# Another Specimen of the new Method to compare transcendental Quantities to each other - On the Comparison of the Arcs of an Ellipse *

Leonhard Euler

**§1** The first specimen I exhibited of this method recently concerned the comparison of the arcs of a circle and a parabola; even though this comparison considered for itself is not new, since this subject was already treated by customary methods a long time ago, it was nevertheless advisable to start from there, in order to fully understand the utility of this new method, which I sketched; especially that it does not only lead to the same truths, which are found by the usual methods, but also does so in a lot easier and more convenient way. For, the usual method requires rather tedious integrations and is of such a nature that, if the arcs of these curves, which are to be compared to each other, could not have been reduced to known quadratures of the circle and the hyperbola, it could not have been applied at all.

**§2** Therefore, it will be seen more clearly from the comparison of the arcs of ellipses and hyperbolas, how much utility this new method has; since

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the rectification of these curves can neither be reduced to the quadrature of the circle nor to logarithms by any means, the usual methods can not be applied here and it is also not obvious how to compare the different arcs of these curves to each other. Hence, since I will show that by means of the new method elliptical and hyperbolic arcs can be compared to each other with the same success as parabolic arcs and the usual methods are completely inept for this, the extraordinary utility of this new method will hence become even more evident.

**§3** But I discovered that by means of this method so elliptic as hyperbolic arcs can be compared in the same way as parabolic arcs and it is no obstruction that the rectification of these curves seems to exceed the power of Analysis completely[1]. Yes, the arcs can even be compared under the same conditions as in the case of the parabola, such that having propounded either an arbitrary elliptic or hyperbolic arc, starting from another certain point of the same curve an arc can be separated, which differs from the first one by a geometrically assignable quantity[2]. But in like manner, starting form a certain point, one will be able to exhibit an arc, which differs from the propounded arc taken twice or trice or arbitrarily often[3] by a geometric quantity.

**§4** Further, it is possible that this difference becomes zero and the found arc becomes equal to the propounded arc itself or even a multiple of it, as it is known to be possible in the case of the parabola. In like manner it happens that it is not possible to exhibit two equal arcs, which are not at the same time already identical[4]; but it will even be more remarkable that so in the case of the ellipse as in the case of the hyperbola having propounded an arbitrary arc always another arc can be assigned, which is equal to the double or triple or any multiple the propounded arc.

**§5** Therefore, as the nature of the comparisons of the different arcs of the ellipse and the hyperbola is similar to the comparisons of the arcs of the parabola, so the lemniscate is detected to be similar to the circle. For, in the

[1]Euler means that the occurring integrals cannot expressed in terms of elementary functions like logarithms etc.

[2]Nowadays we would call this an algebraic quantity.

[3]By this Euler means a twice or trice etc. as long arc.

[4]It is important to note that Euler uses the term ëqual"to describe two different arcs with the same or equal length; they do not have to be identical.

case of the lemniscate as in the case of the circle, if an arbitrary arc was propounded, starting from a certain point, it is possible to separate an arc, which was either equal to the propounded arc or to its double or triple or to an arbitrary multiple. For, on this curve as in the case of the circle no arcs of such a kind are exist, whose difference can be assigned geometrically.

**§6** But what I mentioned here, extends a lot further than to the discussed curves, ellipse, hyperbola and lemniscate, which only constitute the simplest cases of the formulas this method yields. For, having expanded these formulas it will be possible to make a similar comparison in infinitely many other classes of curves. But as the first specimen was based on the expansion of this equation

$$0 = \alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy,$$

so here one has to assume an equation extending further; but it nevertheless has to be possible to define both variables by the extraction of square roots. Therefore, let this equation be propounded

CANONICAL EQUATION

$$0 = \alpha + \gamma(xx+yy) + 2\delta xy + \zeta xxyy$$

**§7** If we solve this equation for $y$ first then for $y$ afterwards, we will obtain

$$y = \frac{-\delta x + \sqrt{\delta\delta xx - (\alpha+\gamma xx)(\gamma+\zeta xx)}}{\gamma+\zeta xx},$$

$$x = \frac{-\delta y + \sqrt{\delta\delta yy - (\alpha+\gamma yy)(\gamma+\zeta yy)}}{\gamma+\zeta yy},$$

where we attributed different signs to the roots, since they are arbitrary; it just has always to be taken into account in the following.

**§8** For the sake of brevity let us put these formulas involving the square roots

$$\sqrt{\delta\delta xx - (\alpha+\gamma xx)(\gamma+\zeta xx)} = X$$

and

$$\sqrt{\delta\delta yy - (\alpha + \gamma yy)(\gamma + \zeta yy)} = Y,$$

that we have

$$y = \frac{-\delta x + X}{\gamma + \zeta xx} \quad \text{or} \quad X = \gamma y + \delta x + \zeta xxy,$$
$$x = \frac{-\delta y + Y}{\gamma + \zeta yy} \quad \text{or} \quad -Y = \gamma x + \delta x + \zeta xyy.$$

**§9** Now let us differentiate the canonical equation and it will be

$$0 = dx(\gamma x + \delta y + \zeta xyy) + dy(\gamma y + \delta x + \zeta xxy),$$

whence we conclude that it will be

$$0 = -Ydx + Xdy \quad \text{or} \quad \frac{dy}{Y} - \frac{dx}{X} = 0.$$

Therefore, since $X$ is a function of $x$ and $Y$ is a function of $Y$, by integrating we will find

$$\int \frac{dy}{Y} - \int \frac{dx}{X} = \text{Const.}$$

**§10** Therefore, vice versa we know, if an integral equation of such a kind was propounded

$$\int \frac{dy}{Y} - \int \frac{dx}{X} = \text{Const.},$$

in which $X$ and $Y$ denote irrational functions of $x$ and $y$ of such a kind that it is

$$X = \sqrt{\delta\delta xx - (\alpha + \gamma xx)(\gamma + \zeta xx)}$$

and

$$Y = \sqrt{\delta\delta yy - (\alpha + \gamma yy)(\gamma + \zeta yy)},$$

that this equation is then solved by the relation between $x$ and $y$ defined by means of the canonical equation.

**§11** But as we found the equation $\frac{dy}{Y}-\frac{dx}{X}=0$, so let us now consider this further extending equation

$$\frac{Qdy}{Y}-\frac{Pdx}{X}=dV$$

and let us investigate, functions of $x$ and $y$ of what kind $P$ and $Q$ must be, that $dV$ admits an integration and hence the difference of the integral formulas

$$\int\frac{Qdy}{Y}-\int\frac{Pdx}{X}=\text{Const.}+V$$

can be exhibited algebraically.

**§13** That this investigation can be done more easily, let us put $xy=u$ and because of $xdy+ydx=du$ we will have $dy=\frac{du}{x}-\frac{ydx}{dx}$; having substituted this value for $dy$ in the differential equation we will find

$$0=dx(\gamma x+\delta y+\zeta xyy)+\frac{du}{x}(\gamma y+\delta x+\zeta xxy)-dx\left(\frac{\gamma yy}{x}+\delta y+\zeta xyy\right)$$

or by multiplying by $x$

$$0=dx(\gamma xx-\gamma yy)+du(\gamma y+\delta x+\zeta xxy)$$

or

$$0=\gamma dx(xx-yy)+Xdu.$$

**§14** Therefore, it will be $\frac{dx}{X}=\frac{du}{\gamma(yy-xx)}$, and because it is $\frac{dy}{Y}=\frac{dx}{X}$, it will also be $\frac{dy}{Y}=\frac{du}{\gamma(yy-xx)}$, whence we will have

$$dV=\frac{(Q-P)du}{\gamma(yy-xx)}.$$

Therefore, first it is plain, if it is $Q=yy$ and $P=xx$ that it will be

$$dV=\frac{du}{\gamma}\quad\text{and}\quad V=\frac{u}{\gamma}=\frac{xy}{\gamma}.$$

Hence having assumed the canonical equation

$$\int\frac{yydy}{Y}-\int\frac{xxdx}{X}=\text{Const.}+\frac{xy}{\gamma}.$$

**§15** But the same integration of the quantity $V$ also succeeds, if any powers of even dimensions of $x$ and $y$ are taken for $P$ and $Q$. In order to see this, let us put $xx+yy=t$ and because of $xy=u$ the canonical equation goes over into this form

$$0=\alpha+\gamma t+2\delta u+\zeta uu,$$

whence it is $t=\frac{-\alpha-2\delta u-\zeta uu}{\gamma}$

**§16** Now let us put $P=x^4$ and $Q=y^4$; it will be

$$dV=\frac{du}{\gamma}(xx+yy)=\frac{tdu}{\gamma}\quad\text{and hence}\quad dV=-\frac{-du}{\gamma\gamma}(\alpha+2\delta u+\zeta uu);$$

whence by integrating it is

$$dV=\frac{-\alpha u}{\gamma\gamma}-\frac{\delta uu}{\gamma\gamma}-\frac{\zeta u^3}{3\gamma\gamma}\quad\text{or}\quad V=-\frac{-xy}{3\gamma\gamma}(3\alpha+3\delta xy+\zeta yyxx).$$

Or because of $\zeta xxyy=-\alpha-\gamma(xx+yy)-2\delta xy$ one will have

$$V=\frac{-xy}{3\gamma\gamma}(2\alpha-\gamma(xx+yy)+\delta xy).$$

**§17** Hence our canonical equation will even solve this integral equation

$$\int\frac{y^4dy}{Y}-\int\frac{x^4dx}{X}=\text{Const.}-\frac{xy}{3\gamma\gamma}(3\alpha+3\delta xy+\zeta xxyy).$$

And by collecting these three cases the canonical equation will even solve this more general differential equation

$$\int\frac{dy(\mathfrak{A}+\mathfrak{B}yy+\mathfrak{C}y^4)}{\sqrt{\delta\delta-(\alpha+\gamma yy)(\gamma+\zeta yy)}}-\int\frac{dx(\mathfrak{A}+\mathfrak{B}xx+\mathfrak{C}x^4)}{\sqrt{\delta\delta-(\alpha+\gamma xx)(\gamma+\zeta xx)}}$$
$$=\text{Const.}+\frac{\mathfrak{B}xy}{\gamma}-\frac{\mathfrak{C}xy}{3\gamma\gamma}(3\alpha+3\delta xy+\zeta xxyy).$$

**§18** To proceed even further, let us put $P=x^6$ and $Q=y^6$ and it will be

$$dV=\frac{du}{\gamma}(y^4+xxyy+x^4)=\frac{du}{\gamma}(tt-uu);$$

therefore, having substituted the value found for $t$ it will be

$$dV=\frac{du}{\gamma^3}(\alpha\alpha+4\alpha\delta u+(4\delta\delta+2\alpha\zeta-\gamma\gamma)uu+4\delta\zeta u^3+\zeta\zeta u^4)$$

and hence by integrating

$$V=\frac{u}{\gamma^3}\left(\alpha\alpha+2\alpha\delta u+\frac{1}{3}(4\delta\delta+2\alpha\zeta-\gamma\gamma)uu+\delta\zeta u^3+\frac{1}{5}\zeta\zeta u^4\right).$$

Hence by means of the canonical equation it will be

$$\int\frac{y^6dy}{Y}-\int\frac{x^6dx}{X}$$

$$=\text{Const.}+\frac{xy}{15\gamma^3}(15\alpha\alpha+30\alpha\delta xy+5(4\delta\delta+2\alpha\zeta-\gamma\gamma)xxyy+15\delta\zeta x^3y^3+3\zeta\zeta x^4y^4).$$

**§19** But now let us also attribute forms of such a kind to our irrational formulas $X$ and $Y$, which can be accommodated to certain cases more easily, and let

$$X=\sqrt{p(A+Cxx+Ex^4)}\quad\text{and}\quad Y=\sqrt{p(A+Cyy+Ey^4)};$$

therefore, it is necessary that it is

$$Ap=-\alpha\gamma,\quad Ep=-\gamma\zeta,\quad Cp=\delta\delta-\gamma\gamma-\alpha\zeta,$$

whence it is

$$\alpha=\frac{-Ap}{\gamma},\quad\zeta=\frac{-Ep}{\gamma}\quad\text{and}\quad\delta=\sqrt{\gamma\gamma+Cp+\frac{AEpp}{\gamma\gamma}}.$$

**§20** Now let $\gamma\gamma=A$ and $p=kk$ and assume $\gamma=-\sqrt{A}$ and it will be

$$\alpha=kk\sqrt{A},\quad\gamma=-\sqrt{A},\quad\zeta=\frac{Ekk}{\sqrt{A}}\quad\text{and}\quad\delta=\sqrt{A+Ckk+Ek^4}$$

and so it will be

$$X=k\sqrt{A+Cxx+Ex^4}\quad\text{and}\quad Y=k\sqrt{A+Cyy+Ey^4}$$

and the canonical equation will result as

$$0=Akk-A(xx+yy)+2xy\sqrt{A(A+Ckk+Ek^4)}+Ekkxxyy.$$

**§21** But hence the variables $x$ and $y$ depend on each other in such a way that it is

$$X = -y\sqrt{A} + x\sqrt{A + Ckk + Ek^4} + \frac{Ekk}{\sqrt{A}}xxy,$$
$$Y = \quad x\sqrt{A} - y\sqrt{A + Ckk + Ek^4} - \frac{Ekk}{\sqrt{A}}xyy,$$

whence it is

$$y = \frac{x\sqrt{A(A + Ckk + Ek^4)} - k\sqrt{A(A + Cxx + Ex^4)}}{A - Ekkxx},$$
$$x = \frac{y\sqrt{A(A + Ckk + Ek^4)} + k\sqrt{A(A + Cyy + Ey^4)}}{A - Ekkyy}.$$

