Investigation of Functions from a given condition of the differentials *

Leonhard Euler

§1 If V denotes an arbitrary function of the two variables x and y, and it is differentiated so that its differential results as

$$dV = Pdx + Qdy,$$

but then these two quantities P and Q are differentiated again, and so it results

$$dP = pdx + rdy \quad \text{and} \quad dQ = sdx + qdy,$$

it is known that it will always be $r = s$. I usually also express this property this way that I say that it is

$$\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right).$$

Of course, by an expression of this kind $\left(\frac{dP}{dy}\right)$ I indicate that the function P is differentiated in such a way that only the quantity y is treated as a variable and the resulting differential is divided by dy, having done which necessarily a finite quantity not containing differentials results.

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§2 Therefore, if the formula $Pdx + Qdy$ was of such a nature that according to this notation in it it is $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$, we hence vice versa conclude that this formula $Pdx + Qdy$ indeed results from the differentiation of a certain function V of x and y. But because this property extends very far, hence it is not possible to conclude more than that the formula $Pdx + Qdy$ is integrable, and nothing is defined in general by this condition alone, whence a peculiar property of the function, from whose differentiation it resulted, could be deduced.

§3 But whenever the function V belongs to a certain class, then having put $dV = Pdx + Qdy$, among the quantities P and Q, except for this general property, there is another particular relation. So we know, if V is a function of no dimension of the two variables x and y, that then except for that general property by which it is $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$ additionally this particular property holds that it is $Px + Qy = 0$. Further, in like manner it was demonstrated, if V was a function of such a kind of x and y, whose number of dimensions was $= n$, that its differential $dV = Pdx + Qdx$ is always of such a nature that it is

$$nV = Px + Qy.$$

Therefore, in this case it is especially remarkable that the integral of the differential formula $Pdx + Qdy$ can be assigned immediately, since it is

$$V = \frac{1}{n}(Px + Qy).$$

§4 Since these things are demonstrated and well known, I recently had the idea to treat questions of this kind in inverse manner and investigate a method, by means of which, if having put $dV = Pdx + Qdy$ it was already found that it is either $Px + Qy = 0$ or $Px + Qy = nV$, it can be found vice versa that V is either a function of no dimension or a homogeneous function, in which the two variables x and y add up to n dimensions everywhere. Of course, not having taken into account those already known properties one must only use the property that it is either $Px + Qy = 0$ or $Px + Qy = nV$ and then by means of legitimate analytical reasons it has to be shown that the function V has this property that it is either of no dimension or a homogeneous function of n dimensions. But it is to be understood that the general property $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$

has to hold always; for, if it would not hold, the equation $dV = Pdx + Qdy$ would even be absurd.

§5 These questions, which seem to be have been hardly considered until now, open a very vast field to extend the boundaries of Analysis even further. For, having propounded the equation $dV = Pdx + Qdy$, one can ask for the nature of the function V, if any relation among the two quantities P and Q or even among the three P, Q and V is propounded. Even though questions of this kind seem to be almost new, there is nevertheless no doubt that the method to resolve them correctly will have the highest utility throughout the whole of Mathematics. For, in the problem of vibrating strings the whole solution is to be attributed to a method of this kind, since the problem can be formulated by a certain relation among the quantities P and Q. Further, I even reduced the whole theory of the motion of fluids in differential formulas of this kind, where certain relations among parts of the differential formulas are prescribed, from which because of the lack of such a method hardly anything can be concluded.

§6 But questions of this kind can also be propounded in another way that only the function V whose nature is in question occurs. For, because having put $dV = Pdx + Qdy$ it is $P = \left(\frac{dV}{dx}\right)$ and $Q = \left(\frac{dV}{dy}\right)$, the first question can be formulated as this:

To find the nature of the function V that it is

$$x\left(\frac{dV}{dx}\right) + y\left(\frac{dV}{dy}\right) = 0,$$

the second on the other hand this way:

To find the nature of the function V that it is

$$x\left(\frac{dV}{dx}\right) + y\left(\frac{dV}{dy}\right) = nV;$$

and in the first question it must be shown that V is a function of no dimension of x and y; in the second on the other hand one has to prove that the function is a homogeneous function of n dimensions. But it is to be noted that in our notation it is

$$dV = dx\left(\frac{dV}{dx}\right) + dy\left(\frac{dV}{dy}\right).$$

§7 Therefore, a further extending problem can be formulated in such a way that having propounded any arbitrary relation among the quantities V, $\left(\frac{dV}{dx}\right)$ and $\left(\frac{dV}{dy}\right)$ it has to be defined, how the function V depends on x and y. Further, questions of this kind can also be extended to functions of three or more variables; even quantities to result from double or triple differentiation can be introduced; of this kind are $\left(\frac{ddV}{dx^2}\right)$, $\left(\frac{ddV}{dxdy}\right)$, $\left(\frac{ddV}{dy^2}\right)$, $\left(\frac{d^3V}{dx^3}\right)$, $\left(\frac{d^3V}{dx^2dy}\right)$ etc., whose meaning is that, for the sake of an example, $\left(\frac{d^3V}{dx^2dy}\right)$ results, if first V is differentiated having taken only x as a variable, and the differential is divided by dx that $\left(\frac{dV}{dx}\right)$ results; but then this quantity is differentiated again, having taken only x as a variable, that its differential divided by dx yields $\left(\frac{ddV}{dx^2}\right)$, which finally differentiated once more, having taken only y as a variable, and divided by dy will give $\left(\frac{d^3V}{dx^2dy}\right)$.

§8 But using this notation it is to be noted that it is

$$\left(\frac{ddV}{dxdy}\right) = \left(\frac{ddV}{dydx}\right)$$

and

$$\left(\frac{d^3V}{dx^2dy}\right) = \left(\frac{d^3V}{dxdydx}\right) = \left(\frac{d^3V}{dydx^2}\right).$$

It does not matter, in which order the quantities x and y, both of which are assumed as the only variable in each differentiation, are distributed, as long as only the prescribed number of differentiations is done. Even if V is a function of three variables x, y, z the same property will hold; for, it will be

$$\left(\frac{d^3V}{dxdydz}\right) = \left(\frac{d^3V}{dxdzdy}\right) = \left(\frac{d^3V}{dydzdx}\right) = \left(\frac{d^3V}{dzdxdy}\right) = \left(\frac{d^3V}{dzdydx}\right);$$

but here I will restrict my investigations only to functions of two variables.

§9 This way we are not only led to unusual still hardly treated questions but also to new notations, which we are still hardly used to; hence this method, whose cultivation we must desire that much, is justly to be considered as a completely new part of Analysis. Therefore, I think it will be quite useful, if I constitute only the first foundations of this method, and because of the novelty of the subject I do not dare to promise hardly the smallest part of the construction to be built on later. But in time without any doubt a lot of and tireless work will be necessary, before this certainly most difficult part of Analysis, can, I do not say, be completed, but be useful to answer many questions in analysis. Therefore, I decided to explain the things which I found until now in this subject in order and carefully, that for others, which are encouraged by the dignity of the subject to invest the same amount of work, I clear the first obstacles out of the way and prepare the minds for this new kind of investigations.

§10 But before I provide this service, I think certain per se perspicuous principles are to be mentioned in advance. At first, of course, if it was $\left(\frac{dV}{dx}\right) = 0$, it is understood that the function V does not depend on x at all, but contains only the other variable y and constants, and so the form $\left(\frac{dV}{dy}\right)$ will also be a function of y only. But if vice versa $\left(\frac{dV}{dy}\right)$ was a function of y only, the quantity V will be an aggregate of a function only of y and of a function only of x; therefore, in this case the form $\left(\frac{dV}{dx}\right)$ will be a function only of x. Further, if it was $dV = RdS$, it is necessary that R is a function of S or S is one of R, whence also V will be a function of S, or of R; since otherwise the integral $\int RdS$ is not defined. Therefore, having set these principles I will resolve the two initially mentioned questions, which marked the starting point of this investigation, afterwards I will proceed to others.

Problem 1

§11 While it is $dV = Pdx + Qdy$, if it was $Px + Qy = 0$, to find, what a function of x and y V is that this condition is fulfilled.

