q)u = \frac{a^7(pp-pq+qq) - 3a^4r^3(pp-qq) + 2ar^6(p+q)^2}{a^6(pp-pq+qq) - r^6(p+q)^2}.$$

Therefore, since the four letters a, p, q and r can be assumed arbitrarily, this solution extends infinitely times further than the preceding one, where only two letters were arbitrary. But nevertheless it is to be noted that just the ratio of the letters p and q enters the calculation such that hence the arbitrary letters are reduced to only three; despite this, because of the restriction concerning the root v, the this solution is to be considered as a particular one such that triples of cubes exist which are not contained in these formulas. But the preceding solution emerges from this one for $p = 0$ such that this one is infinitely many times more general.

§12 But we will obtain a still more general solution, if we assume none of the three cubes as known or in general look for x, y and z that

$$x^3 + y^3 + z^3 = v^3.$$

To this end, put

$$x = pt + u, \quad y = -pt + qu \quad \text{and} \quad z = t - qu,$$

by which positions still nothing is restricted; but after the substitution it will be

$$\begin{aligned} t^3 + \;\; & 3ppttu + \;\; 3ptuu + u^3 = v^3. \\ & + 3ppqttu - 3pqqtuu \\ & - \;\; 3qttu + \;\; 3qqtuu \end{aligned}$$

Now assume $v = t + u$, whence certainly a huge restriction results, and having divided the equation by $3tu$ one will find

$$(pp + ppq - q)t + (p - pqq + qq)u = t + u$$

or

$$\frac{t}{u} = \frac{-1 + p + qq - pqq}{1 + q - pp - ppq};$$

therefore, one will have to take

$$t = n(-pqq + qq + p - 1) \quad \text{and} \quad u = n(-ppq - pp + q + 1),$$

whence one finds

$$\begin{aligned} x &= n(-ppqq + pqq - ppq - p + q + 1), \\ y &= n(p + q - pp + qq - ppq - pqq), \\ z &= n(+ppqq - pqq + ppq + p - q - 1), \\ v &= n(-pqq - ppq - pp + qq + p + q). \end{aligned}$$

But hence $z = -x$ and $v = y$ which case is per se obvious.

§13 But in the following way a further-extending solution is found. Put

$$x = mt + pu, \quad y = nt + qu \quad \text{and} \quad z = -nt + ru$$

and it will be

$$\begin{aligned} x^3 + y^3 + z^3 = m^3t^3 &+ 3mmpttu + 3mpptuu + p^3u^3; \\ &+ 3nnq \qquad + 3nqq \qquad + q^3 \\ &+ 3nnr \qquad - 3nrr \qquad + r^3 \end{aligned}$$

since which sum must be a cube $= v^3$, put

$$v = mt + \frac{mmp + nn(q+r)}{mm}u$$

and dividing by uu it will be

$$\begin{aligned} &3t(mpp + n(qq - rr)) + u(p^3 + q^3 + r^3) \\ &= \frac{3t}{m^3}(mmp + nn(q+r))^2 + \frac{u}{m^6}(mmp + nn(q+r))^3 \end{aligned}$$

and so having neglected the common factor which is arbitrary it will be

$$\begin{aligned} t &= m^6(p^3 + q^3 + r^3) - (mmp + nn(q+r))^3, \\ u &= 3m^3(mmp + nn(q+r))^2 - 3m^6(mpp + n(qq - rr)); \end{aligned}$$

if these forms are divided by the common factor $q + r$ again, it results

$$\begin{aligned} t &= m^6(qq - qr + rr) - 3m^4nnpp - 3mmn^4p(q+r) - n^6(q+r)^2, \\ u &= -3m^6n(q-r) + 6m^5nnp + 3m^3n^4(q+r). \end{aligned}$$

§14 Hence now for x, y, z the following expressions emerge:

$$\begin{aligned} x &= m^7(qq - qr + rr) - 3m^6np(q-r) + 3m^5nnpp - mn^6(q+r)^2, \\ y &= -m^6n(2qq - 2qr - rr) + 6m^5nnpq - 3m^4n^3pp + 3m^3n^4q(q+r) \\ &\quad -3mmn^5p(q+r) - n^7(q+r)^2, \\ z &= +m^6n(-qq - 2qr + 2rr) + 6m^5nnpr + 3m^4n^3pp + 3m^3n^4r(q+r) \end{aligned}$$

$$+3mmn^5p(q+r)+n^7(q+r)^2,$$

the sum of which cubes is again a cube having the root v that

$$v=m^7(qq-qr+rr)-3m^6np(q-r)+3m^5nnpp-3m^4n^3(qq-rr)$$
$$+6m^3n^4p(q+r)+2mn^6(q+r)^2.$$