**§22** Therefore, these values will solve this very far extending integral equation deduced from § 17, if it is multiplied by $-k$,

$$\int \frac{dx(\mathfrak{A} + \mathfrak{B}xx + \mathfrak{C}x^4)}{\sqrt{A + Cxx + Dx^4}} - \int \frac{dy(\mathfrak{A} + \mathfrak{B}yy + \mathfrak{C}y^4)}{\sqrt{A + Cyy + Ey^4}}$$
$$= \text{Const.} + \frac{\mathfrak{B}kxy}{\sqrt{A}} + \frac{\mathfrak{C}kxy}{3A\sqrt{A}}(3Akk + 3xy\sqrt{A(A + Ckk + Ek^4)} + Ekkxxyy)$$
$$= \text{Const.} + \frac{\mathfrak{B}kxy}{\sqrt{A}} + \frac{\mathfrak{C}kxy}{6A\sqrt{A}}(3Akk + 3A(xx + yy) - Ekkxxyy).$$

**§23** Therefore, if a certain curve was of such a nature that to the abscissa $x$ this arc corresponds

$$\int \frac{dx(\mathfrak{A} + \mathfrak{B}xx + \mathfrak{C}x^4)}{\sqrt{A + Cxx + Ex^4}}$$

and this arc is denoted by $\Pi.\ x$ and another arc corresponding to the abscissa $y$ in the same curve

$$\int \frac{dy(\mathfrak{A} + \mathfrak{B}yy + \mathfrak{C}y^4)}{\sqrt{A + Cyy + Ey^4}}$$

is denoted by $\Pi.\ y$, these two arcs will be related to each other as follows

$$\Pi.\ x - \Pi.\ y = \text{Const.} + \frac{\mathfrak{B}kxy}{\sqrt{A}} + \frac{\mathfrak{C}kxy}{6A\sqrt{A}}(3Akk + 3A(xx+yy) - Ekkxxyy),$$

if the abscissas $x$ and $y$ depend on each other in such a way that it is

$$x = \frac{y\sqrt{A(A+Ckk+Ek^4)} + k\sqrt{A(A+Cyy+Ey^4)}}{A - Ekkyy}$$

and

$$y = \frac{x\sqrt{A(A+Ckk+Ek^4)} - k\sqrt{A(A+Cxx+Ex^4)}}{A - Ekkxx}.$$

**§24** But in order to determine that constant the integral equation contains, consider the case in which it is $y = 0$ and $x = k$; if now the arc corresponding to a vanishing abscissa also vanishes, it will be $\Pi.\ k = \text{Const.}$ for this case; having substituted this value one will have

$$\Pi.\ x - \Pi.\ y - \Pi.\ k = \frac{\mathfrak{B}kxy}{\sqrt{A}} + \frac{\mathfrak{C}kxy(kk+xx+yy)}{2\sqrt{A}} - \frac{\mathfrak{C}Ek^3x^3y^3}{6A\sqrt{A}}.$$

Therefore, this way three arcs are found on that curve and one of these arcs exceeds the sum of the remaining two by a geometrically assignable quantity.

**§25** Hence it is plain in general, if the curve was of such a nature that the arc corresponding to the abscissa $x$ is

$$\Pi.\ x = \int \frac{\mathfrak{A}dx}{\sqrt{A+Cxx+Ex^4}}$$

and hence it is $\mathfrak{B} = 0$ and $\mathfrak{C} = 0$, that then the difference of those arcs becomes zero; and therefore, the arcs can be compared to each other in this case in the same way as it was done in the case of circle. But if the term $\mathfrak{B}xx$ or $\mathfrak{C}x^4$ or even both occur in the numerator, then the difference of those three arcs is assignable geometrically and hence the comparison of the arc will succeed as it did in the case of the parabola. But this comparison will be made in the same way I explained in the first specimen for the circle and the parabola.

**§26** Since three arcs enter the calculation, whose abscissas are $x$, $y$ and $k$, it is plain that, as $y$ depends on $x$ and $k$, that so $k$ depends on $x$ and $y$, whence given two variables the third will be determined by these equations

$$x = \frac{y\sqrt{A(A+Ckk+Ek^4)} + k\sqrt{A(A+Cyy+Ey^4)}}{A-Ekkyy},$$

$$y = \frac{y\sqrt{A(A+Ckk+Ek^4)} - k\sqrt{A(A+Cxx+Ex^4)}}{A-Ekkxx},$$

$$k = \frac{x\sqrt{A(A+Cyy+Ey^4)} - y\sqrt{A(A+Cxx+Ex^4)}}{A-Ekkyy}.$$

**§27** If then one gets rid of all irrational quantities, this equation will result

$$EEk^4x^4y^4 = AA(2kkxx + 2kkyy + 2xxyy - k^4 - x^4 - y^4)$$

$$+4ACkkxxyy + 2AEkkxxyy(kk + xx + yy).$$

Since the three abscissas $k$, $x$, $y$ all enter into this equation equally, one will be able to consider their squares $kk$, $xx$, $yy$ as roots of a cubic equation of this kind

$$Z^3 - pZZ + qZ - r = 0,$$

and since it is

$$p = kk + xx + yy,$$

$$q = kkxx + kkyy + xxyy,$$

$$r = kkxxyy,$$

it will be

$$EErr = AA(4q - pp) + 4ACr + 2AEpr$$

or

$$(Ap - Er)^2 = 4AAq + 4ACr.$$

**§28** Therefore, if, having constituted this relation among the coefficients $p$, $q$ and $r$, for $kk$, $xx$ and $yy$ the three roots of this cubic equation are taken

$$Z^3 - pZZ + qZ - r = 0,$$

for the comparison of the arcs of the curve, which we considered (§ 23), it will be

$$\Pi.\ x - \Pi.\ y - \Pi.\ k = \frac{\mathfrak{B}\sqrt{r}}{\sqrt{A}} + \frac{\mathfrak{C}p\sqrt{r}}{2\sqrt{A}} - \frac{\mathfrak{C}Er\sqrt{r}}{6A\sqrt{A}}.$$

**§29** Let the abscissas with the respective signs, $+x$, $-y$, $-k$, be the roots of this cubic equation

$$z^3 + szz + tz - u = 0;$$

it will be

$$\sqrt{r} = u, \quad q = tt + 2su \quad \text{and} \quad p = ss - 2t$$

and

$$(Ass - 2At - Euu)^2 = 4AAtt + 8AAsu + 4ACuu$$

or

$$t = \frac{Ass - Euu}{4A} - \frac{2Asu + Cuu}{Ass - Euu}.$$

But the roots of this equation will be found by means of the trisection of the angle in such a way that having taken $v = \frac{2}{3}\sqrt{ss - 3t}$ and the angle $\Phi$, whose cosine of course is

$$\cos\Phi = \frac{27u + 9st - 2s^3}{2(ss - 3t)\sqrt{ss - 3t}},$$

the roots itself will be

$$x = v\cos\frac{1}{3}\Phi - \frac{1}{3}s, \quad y = v\cos\left(60^\circ + \frac{1}{3}\Phi\right) - \frac{1}{3}s,$$

$$k = v\cos\left(60^\circ - \frac{1}{3}\Phi\right) - \frac{1}{3}s.$$

**§30** But having put these things concerning the roots aside, let us consider the use of the found formula more accurately; and first this remarkable differential equation occurs

$$\frac{dx}{\sqrt{A+Cxx+Dx^4}} = \frac{dy}{\sqrt{A+Cyy+Ey^4}};$$

we know about this equation that it is equivalent to this integral equation

$$x = \frac{y\sqrt{A(A+Ckk+Ek^4)}+k\sqrt{A(A+Cyy+Ey^4)}}{A-Ekkyy};$$

and since it contains a new arbitrary constant $k$, it will indeed be the complete integral equation.

**§31** If for this case we put

$$\int \frac{dx}{\sqrt{A+Cxx+Ex^4}} = \Pi.\ x,$$

since having put $y=0$, it is $x=k$, it will be $\Pi.\ x = \Pi.\ k + \Pi.\ y$. Hence, if it is $k=y$, that it also is

$$x = \frac{2y\sqrt{A(A+Cyy+Ey^4)}}{A-Ey^4},$$

it will be $\Pi.\ x = 2\Pi.\ y$ and hence this value of $x$ solves this differential equation

$$\frac{dx}{\sqrt{A+Cxx+Ex^4}} = \frac{2dy}{\sqrt{A+Cyy+Ey^4}};$$

but since this equation does not contain a new constant, it will only be a particular solution of the propounded differential equation.

**§32** Nevertheless it is even possible to exhibit the complete integral of this differential equation. For, put

$$\frac{dy}{\sqrt{A+Cyy+Ey^4}} = \frac{dz}{\sqrt{A+Czz+Ez^4}}$$

and it will be

$$y = \frac{z\sqrt{A(A+Ckk+Ek^4)} + k\sqrt{A(A+Czz+Ez^4)}}{A - Ekkzz},$$

which value is to be substituted for $y$ in the formula

$$x = \frac{2y\sqrt{A(A+Cyy+Ey^4)}}{A - Ey^4},$$

and this way $x$ will be expressed by $z$ and the new arbitrary constant $k$, which value will be the complete integral of this differential equation

$$\frac{dx}{\sqrt{A+Cxx+Ex^4}} = \frac{2dz}{\sqrt{A+Czz+Ez^4}}.$$

**§33** Let us set $\Pi.\ k = n\Pi.\ y$ and let us assume that the value of $k$ was already found and from the preceding paragraphs we conclude, if one takes

$$x = \frac{y\sqrt{A(A+Ckk+Ek^4)} + k\sqrt{A(A+Cyy+Ey^4)}}{A - Ekkyy},$$

that it will be $\Pi.\ = (n+1)\Pi.\ y$. Therefore, since in the case $n = 1$ it is $k = y$, the value hence found for $x$ will give the value of $k$ for the case $n = 2$, whence the value of $x$ is found, that it will be $\Pi.\ x = 3\pi.\ y$. This value taken for $k$ will then yield the value of $x$ that it is $\Pi.\ x = 4\Pi.\ y$, and this way one can proceed arbitrarily far.

**§34** But having found the value of $x$ that it is $\Pi.\ x = n\Pi.\ y$, it will be a particular integral of this differential equation

$$\frac{dx}{\sqrt{A+Cxx+Ex^4}} = \frac{ndy}{\sqrt{A+Cyy+Ey^4}};$$

but then take

$$z = \frac{x\sqrt{A+Ckk+Ek^4)} + k\sqrt{A(A+Cxx+Ex^4)}}{A - Ekkxx}$$

and so the complete value of the integral $z$ will be obtained for this differential equation

$$\frac{dz}{\sqrt{A+Czz+Ez^4}} = \frac{ndy}{\sqrt{A+Cyy+Ey^4}};$$

for, it will be $\Pi.\, z = \Pi\, k + \Pi.\, x = \Pi.\, k + n\Pi.\, y$.

**§35** Now let us also contemplate the further extending formula in general and represent it as the curved line *akfgpqrst* (Fig. 1) and let this curve be of such a nature that having put an arbitrary abscissa $AK = x$ the arc corresponding to it is

$$ak = \int \frac{dx(\mathfrak{A} + \mathfrak{B}\mathfrak{x}\mathfrak{x} + \mathfrak{C}\mathfrak{x}^4)}{\sqrt{A + Cxx + Ex^4}},$$

which arc we want to denote by $\Pi.\, x$.

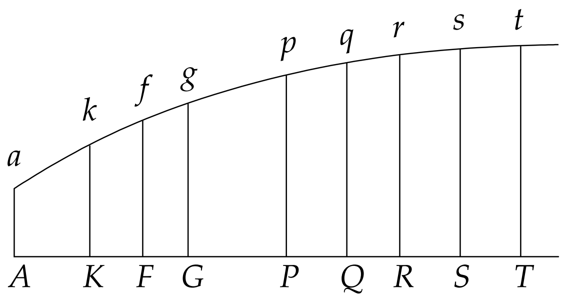


FIG. 1

But it is manifest, the same way this relation among the arc *ak* and its abscissa *AK* was constituted, that one is able to constitute the same relation among an arc and another abscissa, to which the arc can be referred. Hence, even though here $x$ denotes the abscissa corresponding to the arc *ak*, $x$ can nevertheless denote also another certain straight line[5] extending to the arc, as long as this line also vanishes, if also the arc itself vanishes.

**§36** Now let us consider three abscissas and let them be $AK = k$, $AF = f$ and $AG = g$; further let these abscissas depend on each other in such a way that it is

[5]By straight line Euler actually means an abscissa in this context.

$$g = \frac{f\sqrt{A(A+Ckk+Ek^4)}+k\sqrt{A(A+Cff+Ef^4)}}{A-EEkkff},$$

$$f = \frac{g\sqrt{A(A+Ckk+Ek^4)}-k\sqrt{A(A+Cgg+Eg^4)}}{A-EEkkgg},$$

$$k = \frac{g\sqrt{A(A+Cff+Ef^4)}-f\sqrt{A(A+Cgg+Eg^4)}}{A-EEkkgg},$$

and the arcs $ak = \Pi.\ k$, $af = \Pi.\ f$ and $ag = \Pi.\ g$ will be connected by this relation:

$$\Pi.\ g - \Pi.\ f - \Pi.\ k = \text{Arc. } ag - \text{Arc. } af - \text{Arc. } ak = \text{Arc. } fg - \text{Arc. } ak$$

$$= \frac{\mathfrak{B}kfg}{\sqrt{A}} + \frac{\mathfrak{C}kfg(kk+ff+gg)}{2\sqrt{A}} - \frac{\mathfrak{C}Ek^3f^3g^3}{6A\sqrt{A}}.$$

**§37** Therefore, given an arbitrary arc $ak$ starting from the origin $a$ of the curve one will be able to separate an arc $fg$ starting from a certain point $f$, such that the difference of the arcs $fg$ and $ak$ can be assigned geometrically. For, because of the given points $k$ and $f$ the abscissas $k$ and $f$ will be given; using these two the abscissa $g$ is defined by means of the first formula. Or, if the points $k$ and $g$ are given, going backwards from the point $g$ one will even be able to separate an arc $gf$, which differs from the arc $ak$ by a geometric quantity. Or finally, given an arbitrary arc $fg$, one will be able to separate an arc $ak$ starting from the origin $a$ of the curve, which arc differs from that given one by a geometric quantity.