Solution

Therefore, since among the quantities P and Q this relation is prescribed that it is $Px + Qy = 0$, it will be $Q = -P\frac{x}{y}$, which value substituted in the equality

$dV = Pdx + Qdy$ will give

$$dV = P\left(dx - \frac{xdy}{y}\right) = \frac{P(ydx - xdy)}{y};$$

therefore, it is necessary that the formula

$$\frac{P(ydx - xdy)}{y}$$

is integrable. To reduce it to the form RdS, represent it this way

$$dV = Py \cdot \frac{ydx - xdy}{yy}.$$

For, having taken

$$\frac{ydx - xdy}{yy} = dS$$

and $S = \frac{x}{y}$, since it is $dV = PydS$, it is necessary that Py is a function of S and hence V will be a function of S, this means of $\frac{x}{y}$. Therefore, the prescribed property, by which it is $Px + Qy = 0$, indicates a form of the function of this kind that V is any arbitrary function of $\frac{x}{y}$; but hence it is obvious that for V a function of no dimension of x and y results.

COROLLARY 1

§12 Therefore, if because of

$$P = \left(\frac{dV}{dx}\right) \quad \text{and} \quad Q = \left(\frac{dV}{dy}\right)$$

this property of the function V is propounded that it is

$$x\left(\frac{dV}{dx}\right) + y\left(\frac{dV}{dy}\right) = 0,$$

we hence conclude that V is a function of the formula $\frac{x}{y}$ or, what reduces to the same, it is a function of no dimension of x and y.

§13 Therefore, hence vice versa it is confirmed, what is known for a long time already, as often as V was a function of no dimension of x and y, that so often it will also be:

$$x\left(\frac{dV}{dx}\right)+y\left(\frac{dV}{dy}\right)=0.$$

But although this is proven very easily, its converse required a singular proof.

Scholium

§14 The essence of this solution is that I reduced the differential of the function V to this form $dV = RdS$, from which, since it contains the one single differential dS, it obviously follows that V is a function of the quantity S only; but it was $S = \frac{x}{y}$, and it is known that all functions of $\frac{x}{y}$ at the same time are functions of no dimension and vice versa. Therefore, it is understood that the same principle is to be applied in the solution of the following problems. Furthermore, without the letters P and Q the problem could have been so propounded as solved this way: Of course, if one has to find the form of the function V that it is

$$x\left(\frac{dV}{dx}\right)+y\left(\frac{dV}{dy}\right)=0,$$

because it is

$$dV=dx\left(\frac{dV}{dx}\right)+dy\left(\frac{dV}{dy}\right),$$

because of

$$\left(\frac{dV}{dy}\right)=-\frac{x}{y}\left(\frac{dV}{dx}\right),$$

it will be

$$dV=\left(dx-\frac{xdy}{y}\right)\cdot\left(\frac{dV}{dx}\right)=y\left(\frac{dV}{dx}\right)\cdot d.\frac{x}{y};$$

hence it is obvious that V necessarily must be a function of this one quantity $\frac{x}{y}$. But as, if it was $dV = Rdr$, it is correctly concluded that V is a function of r only, so further, if it was

$$dV = Rdr + Sds,$$

we have to conclude that V is a function of the two variables r and s; this principle will be useful in the investigation of the nature of function of three variables, while a certain condition of the differentials is propounded.

Problem 2

§15 While it is

$$dV = Pdx + Qdy,$$

to define the nature of the function V that it is

$$Px + Qy = nV,$$

where n denotes an arbitrary number.

Solution

From the prescribed condition

$$Px + Qy = nV$$

find the one of the quantities P and Q, say

$$Q = \frac{nV}{y} - \frac{Px}{y},$$

which value substituted in the differential equation will give

$$dV = Pdx + \frac{nVdy}{y} - \frac{Pxdy}{y},$$

which in turn immediately gives this form:

$$dV = -\frac{nVdy}{y} = Py\frac{(ydx - xdy)}{yy} = Pyd.\frac{x}{y},$$

which, that the left-hand side is rendered integrable, we want to multiply by y^{-n}, and so it will result

$$d.y^{-n}V = Py^{1-n}d.\frac{x}{y},$$

whence we conclude that $y^{-n}V$ is a function of the quantity $\frac{x}{y}$ or a function of no dimension of the two variables x and y. Therefore, let Z denote an arbitrary function of no dimension, and because it is $y^{-n}V = Z$, it will be $V = y^n Z$; but such an expression contains all homogeneous functions of x and y, whose number of dimensions is $= n$.

COROLLARY 1

§16 Therefore, whenever we know that V is a function of such a nature that it is

$$nV = x\left(\frac{dV}{dx}\right) + y\left(\frac{dV}{dy}\right),$$

we will able to affirm with certainty that V is a homogeneous function in which the two variables add up to n dimensions everywhere.