Indeed, these numbers can also be exhibited in the following way:

$$\begin{array}{lllllll}
x= & +3m^5n^2pp & -3m^6npq & +3m^6npr & +m^7qq & -m^7qr & +m^7rr, \\
 & & & & -mn^6 & -2mn^6 & -mn^6 \\
y= & -3m^4n^3pp & +6m^5n^2pq & -3m^2n^5pr & -2m^6nqq & +2m^6nqr & +m^6nrr, \\
 & & -3m^2n^5 & & +3m^3n^4 & +3m^3n^4 & -n^7 \\
 & & & & -n^7 & -2n^7 & \\
z= & +3m^4n^3pp & +3m^2n^5pq & +6m^5n^2pr & -m^6nqq & -2m^6nqr & +2m^6nrr, \\
 & & & +3m^2n^5 & +n^7 & +3m^3n^4 & +3m^3n^4 \\
 & & & & & +2n^7 & +n^7 \\
v= & +3m^5n^2pp & -3m^6npq & +3m^6npr & +m^7qq & -m^7qr & +m^7rr. \\
 & & +6m^3n^4 & +6m^3n^4 & -3m^4n^3 & +4mn^6 & +3m^4n^3 \\
 & & & & +2mn^6 & & +2mn^6.
\end{array}$$

Having substituted which values it actually is

$$x^3+y^3+z^3=v^3.$$

§**15** If each of these numbers is additionally multiplied by an indefinite coefficient, these formulas will contain six arbitrary letters, which will certainly be reduced to four, whence they seem to extend very far and contain completely all cases; but nevertheless, from the solution in which we attributed a value to v depending on the letters x, y and z, it is understood that these formulas can only be considered as particular solutions. Furthermore, also by other positions other solutions are found which are more suitable for certain cases;

for, then one also has methods to find other particular solutions from any found solution. Nevertheless, by all artifices, if they are not iterated infinitely many times, no solution which can be considered as general can be obtained. Yes, it has even almost been believed that problems of this kind are of such a nature that they do not admit a general solution at all, from which the following solution of the problem, which is indeed general, is most remarkable and seems apt to expand the limits of Diophantine analysis.

PROBLEM

§16 *To find all triples of cubes the sum of which is a cube.*

SOLUTION

Let A, B, C be the roots of the three cubes and D the cube root of their sum that

$$A^3 + B^3 + C^3 = D^3,$$

to which equation this form shall be attributed

$$A^3 + B^3 = C^3 - D^3.$$

Now put

$$A = p + q, \quad B = p - q, \quad C = r - s \quad \text{and} \quad D = r + s,$$

by which position the generality of the solution is not restricted. But hence

$$A^3 + B^3 = 2p^3 + 6pqq \quad \text{and} \quad D^3 - C^3 = 2s^3 + 6rrs$$

and so it will be

$$p(pp + 3qq) = s(ss + 3rr),$$

which equation can only hold, if $pp + 3qq$ and $ss + 3rr$ have a common divisor. But it is known that such numbers have no other divisors than those which are of the same form; but to obtain this, instead of the four letters p, q, r and s introduce six new ones this way

$$p = ax + 3by, \qquad s = 3cy - dx,$$
$$q = bx + \ ay, \qquad r = \ dy + cx,$$

whence the generality of the solution is restricted a lot less. But hence it will be

$$pp + 3qq = (aa + 3bb)(xx + 3yy)$$

and

$$ss + 3rr = (dd + 3cc)(xx + 3yy)$$

and our equation divided by $xx + 3yy$ will take the following form

$$(ax + 3by)(aa + 3bb) = (3cy - dx)(dd + 3cc),$$

by which we already obtained that the letters x and y have only one dimension and can hence be defined in rational terms. For, since

$$\frac{x}{y} = \frac{-3b(aa + 3bb) + 3c(dd + 3cc)}{a(aa + 3bb) + d(dd + 3cc)},$$

put

$$x = -3nb(aa + 3bb) + 3nc(dd + 3cc),$$
$$y = na(aa + 3bb) + nd(dd + 3cc).$$

From these values the letters p, q, r, s etc. are defined in such a way that

$$p = 3n(ac + bd)(dd + 3cc),$$
$$q = n(3bc - ad)(dd + 3cc) - n(aa + 3bb)^2,$$
$$r = n(dd + 3cc)^2 - n(3bc - ad)(aa + 3bb),$$
$$s = 3n(ac + bd)(aa + 3bb).$$

And hence finally the roots of the cubes in question, i.e. A, B, C, D will be

$$A = n(3ac + 3bc - ad + 3bd)(dd + 3cc) - n(aa + 3bb)^2,$$
$$B = n(3ac - 3bc + ad + 3bd)(dd + 3cc) + n(aa + 3bb)^2,$$
$$C = n(dd + 3cc)^2 - n(3ac + 3bc - ad + 3bd)(aa + 3bb),$$
$$D = n(dd + 3cc)^2 + n(3ac - 3bc + ad + 3bd)(aa + 3bb),$$

by which values one obtains that

$$A^3 + B^3 + C^3 = D^3;$$

and since the solution was not restricted in any way, it obviously extends as far as possible and contains all triples of cubes the sum of which is a cube again.