**§38** This case, in which it is $f = k$, should especially be expanded; therefore, if the abscissa $AG = g$ (Fig. 2) is assumed in such a way that it is

$$g = \frac{2k\sqrt{A(A+Ckk+Ek^4)}}{A-Ek^4},$$

while it is $AK = k$,

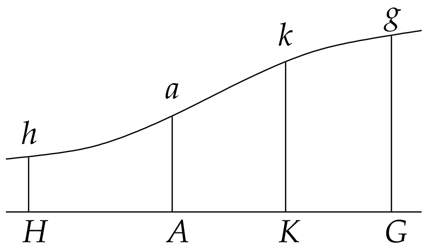

FIG. 2

it will be

$$\text{Arc. } ag - 2\text{Arc. } ak = \frac{\mathfrak{B}kkg}{\sqrt{A}} + \frac{\mathfrak{C}kkg(2kk+gg)}{2\sqrt{A}} - \frac{\mathfrak{C}Ek^6g^3}{6A\sqrt{A}}.$$

If it now was $Ek^4 > A$, the value of $g$ will turn out to be negative, which therefore taken backwards*By this Euler means in negative direction of the x-axis* becomes $AH = h$, such that it is $g = -h$ and $\Pi.\ g = -\Pi.\ h$, while

$$h = \frac{2k\sqrt{A(A+Ckk+Ek^4)}}{Ek^4 - A},$$

and having changed the signs it will be

$$\text{Arc. } ah + 2\text{Arc. } ak = \frac{\mathfrak{B}kkh}{\sqrt{A}} + \frac{\mathfrak{C}kkh(2kk+hh)}{2\sqrt{A}} - \frac{\mathfrak{C}Ek^6h^3}{6A\sqrt{A}}.$$

**§39** Hence it is understood that the abscissa $k$ can obtain a value of such a kind that it is $h = k$; hence, if starting from the point $a$ two identical arcs extending equally to both different sides are taken and it was $AH = AK$, it will also be Arc. $ah$ = Arc. $ak$; hence, if it is $h = k$ or

$$Ek^4 - A = 2\sqrt{A(A+Ckk+Ek^4)}$$

or

$$EEk^8 - 6AEk^4 - 4ACkk - 3AA = 0,$$

it will be

$$3\text{Arc. } ak = \frac{\mathfrak{B}k^3}{\sqrt{A}} + \frac{2\mathfrak{C}k^5}{2\sqrt{A}} - \frac{\mathfrak{C}Ek^9}{6A\sqrt{A}};$$

therefore, the arc corresponding to this abscissa $AK = k$ will be rectifiable absolutely, since it is

$$\text{Arc. } ak = \frac{\mathfrak{B}k^3}{3\sqrt{A}} + \frac{\mathfrak{C}k^5}{2\sqrt{A}} - \frac{\mathfrak{C}Ek^9}{18A\sqrt{A}}.$$

**§40** But that equation, even though it is of degree eight, can be solved conveniently; for, having put its factors

$$(k^4 + \alpha kk + \beta)(k^4 - \alpha kk + \gamma) = 0,$$

one finds

$$\beta + \gamma = \alpha\alpha - \frac{6A}{E}, \quad \beta - \gamma = \frac{4AC}{\alpha EE} \quad \text{and} \quad \beta\gamma = -\frac{3AA}{EE},$$

whence this equation results

$$\alpha^4 - \frac{12A}{E}\alpha\alpha + \frac{36AA}{EE} - \frac{16AACC}{\alpha\alpha E^4} = -\frac{12AA}{EE}$$

and hence

$$\alpha\alpha = \frac{4A}{E} + \sqrt[3]{\frac{16AACC - 64A^3E}{E^4}}$$

and because of

$$\gamma = \frac{\alpha\alpha}{2} - \frac{3A}{E} - \frac{2AC}{\alpha EE}$$

it will be

$$kk = \frac{1}{2}\alpha \pm \sqrt{\frac{2AC}{\alpha EE} + \frac{3A}{E} - \frac{1}{4}\alpha\alpha}$$

or even

$$kk = -\frac{1}{2}\alpha \pm \sqrt{\frac{-2AC}{\alpha EE} + \frac{3A}{E} - \frac{1}{4}\alpha\alpha}.$$

**§41** But it is always true in these curves that to the negative abscissa the same arc taken negatively corresponds. For, since it is

$$\Pi.\ x = \int \frac{dx(\mathfrak{A} + \mathfrak{B}xx + \mathfrak{C}x^4)}{\sqrt{A + Cxx + Ex^4}},$$

if the abscissa $x$ is taken negatively, it will be

$$\Pi.\ (-x) = \int \frac{-dx(\mathfrak{A} + \mathfrak{B}xx + \mathfrak{C}x^4)}{\sqrt{A + Cxx + Ex^4}} = -\Pi.\ x.$$

Therefore, it seems, as often as a real arc corresponds to the abscissa $k$ defined in the preceding paragraph, that then the length of the same arc can be assigned geometrically.

**§42** But I did not dare to affirm that this reasoning, by which I found the absolutely rectifiable arc, is always correct; for, it seems that cases exist, in which it is actually incorrect. For, if it is $\mathfrak{B} = 0$ and $\mathfrak{C} = 0$ and hence

$$\Pi.\ x = \int \frac{\mathfrak{A}dx}{\sqrt{A + Cxx + Dx^4}},$$

in § 39 $3\,\text{Arc.}\,ak = 0$ would result, although the abscissa $k$ does not become $= 0$ as seen from the equation of degree eight exhibited there. But one has to recall that this equation resulted from this one

$$k = \frac{2k\sqrt{A(A + Ckk + Ek^4)}}{Ek^4 - A};$$

since this equation immediately yields the root $k = 0$, this will be the only root, which in this case fulfills all the conditions and the other will simply not.

**§43** And nevertheless this reasoning is not to be considered to be wrong at all in these cases, even if for $k$ another arbitrary root is taken, but it is rather to be understood in such a way that many arcs correspond to the same abscissa; and only one of these arcs, namely the negative one, fulfills all the requirements; and therefore, in this case, even if one sets $h = k$ in § 38, it does hence nevertheless not follow that it is $\text{Arc.}\,ah = \text{Arc.}\,ak$ and hence $\text{Arc.}\,ah + 2\,\text{Arc.}\,ak = 3\,\text{Arc.}\,ak$, since to the same abscissa $h = k$ also other arcs except for $\text{Arc.}\,ak$ correspond, among which there is one, which indeed yields $\text{Arc.}\,ah + 2\,\text{Arc.}\,ak = 0$.

**§44** In order to see this more clearly, let us put $A = 1$, $C = 2$ and $E = 1$ while it is $\mathfrak{B} = 0$ and $\mathfrak{C} = 0$ and it will be $\Pi.\ x = \mathfrak{A} \arctan x$ and $\text{Arc.}\ ak = \mathfrak{A} \arctan k$ and $\text{Arc.}\ ah = \mathfrak{A} \arctan h$; therefore, having put

$$h = \frac{2k\sqrt{1+2kk+k^4}}{k^4-1} = \frac{2k}{k^4-1}$$

it will be $\mathfrak{A}A \arctan h + 2\mathfrak{A} \arctan k = 0$. If one now puts $h = k$, it will be $kk = 3$ and $k = \sqrt{3}$ and one will find $\mathfrak{A}(\arctan \sqrt{3} + 2 \arctan k)$ and $\text{Arc.}\ \sqrt{3} = \text{Arc.}\ 60^\circ$; nevertheless hence it does not follow that $3\mathfrak{A}\,\text{Arc.}\ 60^\circ = 0$, which would certainly be wrong; but since to the tangent of $\sqrt{3}$ also the arc $-120^\circ$ corresponds, this value written in the first arc instead of $\arctan \sqrt{3}$ will yield the truth, namely

$$\mathfrak{A}(-\text{Arc. } 120^\circ + 2\text{Arc. } 60^\circ) = 0.$$

**§45** Therefore, this ambiguity, that several values Arc. $ak$ can correspond to the same quantity $k$, which we assumed as abscissa here, is the reason that, even though in § 38 one puts $h = k$, it is nevertheless not possible to write $3\,\text{Arc.}\ ak$ for $\text{Arc.}\ ah + 2\,\text{Arc.}\ ak$. Nevertheless, even in this case it will be

$$\text{Arc. } ah + 2\text{Arc. } ak = \frac{\mathfrak{B}k^3}{\sqrt{A}} + \frac{3\mathfrak{C}k^5}{2\sqrt{A}} - \frac{\mathfrak{C}Ek^9}{6A\sqrt{A}};$$

for, to the abscissa $h$, even though it is $= k$, nevertheless, except for the arc $ak$, also another arc will correspond, which substituted for Arc. $ah$ solves the equation. Therefore, this ambiguity must carefully be taken into account, that no errors are made.

**§46** But if an ambiguity of this kind does not occur such that only one arc corresponds to the same abscissa, then, having put the abscissa $h = k$, it will also be possible to write Arc. $ak$ for Arc. $ah$ and $3\,\text{Arc.}\ ak$ for $\text{Arc.}\ ah + 2\,\text{Arc.}\ ak$ without hesitation and hence one must not be afraid of any errors, whatever root of the equation of degree eight found in § 39 is taken for $k$. This will be evident in the case, in which it is $\mathfrak{A} = A$, $\mathfrak{B} = 2C$ and $\mathfrak{C} = 3E$, in which it certainly is

$$\Pi.\ x = x\sqrt{A + Cxx + Ex^4}$$

and hence an algebraic quantity and

$$\Pi.\ g - \Pi.\ f - \Pi.\ k = \frac{2Ckfg}{\sqrt{A}} + \frac{3Ekfg(kk+ff+gg)}{2\sqrt{A}} - \frac{EEk^3f^3g^3}{2A\sqrt{A}}.$$

**§47** If one now sets $f = k$, it will be

$$g = \frac{2k\sqrt{A(A+Ckk+Ek^4)}}{A-Ek^4}$$

and hence

$$\sqrt{A(A+Cgg+Eg^4)} = \frac{A(gg-2kk)+Ek^4gg}{2kk}.$$

and

$$\sqrt{A(A+Cgg+Eg^4)} = \frac{A(gg-2kk)+Ek^4gg}{2kk}.$$

Now let $g = -k$ or

$$Ek^4 - A = 2\sqrt{A(A+Ckk+Ek^4)};$$

it will be

$$\sqrt{A(A+Cgg+Eg^4)} = \frac{-A+Ek^4}{2} = \sqrt{A(A+Ckk+Ek^4)};$$

hence it is $\Pi.\ g = -\Pi.\ k$ and

$$-\Pi.\ k = \frac{-2Ck^3}{\sqrt{A}} - \frac{9Ek^5}{2\sqrt{A}} \quad \text{or} \quad 3\Pi.\ k = \frac{k(4ACkk+9AEk^4-EEk^8)}{2A\sqrt{A}}.$$

But it is

$$EEk^8 = 6AEk^4 + 4ACkk + 3AA,$$

what because of $\Pi.\ k = \sqrt{A+Ckk+Ek^4}$ is true.

**§48** But although this curve is rectifiable per se, it nevertheless evidently proves this, what we want, namely that in our formulas also non-rectifiable curves are contained, in which case it is nevertheless possible to assign an absolutely rectifiable arc in the way we explained before. But having found one single rectifiable arc as *ak*, using it one will immediately be able to exhibit infinitely many others of the same nature; for, since starting from a certain point $f$ an arc $fg$ can be separated, whose difference to that given arc is geometric, this arc $fg$ will also be rectifiable. But furthermore, still infinitely many other equally rectifiable arcs will be found in the following way from that rectifiable arc; it will be convenient to explain this in general.

**§49** To simplify our formulas, first, for the sake of brevity, let us put

$$\sqrt{A(A+Ckk+Ek^4)}=K, \quad \sqrt{A(A+Cff+Ef^4)}=F, \quad \sqrt{A(A+Cgg+Eg^4)}=G,$$

that by means of § 36 it is

$$g=\frac{fK+kF}{A-Ekkff}, \quad f=\frac{gK-kG}{A-Ekkgg}, \quad k=\frac{gF-fG}{A-Effgg}.$$

If it now was

$$\Pi.\ x=\int\frac{dx(\mathfrak{A}+\mathfrak{B}xx+\mathfrak{C}x^4)}{\sqrt{A+Cxx+Ex^4}},$$

it will be

$$=\frac{\mathfrak{B}kfg}{\sqrt{A}}+\frac{\mathfrak{C}kfg(kk+ff+gg)}{2\sqrt{A}}-\frac{\mathfrak{C}Ek^3f^3g^3}{6A\sqrt{A}}.$$

**§50** In like manner except for the abscissa $AK=k$ take two other abscissas $AP=p$, $AQ=q$ and having also put

$$\sqrt{A(A+Cpp+Ep^4)}=P \quad \text{and} \quad \sqrt{A(A+Cqq+Eq^4)}=Q$$

and having constituted this relation

$$q=\frac{pK+kP}{A-Ekkpp}, \quad p=\frac{qK-kQ}{A-Ekkqq}, \quad k=\frac{qP-pQ}{A-Eppqq},$$

for the same curve it will be

$$\text{Arc. } pq-\text{Arc. } ak=\frac{\mathfrak{B}kpq}{\sqrt{A}}+\frac{\mathfrak{C}kpq(kk+pp+qq)}{2\sqrt{A}}-\frac{\mathfrak{C}Ek^3p^3q^3}{6A\sqrt{A}}.$$

**§51** Therefore, having subtracted that equation from this one this difference will remain

$$\text{Arc. } pq - \text{Arc. } fg$$
$$= \frac{\mathfrak{B}k(pq - fg)}{\sqrt{A}} + \frac{\mathfrak{C}kpq(kk + pp + qq) - \mathfrak{C}kfg(kk + ff + gg)}{2\sqrt{A}} - \frac{\mathfrak{C}Ek^3(p^3q^3 - f^3g^3)}{6A\sqrt{A}},$$

where the abscissas $f$, $g$, $p$ and $q$ will depend on each other in such a way that it is

$$k = \frac{gF - fG}{A - Effgg} = \frac{qP - pQ}{A - Eppqq} \quad \text{or} \quad \frac{1}{k} = \frac{gF + fG}{A(gg - ff)} = \frac{qP + pQ}{A(qq - pp)},$$

whence at the same time the abscissa $k$ can be eliminated and the relation among $f$, $g$, $p$, $q$ can be defined.