Corollary 1

§17 Let us hence derive more special solutions and first let $d = 0$ and it will be

$$A = 9n(a + b)c^3 - n(aa + 3bb)^2,$$
$$B = 9n(a - b)c^3 + n(aa + 3bb)^2,$$
$$C = 9nc^4 - 3nc(a + b)(aa + 3bb),$$
$$D = 9nc^4 + 3nc(a - b)(aa + 3bb).$$

If here one further sets $b = a$, it will be

$$A = 18nac^3 - 16na^4, \quad B = 16na^4, \quad C = 9nc^4 - 24na^3c$$

and

$$D = 9nc^4;$$

but if $b = -a$, one will find

$$A = -16na^4, \quad B = 18nac^3 + 16na^4, \quad C = 9nc^4$$

and

$$D = 9nc^4 + 24na^3c.$$

COROLLARY 2

§18 Now let us put $c = 0$ and it will be

$$\begin{aligned} A &= n(3b-a)d^3 - n(aa+3bb)^2, \\ B &= n(3n+a)d^3 + n(aa+3bb)^2, \\ C &= nd^4 - nd(3b-a)(aa+3bb), \\ D &= nd^4 + nd(3b+a)(aa+3bb). \end{aligned}$$

If one further sets $b = a$, it will be

$$A = 2nad^3 - 16na^4, \quad B = 4nad^3 + 16na^4, \quad C = nd^4 - 8na^3d, \\ D = nd^4 + 16na^3d;$$

but if $a = -b$, it will be

$$A = 4nbd^3 - 16nb^4, \quad B = 2nbd^3 + 16nb^4, \quad C = nd^4 - 16nb^3d, \\ D = nd^4 + 8nb^3d.$$

COROLLARY 3

§19 Now let $b = 0$ and our formulas will become

$$\begin{aligned} A &= na(3c-d)(dd+3cc) - na^4, \\ B &= na(3c+d)(dd+3cc) + na^4, \\ C &= n(dd+3cc)^2 - na^3(3c-d), \\ D &= n(dd+3cc)^2 + na^3(3c+d). \end{aligned}$$

If one now furthermore sets $d = c$, it will be

$$A = 8nac^3 - na^4, \quad B = 16nac^3 + na^4, \quad C = 16nc^4 - 2na^3c, \\ D = 16nc^4 + 4na^3c;$$

but if $d = -c$, it will be

$$A = 16nac^3 - na^4, \quad B = 8nac^3 + na^4, \quad C = 16nc^4 - 4na^3c, \\ D = 16nc^4 + 2na^3c.$$

COROLLARY 4

§20 Finally, put $a = 0$ and we will obtain

$$\begin{aligned} A &= 3nb(c+d)(dd+3cc) - 9nb^4, \\ B &= 3nb(d-c)(dd+3cc) + 9nb^4, \\ C &= n(dd+3cc)^2 - 9nb^3(c+d), \\ D &= n(dd+3cc)^2 + 9nb^3(d-c). \end{aligned}$$

If ones further sets $d = c$, it will be

$$A = 24nbc^3 - 9nb^4, \quad B = 9nb^4, \quad C = 16nc^4 - 18nb^3c,$$
$$D = 16nc^4;$$

but if $c = -d$, one will have

$$A = -9nb^4, \quad B = 24nbd^3 + 9nb^4, \quad C = 16nd^4,$$
$$D = 16nd^4 + 18nb^3d.$$

COROLLARY 5

§21 If one of the numbers A, B, C becomes negative, which can be done arbitrarily, as if $A = -E$, it will be $B^3 + C^3 = D^3 + E^3$ and so at the same time we solved this problem in most general manner, in which two pairs of cubes are in question, the sum of which is equal to each other. But if two roots result as negative, as, e.g., $A = -E$ and $B = -F$, it will be $C^3 = D^3 + E^3 + F^3$ and so one will again have the solution of our problem.

SCHOLIUM 1

§22 The most simple formulas exhibited in these corollaries are reduced to these two, if in Corollary 3 one writes a for $2a$ and $n = \frac{n}{16}$, and in Corollary 1 $\frac{1}{2}a$ for a:

$$A = nac^3 - na^4, \qquad A = 9nac^3 - na^4,$$
$$B = 2nac^3 + na^4, \qquad B = na^4,$$
$$C = nc^4 - na^3c, \qquad C = 9nc^4 - 3na^3c,$$
$$D = nc^4 + 2na^3c, \qquad D = 9nc^4,$$

the first of which agrees with the one found above in § 8; but the other yields this most simple case $A = 8$, $B = 1$, $C = 6$ and $D = 9$ such that

$$1^3 + 6^3 + 8^3 = 9^3.$$

Scholium 2

§**23** At first sight the general formulas found in the problem seem to extend not further than the formulas exhibited above (§ 14), since both contain five arbitrary letters and those additionally can receive a common coefficient such that they even seem to be more general. Nevertheless, the nature of the solution shows that the formulas found in the problem are the most general ones, while the above ones are very severely restricted. To see this restriction more clearly, from § 13 consider the position

$$v = mt + \frac{mmp + nn(q+r)}{mm}u = mt + pu + \frac{nn}{mm}(q+r)u.$$

But on the other hand $mt + pu = x$ and $y + z = (q+r)u$ such that the position is

$$v = x + \frac{nn}{mm}(y+z).$$

Hence that $x^3 + y^3 + z^3 = v^3$ in that solution it is assumed to be

$$\frac{v-x}{y+z} = \frac{nn}{mm} = \text{square;}$$

and so that does not extend to other cases than those in which $\frac{v-x}{y+z}$ or $\frac{D-A}{B+C}$ is a square number. Therefore, as often as $\frac{D-A}{B+C}$ is not a square, the case is not contained in the above formulas; but that cases of this kind exist is also clear from the example $1^3 + 6^3 + 8^3 = 9^3$ in which neither $\frac{9-1}{6+8}$ nor $\frac{9-6}{1+8}$ nor $\frac{9-8}{1+6}$

becomes a square. But this solution of the problem is not restricted in such a way, since

$$\frac{D-C}{A+B} = \frac{s}{p} = \frac{pp+3qq}{ss+3rr} = \frac{aa+3bb}{dd+3cc},$$

whence from the general solution only the cases in which $\frac{aa+3bb}{dd+3cc}$ is a square number, are contained in the above formulas of § 14; hence the highest generality of our solution shows very clearly.

Scholium 3

§24 But the nature of this problem requires just integer numbers and certainly such which are mutually prime; for, if it was $A^3+B^3+C^3=D^3$, then also all equal multiples and equal submultiples of the numbers A, B, C, D solve the problem; and hence it will suffice to have known only the cases, in which the numbers A, B, C, D are both integer numbers and mutually prime. To this end having taken any either positive or negative numbers for a, b, c, d, hence first form

$$x = 3n(c(dd+3cc) - b(aa+3bb)),$$
$$y = n(d(dd+3cc) + a(aa+3bb))$$

and take such a fraction for n that x and y become mutually prime integers. From these further form

$$p = ax+3by, \quad q = bx-ay, \quad r = dy+cx \quad \text{and} \quad s = 3cy-dx,$$

which must again be lowered by the common divisor, if they have one. Hence finally one will have

$$A = p+q, \quad B = p-q, \quad C = r-s \quad \text{and} \quad D = r+s$$

and so it will be

$$A^3+B^3+C^3 = D^3.$$

And the cases in which one of these numbers becomes negative will at the same time yield all solutions in which the sum of two cubes is equal to the sum of two other cubes.

In this calculation it will be convenient to have a table of the numbers of the form $mm + 3nn$ readily available, whence thereafter the formulas $aa + 3bb$ and $dd + 3cc$ can be taken.

Table of Numbers of the form $mm + 3mm$

The number$n$

m	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0	0	3	12	27	48	75	108	147	192	243	300	363	432	507	588	675	768	867	972
1	1	4	13	28	49	76	109	148	193	244	301	364	433	508	589	676	769	868	973
2	4	7	16	31	52	79	112	151	196	247	304	367	436	511	592	679	772	871	976
3	9	12	21	36	57	84	117	156	201	252	309	372	441	516	597	684	777	876	981
4	16	19	28	43	64	91	124	163	208	259	316	379	448	523	604	691	784	883	988
5	25	28	37	52	73	100	133	172	217	268	325	388	457	532	613	700	793	892	997
6	36	39	48	63	84	111	144	183	228	279	336	399	468	543	624	711	804	903	
7	49	52	61	76	97	124	157	196	241	292	349	412	481	556	637	724	817	916	
8	64	67	76	91	112	139	172	211	256	307	364	427	496	571	652	739	832	931	
9	81	84	93	108	129	156	189	228	273	324	381	444	513	588	669	756	849	948	
10	100	103	112	127	148	175	208	247	292	343	400	463	532	607	688	775	868	967	
11	121	124	133	148	169	196	229	268	313	364	421	484	553	628	709	796	889	988	
12	144	147	156	171	192	219	252	291	336	387	444	507	576	651	732	819	912		
13	169	172	181	196	217	244	277	316	361	412	469	532	601	676	757	844	937		
14	196	199	208	223	244	271	304	343	388	439	496	559	628	703	784	871	964		
15	225	228	237	252	273	300	333	372	417	468	525	588	657	732	813	900	993		
16	256	259	268	283	304	331	364	403	448	499	556	619	688	763	841	931			
17	289	292	301	316	337	364	397	436	481	532	589	652	721	796	877	964			
18	324	327	336	351	372	399	432	471	516	567	624	687	756	831	912	999			
19	361	364	373	388	409	436	469	508	553	604	661	724	793	868	949				
20	400	403	412	427	448	475	508	547	592	643	700	763	832	907	988				
21	441	444	453	468	489	516	549	588	633	684	741	804	873	948					
22	484	487	496	511	532	559	592	631	676	727	784	847	916	991					
23	529	532	541	556	577	604	637	676	721	772	829	892	961						
24	576	579	588	603	624	651	684	723	768	819	876	939							
25	625	628	637	652	673	700	733	772	817	868	925	988							
26	676	679	688	703	724	751	784	823	868	919	976								
27	729	732	741	756	777	804	837	876	921	972									
28	784	787	796	811	832	859	892	931	976										
29	841	844	853	868	889	916	949	988											
30	900	903	912	927	948	975													
31	961	964	973	988															