**§52** In order to do this elimination more easily, note that it also is

$$K = \frac{A(ff + gg - kk) - Ekkffgg}{2fg} = \frac{A(pp + qq - kk) - Ekkppqq}{2pq},$$

whence it is

$$kk = \frac{Apq(ff + gg) - Afg(pp + qq)}{(pq - fg)(A - Efgpq)} = \frac{(gF - fG)^2}{(A - Effgg)^2} = \frac{(qP - pQ)^2}{(A - Eppqq)^2}.$$

Therefore, it will be

$$pq(kk + pp + qq) - fg(kk + ff + gg) = pq(pp + qq) - fg(ff + gg)$$
$$+ \frac{Apq(ff + gg) - Afg(pp + qq)}{A - Efgpq}$$

and hence it is obtained

$$\text{Arc. } pq - \text{Arc. } fg = \frac{\mathfrak{B}k(pq - fg)}{\sqrt{A}} + \frac{\mathfrak{C}k(pq - fg)(ff + gg + pp + qq)}{2\sqrt{A}}$$
$$- \frac{\mathfrak{C}Ek(pq - fg)^2(pq(ff + gg) - fg(pp + qq))}{6(A - Efgpq)\sqrt{A}}.$$

**§53** Therefore, since it is

$$kk = \frac{A(pq(ff+gg) - fg(pp+qq))}{(pq-fg)(A-Efgpq)}$$

and the four abscissas $f$, $g$, $p$, $q$ depend on each other in such a way that it is

$$\frac{gF+fG}{gg-ff} = \frac{qP+pQ}{qq-pp},$$

it is plain, having propounded an arbitrary arc $fg$, that one is able to separate an arc $pq$ starting from another given point $p$ in the curve, which differs from that arc by an algebraically assignable quantity.

**§54** If furthermore, by proceeding further from the point $q$, a point $r$ is taken such that having put the abscissa $AR = r$ it is

$$\frac{gF+fG}{gg-ff} = \frac{rQ+qR}{rr-qq}$$

or

$$\frac{pq(ff+gg)-fg(pp+qq)}{(pq-fg)(A-Efgpq)} = \frac{qr(ff+gg)-fg(qq+rr)}{(qr-fg)(A-Efgqr)} = \frac{qr(pp+qq)-pq(qq+rr)}{(qr-pq)(A-Epqqr)},$$

also Arc. $qr -$ Arc. $fg$ will be an algebraic quantity, which difference added to the first will give

$$\text{Arc. } pr - 2\text{Arc. } fg = \text{algebr. quantity},$$

and so starting from the given point $p$ one is able to separate an arc $pr$, which exceeds the double of the propounded arc $fg$ by an algebraic quantity.

**§55** In like manner, if further the abscissas $AS = s$, $AT = t$ etc. are taken in such a way that it is

$$\frac{gF+fG}{gg-ff} = \frac{sR+rS}{ss-rr} = \frac{tS+sT}{tt-ss} \quad \text{etc.},$$

the arc $ps$ will exceed the triple of the arc $fg$, the arc $pt$ the quadruple of the arc $fg$ etc. by an geometrically assignable quantity. But vice versa given either the arc $pr$ or $ps$ or $pt$ etc. one will be able to find an arc $fg$ starting from the given point $f$, which differs from the half or third or fourth part of the given arc by a geometrically assignable quantity.

**§56** It could also happen that, although the quantities $\mathfrak{B}$ and $\mathfrak{C}$ are not equal to zero, this geometrically assignable difference nevertheless vanishes; yes, one is even always able to define one of the abscissas in such a way that this difference indeed goes over into zero. Therefore, in these cases one will be able to assign two arcs of such a kind in the curve, which are either equal to each other or have a given ratio of two numbers.

**§57** Since these things extend very far and can be applied to all curves, whose arc for an abscissa or another arbitrary line is expressed in terms of $x$ in such a way that it is

$$= \int \frac{dx(\mathfrak{A} + \mathfrak{B}xx + \mathfrak{C}x^4)}{\sqrt{A + Cxx + Ex^4}},$$

it will be convenient to expand these properties for several curves, that the use of this method is seen more clearly. Therefore, it seems to be advisable to explain this comparison of arcs in the case of the ellipse first.

## ON THE COMPARISON OF ARCS IN THE CASE OF THE ELLIPSE

**§58** Therefore, let the elliptic quadrant $ABa$ (Fig. 1) be propounded and let its center be in $A$; put the one semiaxis, which contains all the abscissas, $AB = a$, the other axis $Aa = na$. Therefore, having taken an arbitrary abscissa $AP = x$ the ordinate will be

$$PM = n\sqrt{aa - xx}$$

and its differential

$$= -\frac{nxdx}{\sqrt{aa - xx}},$$

whence the arc corresponding to this abscissa becomes

$$am = \int dx\sqrt{\frac{aa + (nn-1)xx}{a - xx}}.$$

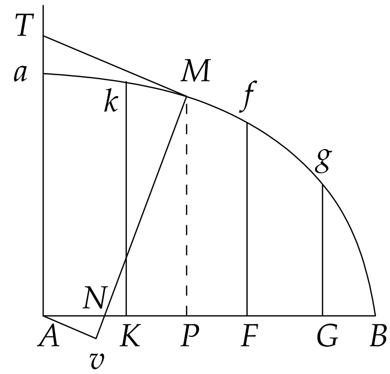


FIG. 3

Set $1 - nn = m$ that it is

$$aM = \int dx\sqrt{\frac{aa - mxx}{aa - xx}}.$$

Since it does not matter, which one of the two semiaxes is greater or smaller, let us assume that $AB$ is smaller; and hence it is $n < 1$ and $m$ is a positive number smaller than 1, and since the focal point lies on the semiaxis $AB$, the distance of this focal point from the center $A$ will be

$$= \sqrt{aa - nnaa} = a\sqrt{m};$$

hence the value of the number $m$ is understood more easily.

Therefore, if the arc corresponding to the arbitrary abscissa $AP = x$ is denoted by $aM = \Pi.\ x$, it will be

$$\Pi.\ x = \int dx \sqrt{\frac{aa - mxx}{aa - xx}},$$

which expression, reduced to our general form, will go over into this one

$$\Pi.\ x = \int \frac{dx(aa - mxx)}{\sqrt{a^4 - (m+1)aaxx + mx^4)}}.$$

And so for this case we will have these values

$$A = a^4, \quad C = -(m+1)aa, \quad E = a, \quad \mathfrak{A} = aa, \quad \mathfrak{B} = -m \quad \text{and} \quad \mathfrak{C} = 0.$$

Therefore, having taken the three abscissas $k$, $x$, $y$ to which the arc $\Pi.\ k$, $\Pi.\ x$, $\Pi.\ y$ correspond such that it is

$$x = \frac{aay\sqrt{a^4 - (m+1)aakk + mk^4} + aak\sqrt{a^4 - (m+1)aayy + my^4}}{a^4 - mkkyy},$$

$$y = \frac{aax\sqrt{a^4 - (m+1)aakk + mk^4} - aak\sqrt{a^4 - (m+1)aaxx + mx^4}}{a^4 - mkkxx},$$

$$k = \frac{aax\sqrt{a^4 - (m+1)aayy + my^4} + aay\sqrt{a^4 - (m+1)aaxx + mx^4}}{a^4 - mxxyy},$$

these three arcs will depend on each other in such a way that it is

$$\Pi.\ x - \Pi.\ y - \Pi.\ k = -\frac{mkxy}{aa}.$$

Therefore, having mentioned these things in advance let us solve the following problems.

## Problem 1

**§59** *Having propounded an arbitrary arc ak of an ellipse (Fig. 3), starting from another certain point f to separate an arc fg such that the difference of the arc ak and fg can be assigned geometrically.*

### Solution

Having drawn the ordinates $kK$, $fF$, $gG$ from the points $k$, $f$, $g$ call the abscissas $AK = k$, $AF = f$, $AG = g$; the ordinates are given, the abscissas on the other hand are in question, and the arcs will be

$$ak = \Pi.\, k, \quad af = \Pi.\, f, \quad ag = \Pi.\, g.$$

Further, for the sake of brevity according to § 49 put

$$\begin{aligned} aa\sqrt{a^4 - (m+1)aakk + mk^4} &= K, \\ aa\sqrt{a^4 - (m+1)aaff + mf^4} &= F, \\ aa\sqrt{a^4 - (m+1)aagg + mg^4} &= G \end{aligned}$$

and constitute this relation among the three abscissas

$$g = \frac{fK + kF}{a^4 - mkkff} \quad \text{or} \quad f = \frac{gK - kG}{a^4 - mkkgg} \quad \text{or} \quad k = \frac{gF - fG}{a^4 - mffgg};$$

having done so one will have

$$\Pi.\, g - \Pi.\, f - \Pi.\, k = \text{Arc. } fg - \text{Arc. } ak = -\frac{mkfg}{aa}.$$

Therefore, having taken the point in such a way that it is

$$AG = g = \frac{fK + kF}{a^4 - mkkff},$$

the difference of the arcs $ak$ and $fg$ can be assigned geometrically. For, it will be

$$\text{Arc. } ak - \text{Arc. } fg = \frac{mkfg}{aa}.$$

Q. E. I.

### COROLLARY 1

**§60** The solution will be the same, if, having propounded the arc $ak$, the point $g$ is given; going backwards starting from that point an arc $gf$ must be separated differing from the given arc by a geometric quantity; for, then the abscissas $k$ and $g$ will be given and hence one will be able to find the value of the third one $f$.

### COROLLARY 2

**§61** Also, given an arbitrary arc $fg$ on the ellipse, one will be able to separate an arc $ak$ starting from the vertex $a$ such that the difference of the arcs $ak$ and $fg$ becomes geometric. Therefore, the rectification of the arc $fg$ will depend on the rectification of a certain arc $ak$ ending at the vertex $a$ of the ellipse.

### COROLLARY 3

**§62** The relation among the three abscissas $k$, $f$, $g$ can also be exhibited in such a way that it is

$$g = \frac{a^4(-kk+ff)}{fK-kF} \quad \text{or} \quad f = \frac{a^4(-kk+gg)}{gK+kG} \quad \text{or} \quad k = \frac{a^4(gg-ff)}{gF+fG};$$

hence comparing them to the preceding ones, it is found

$$\begin{aligned} K &= \frac{a^4(ff+gg-kk)-mkkffgg}{2fg} &&= aa\sqrt{(aa-kk)(aa-mkk)}, \\ F &= \frac{a^4(kk+gg-ff)-mkkffgg}{2kg} &&= aa\sqrt{(aa-ff)(aa-mff)}, \\ G &= \frac{-a^4(ff+gg-kk)+mkkffgg}{2kf} &&= aa\sqrt{(aa-gg)(aa-mgg)}; \end{aligned}$$

but then one will also have

$$fg(gg-ff)K - kg(gg-kk)F - kf(ff-kk)G = 0.$$

### COROLLARY 4

**§63** If the difference of the arcs $ak$ and $fg$ must vanish completely, it is plain that this can only happen, if it is either $k = 0$ or $f = 0$ or $g = 0$. In the first case the arc $ak$ itself and hence the arc $fg$ vanishes, but in the two remaining cases the one of the two endpoints of the arc $fg$ falls on the point $a$ and the arc $fg$ does not only become equal to the arc $ak$ but even identical to it.

### COROLLARY 5

**§64** In order to apply this relation among the abscissas more easily, it will helpful to have noted that in general, if the normal $MN$ is drawn to the point $M$ and from $A$ the perpendicular, which will be parallel to the tangent $MT$, is dropped to it, and one puts $AP = x$, that it will be

$$PM = n\sqrt{aa - xx}, \quad PN = nnx, \quad AN = mx, \quad MN = n\sqrt{aa - mxx},$$

$$AV = \frac{mx\sqrt{aa - xx}}{\sqrt{aa - mxx}}, \quad NV = \frac{mnxx}{\sqrt{aa - mxx}}, \quad MV = \frac{aa}{\sqrt{aa - mxx}},$$

$$MT = \frac{x\sqrt{aa - mxx}}{\sqrt{aa - xx}}, \quad AT = \frac{naa}{\sqrt{aa - xx}} \quad \text{und} \quad AV \cdot MT = mxx.$$

### COROLLARY 6

**§65** Therefore, having put $g$ for $x$ for the point $g$ these values are found

$$g = \frac{a^2k\sqrt{(aa - ff)(aa - mff)} + aaf\sqrt{(aa - kk)(aa - mkk)}}{a^4 - mkkff},$$

$$\sqrt{aa - gg} = \frac{a^3\sqrt{(aa - kk)(aa - ff)} - akfsqrt(aa - mkk)(aa - mff}{a^4 - mkkff},$$

$$\sqrt{aa - mgg} = \frac{a^3\sqrt{(aa - mkk)(aa - mff)} - makf\sqrt{(aa - kk)(aa - ff)}}{a^4 - mkkff}$$

and

$$\sqrt{(aa - gg)(aa - mgg)}$$

$$= \frac{a^4kf(2maa(kk + ff) - (m + 1)(a^4 + mkkff)) + aa(aa(a^4 + mkkff)\sqrt{(aa - mkk)(aa - ff)(aa - mff)}}{(a^4 - mkkff)^2},$$

whence further it is found

$$aa\sqrt{aa-mgg}+mkf\sqrt{aa-gg}=a\sqrt{(aa-mkk)(aa-mff)},$$
$$aa\sqrt{aa-gg}+kf\sqrt{aa-mgg}=a\sqrt{(aa-kk)(aa-ff)}.$$

### CASE 1

**§66** *Having propounded the arc $ak$ of the ellipse (Fig. 4), which arc ends at the one vertex $a$, starting from the other vertex $B$ to separate an arc $Bf$ such that the difference of the arcs $ak$ and $Bf$ is geometric.*

Therefore, the problem is reduced to this case, if the point $g$ is set to be in the vertex $g$ or it is $g=a$, and one has to find the point $f$ or the abscissa $AF=f$. But because of $g=a$ it will be $G=0$ and one will have

$$f=\frac{aK}{a^4-maakk}=a\sqrt{\frac{aa-kk}{aa-mkk}}$$

or having drawn the normal $kN$ to the point $k$ one has to take

$$AF=f=\frac{AB\cdot Kk}{Nk}.$$

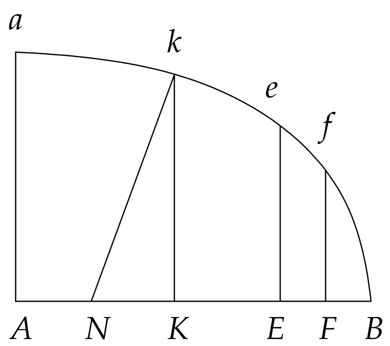


FIG. 4

But having taken this point that way the difference of the arcs will be

$$\text{Arc. } ak-\text{Arc. } Bf=\frac{mkf}{a}=mk\sqrt{\frac{aa-kk}{aa-mkk}}=\frac{AN\cdot Kk}{Nk}.$$

## COROLLARY

**§67** Therefore, it can happen that the point $k$ and $f$ denote the same point and so the quadrant $aeB$ will be dissected into two parts, whose difference must be geometric. For this set $k = f = AE = e$ and it will be

$$e = a\sqrt{\frac{aa-ee}{aa-mee}} \quad \text{or} \quad a^4 - 2aaee + me^4 = 0,$$

whence it is

$$ee = \frac{aa \pm aa\sqrt{1-m}}{m} = \frac{aa(1\pm n)}{m}$$

- because of $m = 1 - nn$. Therefore, hence it will be

$$e = \frac{a}{\sqrt{1\pm n}}.$$

But since it must be $e < a$, it will be

$$e = \frac{a}{\sqrt{1+n}}$$

or

$$AE = \frac{AB^2}{\sqrt{AB^2 + AB\cdot Aa}} \quad \text{and} \quad Ee = \frac{na\sqrt{n}}{\sqrt{1+n}},$$

such that it is

$$AE : Ee = 1 : n\sqrt{n} = AB\sqrt{AB} : Aa\sqrt{Aa}.$$

And in this case it will be

$$\text{Arc. } ae - \text{Arc. } Be = a(1-n) = AB - Aa.$$

## CASE 2

**§68** *Having propounded the arc ak (Fig. 5) ending at the vertex, starting from its other endpoint k to separate an arc kg such that the difference of the arc ak and kg is rectifiable.*

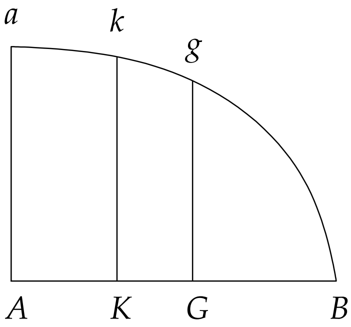


FIG. 5

Therefore, in this case the point $f$ falls on $k$ and it will be $f = k$ and hence also $F = K$; hence it is found

$$AG = g = \frac{2kK}{a^4 - mk^4} = \frac{2aak\sqrt{(aa-kk)(aa-mkk)}}{a^4 - mk^4}.$$

Therefore, having taken the abscissa $AG$ the difference of the arcs will be

$$\text{Arc. } ak - \text{Arc. } kg = \frac{mkkg}{aa} = \frac{2mk^3\sqrt{(aa-kk)(aa-mkk)}}{a^4 - mk^4}.$$

## COROLLARY 1

**§69** Therefore, vice versa an arbitrary arc $ag$ ending at the vertex $a$ can be split into two parts in $k$ in such a way that the difference of the parts $ak - kg$ becomes rectifiable. For, because of the known abscissa $AG = g$ the abscissa in question $AK = k$ must be defined from this equation

$$gg(a^4 - mk^4)^2 = 4a^4kk(aa-kk)(aa-mkk),$$

which goes over into this equation of degree eight

$$mmggk^8 - 4ma^4k^6 - 2ma^4ggk^4 + 4(m+1)a^6k^4 - 4a^8kk + a^8gg = 0.$$

## COROLLARY 2

**§70** But if the factors of this equation are put

$$(mgk^4 - Akk + a^4g)(mgk^4 - Bkk + a^4g) = 0,$$

one finds

$$A + B = \frac{4a^4}{g} \quad \text{and} \quad AB = 4(m+1)a^6 - 4ma^4gg,$$

whence

$$A - B = \frac{4aa}{g}\sqrt{a^4 - (m+1)aagg + mg^4},$$

such that it is

$$A = \frac{2a^4 + 2aa\sqrt{(aa-gg)(aa-mgg)}}{g}$$

and

$$B = \frac{2a^4 - 2aa\sqrt{(aa-gg)(aa-mgg)}}{g}.$$

As a logical consequence it is

$$k^4 = \frac{2a^4kk \pm 2aakk\sqrt{(aa-gg)(aa-mgg)} - a^4gg}{mgg}$$

and

$$kk = \frac{a^4 \pm aa\sqrt{(aa-gg)(aa-mgg)} \pm a^3\sqrt{2aa-(m+1)gg \pm 2\sqrt{(aa-gg)(aa-mgg)}}}{mgg}.$$

## COROLLARY 3

**§71** Therefore, the four roots of $kk$ are

$$\text{I.} \quad kk = \frac{a^4 + aa\sqrt{(aa-gg)(aa-mgg)} + a^3\sqrt{aa-gg} + a^3\sqrt{aa-mgg}}{mgg},$$

$$\text{II.} \quad kk = \frac{a^4 + aa\sqrt{(aa-gg)(aa-mgg)} - a^3\sqrt{aa-gg} - a^3\sqrt{aa-mgg}}{mgg},$$

III. $kk = \dfrac{a^4 - aa\sqrt{(aa-gg)(aa-mgg)} + a^3\sqrt{aa-gg} - a^3\sqrt{aa-mgg}}{mgg},$

IV. $kk = \dfrac{a^4 - aa\sqrt{(aa-gg)(aa-mgg)} - a^3\sqrt{aa-gg} + a^3\sqrt{aa-mgg}}{mgg},$

which, using the ambiguity of the square root sign, can conveniently be represented this way in one single equation

$$kk = \frac{aa}{mgg}(a \pm \sqrt{aa-gg})(a \pm \sqrt{aa-mgg}).$$

## COROLLARY 4

**§72** But the values of $k$ will hence be

$$k = \pm \frac{a}{g\sqrt{m}}\left(\sqrt{\frac{a+g}{2}} \pm \sqrt{\frac{a-g}{2}}\right)\left(\sqrt{\frac{a+g\sqrt{m}}{2}} \pm \sqrt{\frac{a-g\sqrt{m}}{2}}\right),$$

which are eight in total, four positive ones and as many negative ones, which are precisely the negatives of the four positive ones; but it is obvious that only the positive values with $k > g$ can actually be the right ones here. But it certainly is

$$k = \frac{a}{g\sqrt{m}}\left(\sqrt{\frac{a+g}{2}} - \sqrt{\frac{a-g}{2}}\right)\left(\sqrt{\frac{a+g\sqrt{m}}{2}} - \sqrt{\frac{a-g\sqrt{m}}{2}}\right).$$

For, it is

$$\sqrt{\frac{a+g}{2}} + \sqrt{\frac{a-g}{2}} > \sqrt{a}, \quad \sqrt{\frac{a+g}{2}} - \sqrt{\frac{a-g}{2}} < \sqrt{g},$$

$$\sqrt{\frac{a+g\sqrt{m}}{2}} + \sqrt{\frac{a-g\sqrt{m}}{2}} > \sqrt{a}, \quad \sqrt{\frac{a+g\sqrt{m}}{2}} - \sqrt{\frac{a-g\sqrt{m}}{2}} > \sqrt{g\sqrt{m}}.$$

## COROLLARY 5

**§73** If one puts

$$\frac{g}{a} = \cos\eta \quad \text{and} \quad \frac{g\sqrt{m}}{a} = \cos\theta,$$

because of $m > 1$ and $\theta > \eta$ and our formula found for the roots of $k$ will go over into this form

$$k = \pm \frac{a}{\cos\theta}\left(\cos\frac{1}{2}\eta \pm \sin\frac{1}{2}\eta\right)\left(\cos\frac{1}{2}\theta \pm \sin\frac{1}{2}\theta\right)$$

or because of

$$\cos\theta = \cos^2\frac{1}{2}\theta - \sin^2\frac{1}{2}\theta$$

one will have

$$k = \pm a \cdot \frac{\cos\frac{1}{2}\eta \pm \sin\frac{1}{2}\eta}{\cos\frac{1}{2}\theta \pm \sin\frac{1}{2}\theta}.$$

Or the eight values will be

$$k = \pm a \cdot \frac{\cos\left(45^\circ - \frac{1}{2}\eta\right)}{\cos\left(45^\circ - \frac{1}{2}\theta\right)}, \qquad k = \pm a \cdot \frac{\sin\left(45^\circ - \frac{1}{2}\eta\right)}{\cos\left(45^\circ - \frac{1}{2}\theta\right)},$$

$$k = \pm a \cdot \frac{\cos\left(45^\circ - \frac{1}{2}\eta\right)}{\sin\left(45^\circ - \frac{1}{2}\theta\right)}, \qquad k = \pm a \cdot \frac{\sin\left(45^\circ - \frac{1}{2}\eta\right)}{\sin\left(45^\circ - \frac{1}{2}\theta\right)}.$$

## COROLLARY 6

**§74** Out of these values the second

$$k = a \cdot \frac{\sin\left(45^\circ - \frac{1}{2}\eta\right)}{\cos\left(45^\circ - \frac{1}{2}\theta\right)} = a \cdot \frac{\sin\left(45^\circ - \frac{1}{2}\eta\right)}{\sin\left(45^\circ + \frac{1}{2}\theta\right)}$$

always fulfills the condition; for, as it is obvious, it not only is $k > a$, but also $k > g$ or $k < a\cos\eta$. From the first value

$$k = a \cdot \frac{\sin\left(45^\circ + \frac{1}{2}\eta\right)}{\cos\left(45^\circ - \frac{1}{2}\theta\right)}$$

it certainly always is $k > a$, because of $\eta > \theta$; but in order for it to be $k > g$, it has to be

$$\frac{\sin\left(45^\circ + \frac{1}{2}\eta\right)}{\cos\left(45^\circ - \frac{1}{2}\theta\right)} < \cos\eta = \sin(90^\circ - \eta) = 2\sin\left(45^\circ - \frac{1}{2}\eta\right)\cos\left(45^\circ + \frac{1}{2}\eta\right)$$

and hence

$$1 < 2\sin\left(45^\circ - \frac{1}{2}\eta\right)\sin\left(45^\circ + \frac{1}{2}\theta\right)$$

or

$$1 < \cos\frac{1}{2}(\theta+\eta) - \cos\left(90^\circ + \frac{1}{2}(\theta-\eta)\right)$$

or

$$1 < \cos\frac{1}{2}(\theta+\eta) + \sin\frac{1}{2}(\theta-\eta).$$

## PROBLEM 2

**§75** *Having propounded an arbitrary arc $fg$ of an ellipse (Fig. 6), starting from a given point $p$ to separate another arc $pq$ such that the difference of these arcs $fg - pq$ becomes geometric.*