SCHOLIUM 4

§25 From this table the numbers for $aa+3bb$ and $dd+3cc$ can be assumed arbitrarily, whence one will have the values of the letters a, b, c, d, which can be taken so positively as negatively. But if smaller numbers are desired for A, B, C, D, it will be convenient to have taken values of such a kind for $aa+3bb$ and $3cc+dd$ which have a common divisor. Therefore, set

$$aa+3bb=mk \quad \text{and} \quad dd+3cc=nk.$$

But then on the other hand let

$$ac+bd=f \quad \text{and} \quad 3bc-ad=g$$

and hence it will be

$$\begin{aligned} A &= n(3f+g)-mmk, \\ B &= n(3f-g)+mmk, \\ C &= nnk-m(3f+g), \\ D &= nnk+m(3f-g), \end{aligned}$$

where it is to be noted, whatever values were found for f and g, that they can be taken so positively as negatively, because of the ambiguous numbers a, b, c, d; hence for each case one will have the following determinations

$$\begin{aligned} \text{either} \quad f &= \pm(ac+bd), \qquad g = \pm(3bc-ad) \\ \text{or} \quad f &= \pm(ac-bd), \qquad g = \pm(3bc+ad). \end{aligned}$$

But it is plain, if while g stays negative f is taken negatively, that the same numbers will result just in permuted order, whence it suffices to have assumed only positive values for f. Furthermore, it is manifest, if $m=n$ or

$$aa+3bb=dd+3cc,$$

that then it will be $A=-C$ and $D=B$, whence one will have to exclude these cases. Finally, if $f=0$, $A=-B$ and $C=D$, these cases are therefore also to be omitted. Often it also happens that either for a and b or for c and d or for each of both pairs several values results, from which the number of solutions becomes even greater.

EXAMPLE 1

§26 Take $aa + 3bb = 19$, it will be $a = 4$ and $b = 1$, but then take $dd + 3cc = 76$ and it will be

$$\begin{aligned} \text{either} \quad & d = 1, \quad c = 5 \\ \text{or} \quad & d = 7, \quad c = 3 \\ \text{or} \quad & d = 8, \quad c = 2. \end{aligned}$$

But then $m = 1$, $n = 4$ and $k = 19$. But for f and g the following values will result

$$\begin{array}{llllll} \text{I.} \quad f = 21, & g = \pm 11, & \text{II.} \quad f = 19, & g = \pm 19, & \text{III.} \quad f = 19, & g = \pm 19, \\ \text{IV.} \quad f = 5, & g = \pm 37, & \text{V.} \quad f = 16, & g = \pm 26, & \text{VI.} \quad f = 0, & g = \pm 38, \end{array}$$

whence the third and the sixth case are to be excluded. And hence it will be

$$\begin{aligned} A &= 12f + 4g - 19, \\ B &= 12f - 4g + 19, \\ C &= 304 - 3f - g, \\ D &= 304 + 3f - g. \end{aligned}$$

Therefore, hence for the first value $f = 21$ and $g = \pm 11$ one will find

$$\begin{aligned} A &= 233 \pm 44, \\ B &= 271 \mp 44, \\ C &= 241 \mp 11, \\ D &= 367 \mp 11, \end{aligned}$$

therefore,

$$\begin{array}{lll}
\text{for the upper signs} & \text{for the lower signs} & \\
A = 277, & A = 189, \quad \text{or} & A = 3, \\
B = 227, & B = 315, & B = 5, \\
C = 230, & C = 252, & C = 4, \\
D = 356, & D = 378, & D = 6.
\end{array}$$

But the cases II and III dividing the formulas by 19, because of $f = 1 \cdot 19$ and $g = \pm 1 \cdot 19$ will give

$$\begin{aligned}
A &= 11 \pm 4, \\
B &= 13 \mp 4, \\
C &= 13 \mp 1, \\
D &= 19 \mp 1,
\end{aligned}$$

therefore,

$$\begin{array}{lll}
\text{either} & & \text{or} \\
A = 15, \quad \text{or} & A = 5, & A = 7, \\
B = 9, & B = 3, & B = 17, \\
C = 12, & C = 4, & C = 14, \\
D = 18, & D = 6, & D = 20.
\end{array}$$

Case IV in which $f = 5$ and $g = \pm 37$ gives

$$\begin{aligned}
A &= 41 \pm 148, \\
B &= 79 \mp 148, \\
C &= 289 \mp 37, \\
D &= 319 \mp 37,
\end{aligned}$$

therefore,

$$\begin{array}{llll} & \text{either} & & \text{or} \\ A = \ \ 189, & \text{or} \ A = \ \ 63, & & A = -107, \\ B = - \ 69, & \ \ B = -23, & & B = \ \ 227, \\ C = \ \ 252, & \ \ C = \ \ 84, & & C = \ \ 326, \\ D = \ \ 282, & \ \ D = \ \ 94, & & D = \ \ 356. \end{array}$$

Case V in which $f = 16$ and $g = \pm 26$ gives

$$\begin{aligned} A &= 173 \pm 104, \\ B &= 211 \mp 104, \\ C &= 256 \mp \ 26, \\ D &= 352 \mp \ 26, \end{aligned}$$

therefore,

$$\begin{array}{lll} \text{either} & & \text{or} \\ A = 277, & A = \ \ 69, & \text{or} \ \ A = \ \ 23, \\ B = 107, & B = 315, & \quad B = 105, \\ C = 230, & C = 282, & \quad C = \ \ 94, \\ D = 326, & D = 378, & \quad D = 126. \end{array}$$