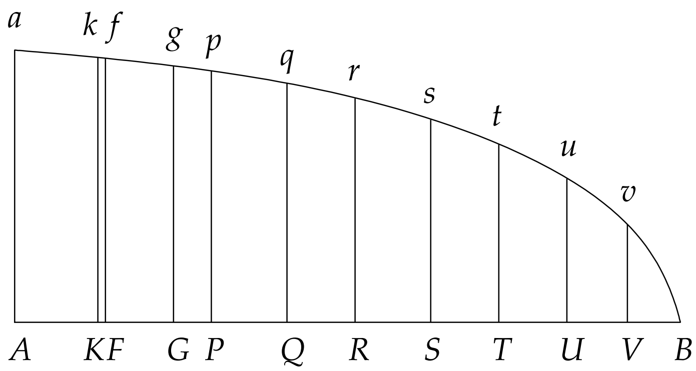


FIG. 6

### SOLUTION

Having drawn the ordinates $fF$, $gG$, $pP$, $qQ$, let the abscissas be $AF = f$, $AG = g$, $AP = p$ and $AQ = q$, then take the arc *ak* starting from the vertex *a*; and this arc *ak* has to exceed the given arc $fg$ by a geometric quantity; and having put the abscissa $AK = k$ and for the sake of brevity having set

$$K = aa\sqrt{(aa-kk)(aa-mkk)},$$

$$F = aa\sqrt{(aa-ff)(aa-mff)}, \quad G = aa\sqrt{(aa-gg)(aa-mgg)},$$

$$P = aa\sqrt{(aa-pp)(aa-mpp)} \quad \text{and} \quad Q = aa\sqrt{(aa-qq)(aa-mqq)}$$

first it will be

$$k = \frac{gF - fG}{a^4 - mffgg} = \frac{a^4(gg-ff)}{gF+fG};$$

hence one finds $k$ such that it is

$$\text{Arc. } ak - \text{Arc. fg} = \frac{mkfg}{aa}.$$

But then determine the abscissa $q$ by means of the preceding problem in such a way that it is

$$q = \frac{pK + kP}{a^4 - mkkpp} = \frac{a^4(pp-kk)}{pK - kP},$$

and it will be

$$\text{Arc. } ak - \text{Arc. } pq = \frac{mkpq}{aa};$$

subtract the first equation from this last equation; it will remain

$$\text{Arc. } fg - \text{Arc. } pq = \frac{mk}{aa}(pq - fg).$$

Q. E. I.

## COROLLARY 1

**§76** Since $k$ depends on the abscissas $p$ and $q$ in the same way as it depends on $f$ and $g$, it will be

$$k = \frac{qP - fG}{a^4 - mppqq} = \frac{a^4(qq-pp)}{qP+pQ}$$

and hence the abscissa $q$ must be defined using the given ones $f$, $g$ and $p$ by means of this equation

$$\frac{gF - fG}{a^4 - mffgg} = \frac{qP - pQ}{a^4 - mppqq}$$

or even from this equation

$$\frac{gg-ff}{gF+fG}=\frac{qq-pp}{qP+pQ};$$

and hence one finds

$$q=\frac{Fgp(pp-gg)+Gfp(pp-ff)-Pfg(gg-ff)}{Ff(pp-gg)+Gg(pp-ff)-Pp(gg-ff)}.$$

COROLLARY 2

**§77** Even the abscissas $p$ and $q$ depend on the abscissa $k$ in such a way that it is

$$\begin{aligned}
aa\sqrt{aa-mqq}+mkp\sqrt{aa-qq}&=a\sqrt{(aa-mkk)(aa-mpp)},\\
aa\sqrt{aa-qq}+kp\sqrt{aa-mqq}&=a\sqrt{(aa-kk)(aa-pp)},\\
aa\sqrt{aa-mpp}-mkq\sqrt{aa-pp}&=a\sqrt{(aa-mkk)(aa-mqq)},\\
aa\sqrt{aa-pp}-kp\sqrt{aa-mpp}&=a\sqrt{(aa-kk)(aa-qq)},\\
aa\sqrt{aa-mkk}-mpq\sqrt{aa-kk}&=a\sqrt{(aa-mpp)(aa-mqq)},\\
aa\sqrt{aa-kk}-pq\sqrt{aa-mkk}&=a\sqrt{(aa-pp)(aa-qq)}.
\end{aligned}$$

COROLLARY 3

**§78** If the difference of the arcs $fg$ and $pq$ must vanish, it is necessary that it is either $k=0$ or $pq=fg$. But if it is $k=0$, because of

$$k=\frac{a^4(gg-ff)}{gF+fG}=\frac{a^4(qq-pp)}{qP+pQ},$$

so the arc $fg$ as the arc $pq$ vanishes. But if it is $pq=fg$, because of

$$\begin{aligned}
aa\sqrt{aa-mkk}-mpq\sqrt{aa-kk}&=a\sqrt{(aa-mpp)(aa-mqq)},\\
aa\sqrt{aa-mkk}-mfg\sqrt{aa-kk}&=a\sqrt{(aa-mff)(aa-mqq)}
\end{aligned}$$

it will be

$$(aa - mpp)(aa - mqq) = (aa - mff)(aa - mgg)$$

and because of

$$aa\sqrt{aa-kk} - pq\sqrt{aa-mkk} = a\sqrt{(aa-pp)(aa-qq)},$$
$$aa\sqrt{aa-kk} - fg\sqrt{aa-mkk} = a\sqrt{(aa-ff)(aa-qq)}$$

it will be

$$(aa - pp)(aa - qq) = (aa - ff)(aa - gg),$$

whence it is plain that it is either $q = g$ and $p = f$ or $q = f$ and $p = g$; but in each of both cases the arc $pq$ does not only become equal but even identical to the arc $fg$.

## COROLLARY 4

**§79** If it could happen that the arc $pq$ vanishes while the arc $fg$ remains finite, this arc would become rectifiable. But, while the arc $pq$ vanishes, because of $q = p$ $k$ results to be $= 0$ and hence it also is $f = g$; hence also the arc $fg$ vanishes.

## COROLLARY 5

**§80** If the arc $pq$ must end at the other vertex $B$ that it is $q = a$, we will have this equation

$$a^2\sqrt{1-m} = \sqrt{(aa-mkk)(aa-mpp)}$$

or

$$a^4 - aakk - aapp + mkkpp = 0 \quad \text{and} \quad kk = \frac{aa(aa-pp)}{aa-mpp}.$$

This value substituted in the equation

$$aa\sqrt{aa-kk} - fgaa - mkk = a\sqrt{(aa-ff)(aa-gg)}$$

yields

$$0 = a^6 + 2(m-1)a^3 fgp - a^4(ff+gg+pp)$$
$$+maa(ffgg+ffpp+ggpp) - mffggpp;$$

this case reduces to the case of the preceding problem, if only the vertices $a$ and $B$ are permuted and instead of the abscissas the ordinates are introduced.

## COROLLARY 6

**§81** One should also note the case, in which the point $p$ is assumed in the point $g$, such that the arc $pq$ becomes contiguous to the arc $fg$ and it is

$$\text{Arc. } fg - \text{Arc. } gq = \frac{mkg}{aa}(q-f)$$

- because of $p = g$. Therefore, since it also is $P = G$, it will be

$$\frac{gF+fG}{gg-ff} = \frac{qG+gQ}{qq-gg},$$

whence the abscissa $q$ is determined. Or having taken

$$k = \frac{gF-fG}{a^4 - mffgg} = \frac{a^4(gg-ff)}{gF+fG}$$

it will be

$$q = \frac{gK+kG}{a^4 - mkkgg} = \frac{a^4(gg-kk)}{gK-kG}.$$

But hence one finds

$$q = \frac{gg}{f} - \frac{a^4(gg-ff)^2}{f} \cdot \frac{a^4 - mg^4}{2FGfg + a^4(a^4(ff+gg) - 2(m+1)aaffgg - mg^4(gg-3ff))}$$

or

$$q = \frac{2FGg(a^4-mg^4) - a^4 f((a^4+mg^4)^2 - 2(m+1)aagg(a^4+mg^4) + 4ma^4g^4)}{a^4((a^4-mg^4)^2 - 4mffgg(aa-gg)(aa-mgg))}$$

or

$$q = \frac{2FGg(a^4-mg^4) - a^4(mg^4 - 2aagg + a^4)(mg^4 - 2maagg + a^4)}{a^4(a^4-mg^4)^2 - 4ma^4ffgg(aa-gg)(aa-mgg)}.$$

## PROBLEM 3

**§82** *Having propounded an arbitrary arc $fg$ of the ellipse, starting from the point $p$ to separate an arc $pqr$, which differs from the double of the arc $fg$ by a geometrically assignable quantity.*

### SOLUTION

First, using the abscissas $AAF = f$, $AG = g$ of the points $f$ and $g$ and the quantities $F$ and $G$ derived from them find the abscissa

$$AK = k = \frac{gF - fG}{a^4 - mffgg} = \frac{a^4(gg - ff)}{gF + fG},$$

that one has

$$\text{Arc. } ak - \text{Arc. } = \frac{mkfg}{aa}.$$

Then find the abscissa $AQ = q$ to the abscissa $AP = p$ of the point $p$ such that it is

$$q = \frac{pK + kP}{a^4 - mkkpp} = \frac{a^4(pp - kk)}{pK - KP},$$

while the capitals $L$ and $P$ always denote functions of such a kind of the lower case letters $k$ and $p$ that, if the lower case letter was $x$, the value of the corresponding capital letter will be

$$X = \sqrt{(aa - xx)(aa - mxx)};$$

and it will be

$$\text{Arc. } ak - \text{Arc. } pq = \frac{mkpq}{aa},$$

whence we obtain

$$\text{Arc. } fg - \text{Arc. } pq = \frac{mk}{aa}(pq - fg).$$

In like manner, if now the point $q$ is considered as given and from it the point $r$ is found, that its abscissa is

$$AR = r = \frac{qK + jQ}{a^4 - mkkqq} = \frac{a^4(qq - kk)}{qK - kQ},$$

we will have

$$\text{Arc. } fg - \text{Arc. } qr = \frac{mk}{aa}(qr - fg).$$

Hence by adding these formulas we will find

$$2\text{Arc. } fg - \text{Arc. } pqr = \frac{mk}{aa}(pq + qr - 2fg)$$

and so starting from a given point $p$ we separated the arc $pr$, which differs from the double of the arc $fg$ by an algebraic quantity.

### COROLLARY 1

**§83** Since it is

$$k = \frac{a^4(gg - ff)}{gF + fG} \quad \text{and} \quad k = \frac{a^4(qq - pp)}{qP + pQ}$$

and in the same way it also is

$$k = \frac{a^4(rr - qq)}{rQ + qR},$$

we will have these equations

$$\frac{gF + fG}{gg - ff} = \frac{qP + pQ}{qq - pp} = \frac{rQ + qR}{rr - qq},$$

whence from the given abscissas $f$, $g$ and $p$ the two remaining abscissas $q$ and $r$ are defined.

### COROLLARY 2

**§84** If the arc $fg$ starts at the vertex $a$ that it is $f = 0$, it will be $k = g$, whence

$$q = \frac{pG + gP}{a^4 - mggpp} = \frac{a^4(pp - gg)}{pG - gP} \quad \text{and} \quad r = \frac{qG + gQ}{a^4 - mggqq} = \frac{a^4(qq - gg)}{qG - gQ}.$$

And if furthermore the point $p$ is given in the other vertex $A$ that it is $p = a$ and $P = 0$, it will be

$$q = \frac{G}{a^3 - magg} = \frac{a\sqrt{(aa-gg)(aa-mgg)}}{aa-mgg};$$

hence

$$aa - qq = \frac{aagg(1-m)(aa-mgg)}{(aa-mgg)^2} = \frac{(1-m)aagg}{aa-mgg}$$

and

$$aa - mqq = \frac{a^4(1-m)(aa-mgg)}{(aa-mgg)^2} = \frac{(1-m)a^4}{aa-mgg}, \quad \text{whence it is} \quad Q = \frac{-(1-m)a^5g}{aa-mgg},$$

since the ordinate must fall onto the lower part, and it will be

$$r = \frac{a(a^4 - 2aagg + mg^4)}{a^4 - 2maagg + mg^4}.$$

## Corollary 3

**§85** Therefore, in this case having taken $r$ in the upper quadrant (Fig. 7) that having put the abscissa $AG = g$ it is

$$AR = r = \frac{a(a^4 - 2aagg + mg^4)}{a^4 - 2maagg + mg^4}$$

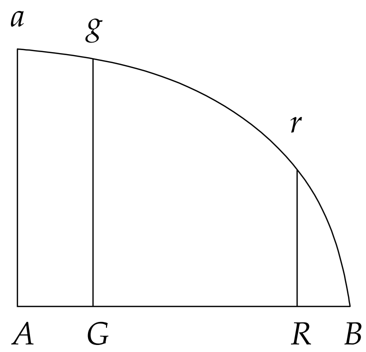


Fig. 7

or

$$BR = a - r = \frac{2(1-m)a^3gg}{a^4 - 2maagg + mg^4},$$

it will be

$$2\text{Arc. } ag - \text{Arc. } Br = \text{algebr. Quant.} = \frac{mg}{aa}(aq + rq) = \frac{mgq}{aa}(a + r)$$

and hence

$$2\text{Arc. } ag - \text{Arc. } Br = \frac{2mg(aa - gg)\sqrt{(aa-gg)(aa-mgg)}}{a^4 - 2maagg + mg^4}.$$

## COROLLARY 4

**§86** If the points $g$ and $r$ must coalesce into one point that it is $r = g$, the value of the common abscissa $AG = AR = g$ must be determined from this equation of degree five

$$mg^5 - mag^4 - 2maag^3 + 2a^3gg + a^4g - a^5 = 0.$$

So, if it is $m = \frac{1}{2}$ and $a = 1$, one will have

$$g^5 - g^4 - 2g^3 + 4gg + 2g - 2 = 0.$$

If it would be $m = \frac{4}{3+\sqrt{2}}$, $g = \frac{a}{\sqrt{2}}$ would result and it would also be

$$2\text{Arc. } ag - \text{Arc. } Bg = a\sqrt{\frac{2+2\sqrt{2}}{3+\sqrt{2}}}.$$

# PROBLEM 4

**§87** *Having propounded an arbitrary arc $fg$ of an ellipse (Fig. 6) to find an arc $pqr$ which is precisely twice as long.*