Therefore, lo and behold the many triples of cubes found from one position:

$$\begin{array}{ll} 227^3 + 230^3 + 277^3 = 356^3, & 107^3 + 356^3 = 227^3 + 326^3, \\ 107^3 + 230^3 + 277^3 = 326^3, & 23^3 + \ 94^3 = \ 63^3 + \ 84^3, \\ 23^3 + \ 94^3 + 105^3 = 126^3, & \\ 7^3 + \ 14^3 + \ 17^3 = \ 20^3, & \\ 3^3 + \ \ 4^3 + \ \ 5^3 = \ \ 6^3, & \end{array}$$

whence one concludes

$$356^3 - 227^3 = 230^3 + 277^3 = 326^3 - 107^3,$$

likewise

$$126^3 - 105^3 = 63^3 + 84^3 = 23^3 + 94^3.$$

EXAMPLE 2

§**27** Let $aa + 3bb = 28$; it will be

$$\begin{aligned} \text{either} \quad a &= 1, \quad b = 3, \\ \text{or} \quad a &= 4, \quad b = 2, \\ \text{or} \quad a &= 5, \quad b = 1; \end{aligned}$$

but then let $dd + 3cc = 84$; it will be

$$\begin{aligned} \text{either} \quad d &= 3, \quad c = 5, \\ \text{or} \quad d &= 6, \quad c = 4, \\ \text{or} \quad d &= 9, \quad c = 1; \end{aligned}$$

and hence $k = 28$, $m = 1$ and $n = 3$; but then for f and g the following values will result

$$\begin{array}{llllll} \text{I.} \quad f = 14, & g = \pm 42, & \text{II.} \quad f = 4, & g = \pm 48, & \text{III.} \quad f = 22, & g = \pm 30, \\ \text{IV.} \quad f = 14, & g = \pm 42, & \text{V.} \quad f = 28, & g = \pm 0, & \text{VI.} \quad f = 26, & g = \pm 18, \end{array}$$

where it is to be noted that these values I and IV of which are identical result only from the position $a = 1$ and $b = 3$ and the two remaining ones produce the same. Therefore, hence we will have

$$\begin{aligned} A &= 9f + 3g - 28, \\ B &= 9f - 3g + 28, \\ C &= 252 + 3f - g, \\ D &= 252 + 3f - g, \end{aligned}$$

whence the first and fourth, dividing by 14, will give

$$
\begin{aligned}
A &= 7 \pm 9,\\
B &= 11 \mp 9,\\
C &= 15 \mp 3,\\
D &= 21 \mp 3,
\end{aligned}
$$

therefore,

$$
\begin{array}{rrrr@{\qquad}rrrr}
 & \text{either} & & & & \text{or} & & \\
A = & 16 = & 8, & & A = & -\ 2 = & -\ 1, \\
B = & 2 = & 1, & & B = & 20 = & 10, \\
C = & 12 = & 6, & & C = & 18 = & 9, \\
D = & 18 = & 9, & & D = & 24 = & 12.
\end{array}
$$

The second case on the other hand, dividing by 4, gives

$$
\begin{aligned}
A &= 2 \pm 36,\\
B &= 16 \mp 36,\\
C &= 60 \mp 12,\\
D &= 66 \mp 12,
\end{aligned}
$$

therefore,

$$
\begin{array}{rrr@{\qquad}rrr}
 & \text{either} & & & \text{or} & \\
A = & 38 = & 19, & A = & -\ 34 = & -\ 17, \\
B = & -\ 20 = & -\ 10, & B = & 52 = & 26, \\
C = & 48 = & 24, & C = & 72 = & 36, \\
D = & 54 = & 27, & D = & 78 = & 39.
\end{array}
$$

The third case divided by 2 gives

$$
\begin{aligned}
A &= 85 \pm 45, \\
B &= 113 \mp 45, \\
C &= 93 \mp 15, \\
D &= 159 \mp 15,
\end{aligned}
$$

therefore,

$$
\begin{array}{rr}
\text{either} & \text{or} \\
A = 130 = 65, & A = 40 = 20, \\
B = 68 = 34, & B = 158 = 79, \\
C = 78 = 39, & C = 108 = 54, \\
D = 144 = 72, & D = 174 = 87.
\end{array}
$$

The fifth cases divided by 28 gives

$$
\begin{aligned}
A &= 8 = 4, \\
B &= 10 = 5, \\
C &= 6 = 3, \\
D &= 12 = 6.
\end{aligned}
$$

Finally, the sixth cases divided by 2 gives

$$
\begin{aligned}
A &= 103 \pm 27, \\
B &= 131 \mp 27, \\
C &= 87 \mp 9, \\
D &= 165 \mp 9,
\end{aligned}
$$

therefore,

$$\begin{array}{ll} \text{either} & \text{or} \\ A = 130 = 65 = 5, & A = \ \ 76 = 38, \\ B = 104 = 54 = 4, & B = 158 = 79, \\ C = \ \ 78 = 39 = 3, & C = \ \ 96 = 48, \\ D = 156 = 78 = 6, & D = 174 = 87. \end{array}$$