## SOLUTION

Therefore, in the solution of the preceding problem it has to be

$$pq + qr - 2fg = 0,$$

and then it will be $2\,\text{Arc.}\,fg = \text{Arc.}\,pqr$. But here, because of the given arc $fg$, except for the semiaxis $AB = a$ and $Aa = a\sqrt{1-m}$ also the abscissas $AF = f$ and $AG = g$ are given together with the values $F$ and $G$ derived from them, whence one has to find

$$k = \frac{a^4(gg - ff)}{gF + fG};$$

and at the same time the value derived from it will be

$$K = \frac{a^4(ff + gg - kk) - mkkffgg}{2fg}$$

(by corollary 3 of problem 1). But in like manner the abscissas $p$ and $q$ depend on $k$ that it is

$$K = \frac{a^4(qq + rr - kk) - mkkppqq}{2pq},$$

and likewise from the abscissas $q$ and $r$ it will be

$$K = \frac{a^4(qq + rr - kk) - mkkqqrr}{2qr}.$$

But from the equation $pq + qr = 2fg$ it is $q = \frac{2fg}{p+q}$, whence we will obtain these two equations

$$K = \frac{a^4(pp - kk)(p + r)^2 + 4a^4ffgg - 4mffggkkpp}{4fgg(p + r)},$$

$$K = \frac{a^4(rr - kk)(p + r)^2 + 4a^4ffgg - 4mffggkkrr}{4fgr(p + r)},$$

from which the two abscissas $p$ and $r$ determining the arc $pr$ in question can be defined. Therefore, hence, by eliminating $K$ and dividing by $p - r$, first we find

$$a^4pq(p+r)^2 + a^4kk(p+r)^2 - 4a^4ffgg - 4mffggkkpr = 0.$$

Further, by adding those equations we will have

$$2K = \frac{a^4pr(p+r)^3 - a^4kk(p+r)^3 + 4a^4ffgg(p+r) - 4mffggkkpr(p+q)}{4fgpr(p+q)}.$$

But from that equation it is

$$a^4(p+q)^2 = \frac{4ffgg(a^4+mkkpq)}{pr+kk},$$

which value substituted in this one yields

$$8Kfgpr = \frac{4ffgg(pr-kk)(a^4+mkkpr)}{pr+kk} + 4a^4ffgg - 4mffggkkpr$$

or

$$\frac{2Kpr(pr+kk)}{fg} = 2a^4pr - 2mk^4pr;$$

hence one finds

$$pr = \frac{(a^4-mk^4)fg - Kkk}{K} = \frac{ffgg(2a^4-mk^4) - a^4kk(ff+gg-kk)}{a^4(ff+gg-kk) - mffggkk}$$

and

$$(p+q)^2 = \frac{4fg}{a^4}(K+mfgkk) = \frac{2(a^4(ff+gg-kk)+2mffggkk)}{a^4}.$$

Therefore,

$$p+r = \frac{\sqrt{2(a^4(ff+gg-kk)+mffggkk)}}{aa}.$$

Further

$$r-p = \frac{\sqrt{2(a^8(gg-ff)^2 - a^8k^4 + 2ma^4ffggk^4 - mmf^4g^4k^4)}}{aa\sqrt{a^4(ff+gg-kk)-mffggkk}}$$

or

$$r-p=\frac{\sqrt{2(a^8(gg-ff)^2-k^4(a^4-mffgg)^2)}}{aa\sqrt{a^4(ff+gg-kk)-mffggkk}}.$$

But since it is

$$a^4(gg-ff)=k(gF+fG) \quad \text{and} \quad a^4-mffgg=\frac{gF-fG}{k},$$

it will be

$$r-p=\frac{2k}{aa}\sqrt{\frac{FG}{K}},$$

whence because of

$$r+p=\frac{\sqrt{2(a^4(ff+gg-kk)+mffggkk)}}{aa}=\frac{2}{aa}\sqrt{fg(K+mfgkk)}$$

each of both abscissas $p$ and $r$ become known. Q.E.I.

## COROLLARY 1

**§88** Since it is

$$k=\frac{gF-fG}{a^4-mffgg}$$

and

$$K=\frac{(a^4+mffgg)FG-a^6fg(1maa(ff+gg)-(m+1)(a^4+mffgg))}{(a^4-mffgg)^2},$$

it will be

$$r+p=\frac{2}{aa}\sqrt{\frac{fgFG-ma^4ffgg(ff+gg)+(m+1)a^6ffgg}{a^4-mffgg}},$$

$$r-p=\frac{2(gF-fG)}{aa}\sqrt{\frac{FG}{(a^4+mffgg)(FG+(m+1)a^6fg)-2ma^8fg(ff+gg)}}.$$

## COROLLARY 2

**§89** If the given arc $fg$ ends at the vertex $a$ that it is $f = 0$ and $F = a^4$, $p + r = 0$ and $r - p = 2g$ results, whence it is $p = -g$ and $r = g$; therefore, the twice as long arc is equally extended to both sides starting from $a$ and consists of two equal halves, the arc $fg$ and the arc $ag$. The same happens, if the given arc ends at the other vertex $B$ that it is is $g = a$ and $G = 0$; for, then it is $r - p = 0$ and $r + p = 2f$ and hence $r = p = f$.

## COROLLARY 3

**§90** As in these cases, where the propounded arc $fg$ ends at the other vertex, the twice as long arc is obvious per se, so, if the propounded ends at none of the two vertices, it is very difficult to assign the twice as long arc; it is not even possible to separate this arc into two parts geometrically.

## COROLLARY 4

**§91** Hence, it is also plain, if vice versa the arc $pr$ is given, that one is able to find the arc $fg$, which will be exactly half of the arc $pr$; but this will require very cumbersome calculations. But if the twice as long arc $pqr$ is equal to the quadrant of the ellipse or it is $p = 0$ and $r = a$, it will not be difficult to assign the arc equal to the half of that arc. For, first it will be

$$q = k \quad \text{and} \quad k = a\sqrt{\frac{1 - \sqrt{1-m}}{m}}$$

and this way so $k$ is known; further, it is

$$K = a^4\sqrt{\frac{1-m}{m}(1 - \sqrt{1-m})}.$$

Further, it is

$$2fg = ak \quad \text{and} \quad ff + gg = \frac{Kk}{a^3} + kk + \frac{mk^4}{4aa}.$$

But it is

$$m = \frac{2aakk - a^4}{k^4} \quad \text{and hence} \quad ff + gg = \frac{2kk + 3aa}{4};$$

therefore, it is

$$g + f = \frac{1}{2}\sqrt{2kk + 3aa + 4ak}$$

and

$$g - f = \frac{1}{2}\sqrt{2kk + 3aa - 4ak}$$

and hence

$$f = \frac{1}{4}\sqrt{3aa + 4ak + 2kk} - \frac{1}{4}\sqrt{3aa - 4ak + 2kk},$$
$$g = \frac{1}{4}\sqrt{3aa + 4ak + 2kk} + \frac{1}{4}\sqrt{3aa - 4ak + 2kk}.$$

COROLLARY 5

**§92** If the one semiaxis is put $Aa = b$ while the other is $AB = a$ that it is $m = \frac{aa-bb}{aa}$, it will be $k = a\sqrt{\frac{a}{a+b}}$ for this case; having substituted this value one will have

$$g \pm f = \frac{a}{2}\sqrt{\frac{5a+3b}{a+b} \pm 4\sqrt{\frac{a}{a+b}}};$$

hence it is

$$f = \frac{a}{2}\sqrt{\frac{5a - 3b - \sqrt{9aa + 14ab + 9bb}}{2(a+b)}},$$
$$g = \frac{a}{2}\sqrt{\frac{5a - 3b + \sqrt{9aa + 14ab + 9bb}}{2(a+b)}}$$

and so the abscissas for each of the two endpoints of the arc $fg$, which is the half of the whole arc of the quadrant, are found.

## Corollary 6

**§93** Therefore, in this case it will be

$$ff + gg = \frac{aa(5a+3b)}{4(a+b)} = aa + \frac{aa(a-b)}{4(a+b)}$$

and also

$$fg = \frac{aa}{2}\sqrt{\frac{a}{a+b}} \quad \text{and} \quad 2fg = aa\sqrt{\frac{a}{a+b}};$$

if, for the sake of an example, it is $a = 25$ and $b = 119$, one will find

$$f = \frac{25}{3\sqrt{2}} \quad \text{and} \quad g = \frac{125}{4\sqrt{2}}.$$

## Scholium

**§94** Hence we obtained the solution of this elegant problem:

*Having propounded the elliptical quadrant $BAa$ (Fig. 8), to separate an arc $fg$ on it geometrically, which is precisely half of the whole arc of the quadrant $afgB$.*

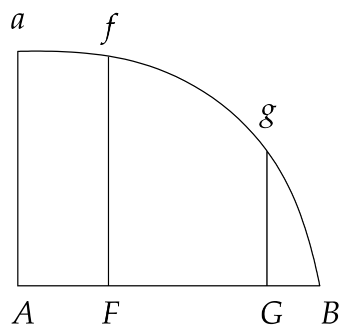


Fig. 8

For, having put the semiaxes $AB = a$ and $Aa = b$ the abscissas for the points $f$ and $g$ in question will be

$$AF = \frac{a}{2}\sqrt{\frac{5a+3b-\sqrt{9aa+14ab+9bb}}{2(a+b)}},$$

$$AG = \frac{a}{2}\sqrt{\frac{5a+3b+\sqrt{9aa+14ab+9bb}}{2(a+b)}},$$

whence for the same points one finds the ordinates

$$Ff = \frac{b}{2}\sqrt{\frac{3a+5b+\sqrt{9aa+14ab+9bb}}{2(a+b)}},$$

$$Gg = \frac{b}{2}\sqrt{\frac{3a+5b-\sqrt{9aa+14ab+9bb}}{2(a+b)}}.$$

## Problem 5

**§95** *To split the given arc pr of the ellipse (Fig. 6) into two parts pq and qr such that the difference of these parts, pq − qr, is assignable geometrically.*

### Solution

Having put $AP = p$, $AQ = q$ and $AR = r$ as in the preceding problem, while the semiaxes are $AB = a$ and $AA = a\sqrt{1-m}$, find the arc $ak$ starting from the vertex $a$ that having put its abscissa $AK = k$ it is

$$k = \frac{qP - pQ}{a^4 - mppqq} = \frac{a^4(qq-pp)}{qP+pQ},$$

and it will be

$$\text{Arc. } ak - \text{Arc. } pq = \frac{mkpq}{aa}.$$

But then let it also be

$$k = \frac{rQ - qR}{a^4 - mqqrr} = \frac{a^4(rr-qq)}{rQ+qR};$$

and it will also be

$$\text{Arc. } ak - \text{Arc. } qr = \frac{mkqr}{aa}$$

and hence

$$\text{Arc. } pq - \text{Arc. } qr = \frac{mkq}{aa}(r-p).$$

Therefore, since the abscissas $p$ and $r$ are given along with the quantities derived from them, $P$ and $R$, the abscissa of the point $q$ in question must be determined from this equation

$$\frac{qP+pQ}{qq-pp} = \frac{rQ+qR}{rr-qq}$$

or

$$Pq(rr-qq) - Rq(qq-pp) = Q(p+r)(qq-pr),$$

which equation squared and then divided by $(qq-pp)(rr-qq)$ gives

$$a^4((p+r)^2-2qq) - 2(m+1)aaprqq + mqq(qq(p+r)^2-2pprr) = 2qqPR : a^4$$

or

$$q^4 = \frac{2qq\left(\frac{PR}{a^4} + mpprr + (m+1)aapr + a^4\right) - a^4(p+r)^2}{m(p+r)^2},$$

by which equation the value of the abscissas $q$ can be defined. Q.E.I.

## Corollary 1

**§96** If the whole quadrant must be divided into two parts, whose difference is geometric, one has to put $p=0$ and $r=a$; hence it is $P=a^4$ and $R=0$ and hence

$$q^4 = \frac{2aaqq-a^4}{m} \quad \text{and} \quad qq = \frac{aa(1-\sqrt{1-m})}{m} \quad \text{and} \quad q = a\sqrt{\frac{1-\sqrt{1-m}}{m}},$$

which is the same determination we found above in the corollary of case 1 in problem 1.

### Corollary 2

**§97** If the one of the abscissas $p$ and $r$ is negative and equal to the other one or it is $p+r=0$, one will immediately have either $q=0$ or

$$Prr - Pqq - Rqq + Rp = 0 \quad \text{or} \quad qq = \frac{Prr+Rpp}{P+R} \quad \text{and hence} \quad P+R=0.$$

But it is obvious, if each of both ordinates $Pp$ and $Rr$ were positive, that it will be $R=P$ and then it its $q=0$.

## Problem 9

**§98** *If the ellipse $ADBFA$ (Fig 9.) was dissected by the diameter $ECF$, to split the half of the circumference $EBF$ in the point $M$ in such a way that the difference of the parts $EM$ and $FM$ is assignable geometrically.*

### Solution

Even though this problem is contained in the preceding one, it it nevertheless not possible to deduce the solution that problem, since it is so $P+r=0$ as $P+R=0$; therefore, one must the find the solution in a peculiar way.

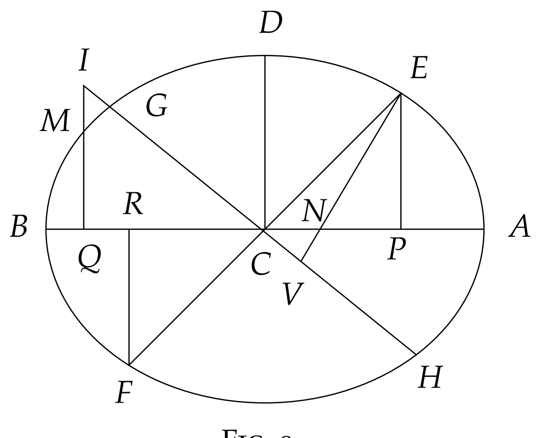


Fig. 9

Therefore, having put the semiaxes $CA=a$, $CD=b=a\sqrt{1-m}$ for the one endpoint $E$ of the propounded arc let the abscissa be $CP=p$; the ordinate will be $PE=\frac{b}{a}\sqrt{aa-pp}$, which coordinate taken negatively will extend to the other endpoint $F$; but let these coordinates be $r$ and $\frac{b}{a}\sqrt{aa-rr}$ such that it is $r=-p$ and $\sqrt{aa-rr}=-\sqrt{aa-pp}$. Since now having taken the new abscissa $k$ and having put the abscissa in question $CQ=q$ from corollary 2 of problem 2 we have

$$aa\sqrt{aa-kk}-pq\sqrt{aa-mkk}=a\sqrt{(aa-pp)(aa-qq)},$$
$$aa\sqrt{aa-kk}-qr\sqrt{aa-mkk}=a\sqrt{(aa-qq)(aa-rr)},$$

this last equation, because of

$$r=-p \quad \text{and} \quad \sqrt{aa-rr}=-\sqrt{aa-pp},$$

goes over into this one

$$aa\sqrt{aa-kk}+pq\sqrt{aa-mkk}=-a\sqrt{(aa-p)(aa-qq)},$$

which added to the first gives

$$2aa\sqrt{aa-kk}=0 \quad \text{and hence} \quad k=a;$$

this value substituted in the other gives

$$-pq\sqrt{1-m}=\sqrt{(aa-pp)(aa-qq)}$$

and hence

$$\frac{-q}{\sqrt{aa-qq}}=\frac{\sqrt{aa-pp}}{p\sqrt{1-m}},$$

as a logical consequence it is

$$q=-\frac{a\sqrt{aa-pp}}{\sqrt{aa-mpp}},$$

where the negative sign indicates that $q$ must be taken in the negative part of the abscissas[6]. Draw the normal $EN$ to the curve in $E$; then will have

$$\frac{PE}{EN}=\frac{\sqrt{aa-pp}}{\sqrt{aa-mpp}}.$$

[6]Nowadays we would call this the negative $x$-axis.

Therefore, it is $CQ = \frac{a \cdot PE}{EN}$. Further, let $GH$ be the conjugated diameter, which the normal $EN$ intersects in $V$; it will be

$$\frac{PE}{EN} = \frac{CV}{CN} = \frac{CQ}{CI},$$

having elongated $CG$ to the point of intersection with the ordinate $QM$ in $I$. Hence, because of $CQ = \frac{a \cdot CQ}{CI}$ $CI = a = CA$ results. Hence this simple construction follows: Continue the conjugated diameter $GH$ beyond the point $G$ to $I$ that it is $CI = CA$; from $I$ drop the perpendicular $IQ$ to the axis $AB$; this perpendicular will intersect the ellipse in the point $M$ in question. But because of $k = a$ it will be

$$\text{Arc. } EM - \text{Arc. } FM = -\frac{2mpq}{a} = \frac{2mp \cdot PE}{EN} = \frac{2CN \cdot CV}{CN} = 2CV$$

- because of $CN = mp$. Q. E. I.

## COROLLARY 1

**§99** If, using the same two equations, by eliminating $k$ the preceding problem is solved in general, one will obtain the following equation

$$mq^4(r\sqrt{aa-pp} - p\sqrt{aa-rr})^2 - 2aaqq(aa+mpr)(aa-pr-\sqrt{(aa-pp)(aa-rr)})$$
$$+a^6(\sqrt{aa-pp} - \sqrt{aa-rr})^2 = 0,$$

whence by resolution we obtain

$$qq = \frac{aa(aa-pr-\sqrt{(aa-pp)(aa-rr)})(aa+mpr \pm \sqrt{(aa-mpp)(aa-mrr)})}{m(r\sqrt{rr-pp} - p\sqrt{aa-rr})^2},$$

$$q = \frac{a\left(\sqrt{\frac{a+r)(a-p)}{2}} - \sqrt{\frac{a-r)(a+p)}{2}}\right)\left(\sqrt{\frac{a+p\sqrt{m})(a+r\sqrt{m})}{2m}} \pm \sqrt{\frac{a-p\sqrt{m})(a-r\sqrt{m})}{2m}}\right)}{r\sqrt{aa-pp} - p\sqrt{aa-rr}}.$$

## Corollary 2

**§100** Although this solution in principle does not differ from the solution of the preceding problem, it nevertheless solves the present problem immediately. For, if we put

$$r = -p \quad \text{and} \quad \sqrt{aa - rr} = -\sqrt{aa - pp},$$

the first equation of the preceding corollary goes over into this form

$$-2aaqq(aa - mpp) \cdot 2aa + a^6(2\sqrt{aa - pp})^2 = 0$$

or

$$qq = \frac{aa(aa - pp)}{aa - mpp}.$$

## Corollary 3

**§101** If we eliminate $q$ from the first two equations, we will obtain

$$q = \frac{aa(\sqrt{aa - pp} - \sqrt{aa - rr})\sqrt{aa - kk}}{(r\sqrt{aa - pp} - p\sqrt{aa - rr})\sqrt{aa - mkk}}$$

and

$$\sqrt{aa - qq} = \frac{a(r - p)\sqrt{aa - kk}}{r\sqrt{aa - pp} - p\sqrt{aa - rr}};$$

hence it is

$$a^4(aa - kk)(\sqrt{aa - pp} - \sqrt{aa - rr})^2 + a^2(aa - kk)(aa - mkk)(r - p)^2$$
$$= aa(aa - mkk)(r\sqrt{aa - pp} - p\sqrt{aa - rr})^2$$

or

$$mk^4(r - p) = 2kk(aa - mpr)(aa - pr - \sqrt{(aa - pp)(aa - rr)})$$
$$- aa(aa - pr - \sqrt{(aa - pp)(aa - rr)})^2,$$

whence it is

$$kk = \frac{(aa - pr - \sqrt{(aa-pp)(aa-rr)})(aa - mpr - \sqrt{(aa-mpp)(aa-mrr)})}{m(r-p)^2},$$

and hence it is concluded

$$k = \frac{\left(\sqrt{\frac{(a+r)(a-p)}{2}} - \sqrt{\frac{(a-r)(a+p)}{2}}\right)\left(\sqrt{\frac{(a+r\sqrt{m})(a-p\sqrt{m})}{2m}} - \sqrt{\frac{(a-r\sqrt{m})(a+p\sqrt{m})}{2m}}\right)}{r-p}.$$

## COROLLARY 4

**§102** Hence it will be

$$kq = \frac{aa(aa - pr - \sqrt{(aa-pp)(aa-rr)})(\sqrt{aa-mpp} - \sqrt{aa-mrr})}{m(r-p)(r\sqrt{aa-pp} - p\sqrt{aa-rr}}.$$

Hence, since the difference of the arcs $pq$ and $qr$ is $= \frac{mkq}{aa}(r-p)$, we will have in general

$$\text{Arc. } pq - \text{Arc. } qr = \frac{(aa - pr - \sqrt{(aa-pp)(aa-rr)})(\sqrt{aa-mpp} - \sqrt{aa-mrr})}{r\sqrt{aa-pp} - p\sqrt{aa-rr}},$$

if the point $q$ is defined using the result from corollary 1, of course. Therefore, it will be

$$\text{Arc. } pq - \text{Arc. } qr = \frac{(\sqrt{aa-pp} - \sqrt{aa-rr})(\sqrt{aa-mpp} - \sqrt{aa-mrr})}{r+p}$$

and

$$q = \frac{\left(\sqrt{\frac{(a+r)(a+p)}{2}} - \sqrt{\frac{(a-r)(a-p)}{2}}\right)\left(\sqrt{\frac{(a+p\sqrt{m})(a+r\sqrt{m})}{2m}} - \sqrt{\frac{(a-p\sqrt{m})(a-r\sqrt{m})}{2m}}\right)}{p+r}.$$

## PROBLEM 7

**§103** *Having propounded the arc $fg$ of an ellipse (Fig. 6), starting from the point $p$ to separate the arc $pqrs$, which differs from the triple of that arc $fg$ by a geometrically assignable quantity.*

### SOLUTION

Let, as before, the abscissas of the given points $f$, $g$ and $p$ be $AF = f$, $AG = g$, $AP = p$ and first find the arc $ak$, whose abscissa is

$$AK = k = \frac{gF - fG}{a^4 - mffgg} = \frac{a^4(pp - ff)}{gF + fG},$$

that it is

$$\text{Arc. } ak - \text{Arc. } fg = \frac{mkfg}{aa}.$$

Then find the point $q$ that it is

$$AQ = q = \frac{pK + kP}{a^4 - mkkpp} = \frac{a^4(pp - kk)}{pK - kP}$$

and hence

$$Q = \frac{a^4(qq - pp) - kk(a^4 - mppqq)}{2kp} = \frac{pq(qq - pp)K - kq(qq - kk)P}{kp(pp - kk)},$$

and it will be

$$\text{Arc. } fg - \text{Arc. } pq = \frac{mk}{aa}(pq - fg).$$

Further, in like manner find the point $r$ that it is

$$AR = r = \frac{qK + kQ}{aa - mkkqq} = \frac{a^4(qq - kk)}{qK - kQ}$$

and

$$R = \frac{a^4(rr - qq) - kk(a^4 - mqqrr)}{2kq} = \frac{qr(rr - qq)K - kr(rr - kk)Q}{kq(qq - kk)},$$

and since it is

$$\text{Arc. } fg - \text{Arc. } qr = \frac{mk}{aa}(qr - fg),$$

it will be

$$2\text{Arc. } fg - \text{Arc. } pqr = \frac{mk}{aa}(pq + qr - 2fg).$$

Hence let us define the point $s$ in the same way that the abscissa is

$$AS = s = \frac{rK + kR}{a^4 - mkkrr} = \frac{a^4(rr - kk)}{rK - kR}$$

and

$$S = \frac{a^4(ss - rr) - kk(a^4 - mrrss)}{2kr} = \frac{rs(ss - rr)K - ks(ss - kk)R}{kr(rr - kk)},$$

and since it will be

$$\text{Arc. } fg - \text{Arc. } rs = \frac{mk}{aa}(rs - fg),$$

one will have

$$3\text{Arc. } fg - \text{Arc. } pqrs = \frac{mk}{aa}(pq + qr + rs - 3fg).$$

Q. E. I.

### COROLLARY 1

**§104** Proceeding in the same way it is obvious that one, starting from a given point $p$, is able to define an arc $pt$ which differs from the quadruple of the given arc $fg$ by an algebraic quantity, and that the operation can be continued this way arbitrarily far.

### COROLLARY 2

**§105** If the given arc $fg$ is equal to the whole quadrant that it is $f = 0$ and $g = a$ and hence $F = a^4$ and $G = 0$, it will be $k = a$ and $K = 0$. Hence one finds

$$q = \frac{P}{a(aa - mpp)} = a\sqrt{\frac{aa - pp}{aa - mpp}}$$

and

$$Q = \frac{-q(qq - aa)}{p(pp - aa)}P = \frac{-(aa - qq)PP}{ap(aa-, pp)(aa - pp)} = -\frac{a^3(aa - qq)}{p};$$

but it is

$$aa - qq = \frac{a(1 - m)pp}{aa - mpp}, \quad \text{whence it is} \quad Q = \frac{-(1 - m)a^5 p}{aa - mpp}.$$

Further

$$r = \frac{Q}{a(aa - mqq)} = -p$$

and

$$R = -aa\sqrt{(aa - pp)(aa - mpp)} = -P.$$

Finally, it will be

$$s = \frac{-P}{a(aa - mpp)} = -a\sqrt{\frac{aa - pp}{aa - mpp}} = -q \quad \text{and} \quad S = -Q = \frac{(1 - m)a^5 p}{aa - mpp}$$

and it will be

$$3\text{Arc. } fg - \text{Arc. } pqrs = \frac{m}{a}pq = mp\sqrt{\frac{aa - pp}{aa - mpp}}.$$

## COROLLARY 3

**§106** One will also be able to define the point $p$ in such a way that it is

$$pq + qr + rs = 3fg,$$

in which case the arc $pqrs$ will become exactly equal to the triple of the given arc $fg$. And so one will further be able to find an arc, which has a certain ratio to the given arc $fg$.

## Scholium

**§107** All these problems, which I treated here for the ellipse, can likewise be solved for the hyperbola; so, given an arbitrary arc of a hyperbola, starting from a certain point of the same hyperbola one will be able to separate an arc, which differs either from the given arc or from the double of the given arc or the triple or from any other multiple of it by a geometrically assignable quantity. Further, it will also be possible to assume this point in such a way that the difference vanishes completely, in which case, given an arbitrary arc of the hyperbola, one will be able to assign another arc, which is equal either to the double or the triple or to any other multiple of the given arc. Hence it is perspicuous, if having propounded an arc another arc was found, which has a ratio of $\mu$ to 1 to the given one, and in like manner another arc is found, which has a ratio of $\nu$ to 1 to the given arc, that then this way one has two arcs of the hyperbola, which have a ratio of $\mu$ to $\nu$ to each other, and so one will be able to exhibit infinitely many pairs of arcs, which have a given ratio to each other. And problems of this kind can not only be solved for the hyperbola but also for all other arbitrary curves, which are of such a nature that the arc corresponding to the abscissa or to the variable $x$ are contained in this form

$$\int \frac{dx(\mathfrak{A}+\mathfrak{B}xx+\mathfrak{C}x^4)}{\sqrt{A+Cxx+Ex^4}},$$

which can also be extended a lot further by the rules given at the beginning in such a way that it is reduced to this form

$$\int \frac{dx(\mathfrak{A}+\mathfrak{B}xx+\mathfrak{C}x^4+\mathfrak{D}x^6+\mathfrak{E}x^8+\text{etc.})}{\sqrt{A+Cxx+Ex^4}};$$

but I think that for now one does not have to spend any more time on neither the hyperbola nor on other other curves of this kind.