Therefore, from this example the following formulas result:

$$\begin{array}{lcl} 1^3 + \ 6^3 + \ 8^3 = \ 9^3, & \text{and} & 1^3 + 12^3 = \ 9^3 + \ \ 10^3, \\ 34^3 + 39^3 + 65^3 = 72^3, & & 10^3 + 27^3 = 19^3 + 24^3, \\ 3^3 + \ 4^3 + \ 5^3 = \ 6^3, & & \\ 38^3 + 48^3 + 79^3 = 87^3, & & \end{array}$$

and hence it follows

$$87^3 - 79^3 = 20^3 + 54^3 = 38^3 + 48^3.$$

Therefore, it is plain that from each assumed example several formulas of this kind are obtained among which the same occurs more often; as the case

$$3^3 + 4^3 + 5^3 = 6^3$$

occurs twice in this and the preceding example.

§**28** Therefore, lo and behold the general solution of the problem in which four rational numbers A, B, C, D of such a kind are in question that $A^3 + B^3 + C^3 = D^3$, or, which is the same, in which four rational numbers p, q, r and s are in question that

$$p(pp + 3qq) = s(ss + 3rr).$$

Since these problems using ordinary methods can only be solved particularly, it is obvious that these ordinary methods still have a huge defect and hence still desire a notable perfection. But then there is no doubt that, what we showed here for one single problem, can be achieved in infinitely many others with the same success. It is certainly clear in general that an equation of this kind

$$\alpha p(nnp + nqq) = \beta s(mss + nrr)$$

or even this further-extending one

$$(\alpha p + \beta q + \gamma r + \delta s + \varepsilon)(mpp + nqq) = (ap + bq + cr + ds + e)(mrr + nss)$$

can be solved in rational terms most generally by putting

$$p = nfx + gy, \qquad q = mfy - gx$$

and

$$r = nhx + ky, \qquad s = mhy - kx;$$

for, it will be

$$mpp + nqq = (gg + mnff)(nxx + myy)$$

and

$$mrr + nss = (kk + mnhh)(nxx + myy),$$

whence the equation divided by $nxx + myy$ will contain the unknowns x and y of just one dimension, from which therefore without any restriction their values will be determined rationally.

§**29** Therefore, not without a reason one can suspect that also of other Diophantine problems, of which still just particular solutions have been found, there are also general solutions and the difference mentioned above derived from the generality and the particularity is not essential; hence it is plain how huge increments are still desired in Diophantine analysis. If one succeeds to get to them in any time, there is no doubt that hence the whole analysis so of the finite as the infinite would obtain auxiliary tools not to be condemned. For, since in integral calculus the principal artifice consists in this that irrational differential formulas are transformed into rational ones, as this artifice has been transferred from Diophantine analysis to this calculus, so greater auxiliary tools are justly be expected; hence the eagerness spent on the development of this branch of analysis, which considered for itself might seem to be somewhat pointless, is not to be considered to be invested without any benefit at all.

§ **30** Here further another condition not less worth one's attention deserves it to be noted, i.e. that more often in Diophantine analysis problems of such a kind occur which seem to admit a general solution by usual methods, although that solution is just a particular one; in these cases particular artifices must be applied that the restriction by which the usual method is limited is got rid of. As if two cubes in integer numbers are in question the sum of which is a square, a solution not restricted in any way seems to be obtained, if the equation

$$x^3 + y^3 = zz$$

is resolved in such a way that one puts

$$x = \frac{pz}{r} \quad \text{and} \quad y = \frac{qz}{r}.$$

For, it will be $(p^3 + q^3)z = r^3$ and hence

$$z = \frac{r^3}{p^3 + q^3}$$

and

$$x = \frac{prr}{p^3 + q^3} \quad \text{and} \quad y = \frac{qrr}{p^3 + q^3}.$$

Hence for x and y to become integer numbers, set $r = (p^3 + q^3)$ that one has

$$x = nnp(p^3 + q^3) \quad \text{and} \quad y = nnq(p^3 + q^3),$$

and it will be

$$x^3 + y^3 = n^6(p^3 + q^3)^4 = \text{square}.$$

§**31** But even though this solution seems to be general, nevertheless only such numbers are found for x and y which have the common factor $p^3 + q^3$ such that hence it seems to be concluded that there are no other mutually prime numbers which substituted for x and y solve the question. Nevertheless, in the case $x = 1$ and $y = 2$ it is perspicuous that $x^3 + y^3 =$ a square. But even if this case can be derived from our formulas putting $p = 1$, $q = 2$ and $n = \frac{1}{3}$, hence it results $x = \frac{1}{9} \cdot 9 = 1$ and $y = \frac{2}{9} \cdot 9 = 2$, nevertheless, that hence other cases of this kind are found, it is necessary that for p and q numbers of such a kind are taken the sum of which is a square, say $= ss$, that thereafter one can set $n = \frac{1}{s}$, whence $x = p$ and $y = q$ will result; this way that what its in question is already postulated as known, i.e. that two cubes can be assigned the sum of which is a square. Therefore, let us see how to occur this inconvenience in the following problem.

Problem

§**31** *To find to mutually prime integer numbers the cubes of which added give a square.*

Solution

Let x and y be the numbers in question that it must be

$$x^3 + y^3 = \text{square}.$$

Therefore, it must be $(x+y)(xx - xy + yy) =$ a square. But about these factors I note that they are either mutually prime or admit three as a common divisor, whence the solution will be two-parted, but which will be combined into one in such a way that each of both factors $x + y$ and $xx - xy + yy$ must either be a square or a tripled square.

I. First let each of both factors be a square and put

$$xx - yy + yy = (pp - pq + qq)^2$$

and it will be either

$$x = pp - 2pq \quad \text{and} \quad y = pp - qq$$

or

$$x = 2pq - pp \quad \text{and} \quad y = qq - pp.$$

Therefore, in the first case it is necessary that $x + y = 2pp - 2pq - qq$ is a square. Since this form is $= 3pp - (p+q)^2$, if one puts it $= rr$, it would be necessary that $3pp = (p+q)^2 + rr =$ the sum of two squares which is impossible. Therefore, the other case remains in which

$$x + y = qq + 2pq - 2pp = (q+p)^2 - 3pp = \text{a square},$$

which is satisfied putting

$$p = 2mn \quad \text{and} \quad q = 3mm - 2mn + nn,$$

$$x = 2pq - pp = 4mn(3mm - 3mn + nn),$$

$$y = qq - pp = (3mm + nn)(3mm - 4mn + nn) = (m-n)(3m-n)(3mm+nn).$$

II. Then on the other hand put

$$xx - xy + yy = \text{a triple of a square} = 3(pp - pq + pp)^2,$$

which is satisfied in three ways:

$$\begin{array}{rll} \text{I.} & x = 2pp - 2pq - \ \ qq, & y = \quad pp + 2pq - 2qq, \\ \text{II.} & x = 2pp - 2pq - \ \ qq, & y = \quad pp - 4pq - 2qq, \\ \text{III.} & x = \ \ pp + 2pq - 2qq, & y = -\,pp + 4pq - \ \ qq. \end{array}$$

In the first case $x + y = 3pp - 3qq =$ a tripled square or $pp - qq =$ a square whence

$$p = mm + nn \quad \text{and} \quad q = 2mn$$

and hence

$$x = 2(m^4 - 2m^3n - 2mn^3 + n^4),$$

and hence

$$y = m^4 + 4m^3n - 6mmnn + 4mn^3 + n^4.$$

In the second case $x + y = 3pp - 6pq =$ a tripled square, therefore, $pp - 2pq =$ a square which is satisfied putting

$$p = 2mn \quad \text{and} \quad q = mm - nn,$$

whence it results

$$x = \quad 3m^4 + 6mmnn - n^4,$$
$$y = -3m^4 + 6mmnn + n^4.$$

In the third case $x + y = 6pq - 3qq = 3\Box$ and $2pq - qq = \Box$, whence

$$p = mm + nn \quad \text{and} \quad q = 2mn$$

and hence

$$x = -3m^4 + 6mmnn + n^4,$$
$$y = \quad 3m^4 + 6mmnn - n^4,$$

which agree with those.
Therefore, lo and behold the three solutions of the propounded problem:

$$\text{I.} \quad \begin{cases} x = 4mn(3mm - 3mn + nn), \\ y = (m-n)(3m-n)(3mm+nn), \end{cases}$$

$$\text{II.} \quad \begin{cases} x = 2(m^4 - 2m^3n - 2mn^3 + n^4), \\ y = m^4 + 4m^3n - 6mmnn + 4mn^3 + n^4, \end{cases}$$

$$\text{III.} \quad \begin{cases} x = \quad 3m^4 + 6mmnn - n^4, \\ y = -3m^4 + 6mmnn + n^4, \end{cases}$$

where certainly the second is detected to be contained in the third which agrees with the fourth, such that the second more complicated one can be omitted.

Corollary 1

§**33** If these formulas found for x and y are multiplied by a square number, they likewise answer the question; so the sum of two cubes $x^3 + y^3$ becomes a square number whence somehow non mutually prime numbers are obtained. But in like manner, if vice versa these formulas have a common quadratic divisor, divided by it they will likewise answer the question, whence mutually prime numbers are found for x and y, as they are in question here. Therefore, we will have two formulas for this task:

$$\text{I.}\quad \begin{cases} x = 4mn(3mm - 3mn + nn), \\ y = (m - n)(3m - n)(3mm + nn), \end{cases}$$

$$\text{II.}\quad \begin{cases} x = \quad 3m^4 + 6mmnn - n^4, \\ y = -3m^4 + 6mmnn + n^4. \end{cases}$$

COROLLARY 2

§**34** It is evident that there are infinitely many cases in which the one of these formulas receives a negative value; this happens in the first, if either m or n is a negative number, or if n is contained within the limits m and $3m$, but in the second formulas if either $\frac{nn}{mm}$ is greater than $3 + 2\sqrt{3}$ or $\frac{mm}{nn}$ is smaller than $2\sqrt{3} - 3$. Therefore, in these cases two cubes are found the difference of which is a square.