COROLLARY 2

§17 Therefore, if one puts

$$dV = Pdx + Qdy,$$

also P and Q will be homogeneous functions of x and y, but their number of dimensions is $n-1$, lower by one order, of course.

COROLLARY 3

§18 Hence, if P and Q were homogeneous functions of the two variables x and y, whose number of dimensions is the same, say $= n-1$, and if it furthermore was

$$\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right),$$

such that $Pdx + Qdy$ is a complete differential formula, then its integral will be assigned most easily. It will be

$$\int (Pdx + Qdy) = \frac{Px + Qy}{n}.$$

Therefore, as long as n does not vanish, an integral of formulas of this kind can be exhibited without any other operation.

Scholium

§19 Therefore, lo and behold the solution of both questions I mentioned initially; since they already contain a specimen of this method, which is to be applied in this kind of questions, it will be possible to apply the same to the solution of other similar problems. Therefore, because here a certain relation among the function V and the quantities derived from it, $P = \left(\frac{dV}{dx}\right)$ and $Q = \left(\frac{dV}{dy}\right)$, is propounded, from which the nature of the function V must be defined, in order to treat the questions of this kind in a certain order, since so P and Q as V are functions of x and y, at first I will assume that the form of the one of the letters P and Q is given; further, I will proceed to problems of such a kind, in which a certain relation among P and Q is prescribed; but then to the others, where a certain relation must intercede among P and V or among Q and V. Finally, I will assume that the given relation is extended to all three quantities V, P and Q, as it was done in the second problem. But because here we only take into account the quantities V, $\left(\frac{dV}{dx}\right)$, $\left(\frac{dV}{dy}\right)$, it is evident that investigations of this kind can be extended a lot further, while the prescribed relation contains other quantities derived from V, as for example $\left(\frac{ddV}{dx^2}\right)$, $\left(\frac{ddV}{dxdy}\right)$ and $\left(\frac{ddV}{dy^2}\right)$. But so much is missing that I dare to promise solutions of all problems of this kind, that I will rather expand only those problems which are easier. For, it will soon be plain that innumerable problems of such a kind can be propounded whose solutions exceed these first attempts by far, and, before this new branch of Analysis was further developed, cannot be hoped for.

Problem 3

§20 While it is

$$dV = Pdx + Qdy,$$

if P was a function of x only to define the nature of the function V.

SOLUTION

It is known that from the part Pdx of the differential the function V can already be found; for, having considered y as a constant, integrate the differential Pdx and assume the arbitrary constant to be added to involve the other variable y somehow. Therefore, because P is a function of x only, $\int Pdx$ will also be a function of such a kind, which we want to be $= X$, and represent the constant to be added by an arbitrary function Y of y only. Therefore, hence $V = X + Y$ will result, or the nature in question of the function V will is this that V is the aggregate of two functions, the one of x only, the other of y only.

COROLLARY

§21 Therefore, because hence it is

$$dV = dX + dY,$$

it is obvious, if P was a function of x only, that then Q will be a function of y only, which is certainly a very well known property.

PROBLEM 4

§22 While it is

$$dV = Pdx + Qdy,$$

if P was a function of y only, to define the nature of the function V.

SOLUTION

Since P is a function of y only, from the part Pdx of the differential alone, having considered y as a constant, the function V is defined in such a way that it is $V = Px + Y$, where Y denotes an arbitrary function of y. Hence the nature in question of the function V will be that, while P and Y denote arbitrary functions of y, the form of the function V always is of this kind $V = Px + Y$.

COROLLARY 1

§23 Therefore, if $P = \left(\frac{dV}{dx}\right)$ is a function of y only, since it is

$$dV = Pdx + xdP + dY,$$

it will be

$$Q = \frac{xdP + dY}{dy},$$

or because of $\frac{dP}{dy} = \left(\frac{ddV}{dxdy}\right)$ it will be

$$Q = x\left(\frac{ddV}{dxdy}\right) + \frac{dY}{dy}.$$

COROLLARY 2

§24 Therefore, if $\left(\frac{dV}{dx}\right)$ was a function of y only, it is necessary that it is

$$\left(\frac{dV}{dy}\right) = x\left(\frac{ddV}{dxdy}\right) + \frac{dY}{dy},$$

where $\frac{dY}{dy}$ denotes an arbitrary function only of y. Hence one can vice versa conclude, if it was

$$\left(\frac{dV}{dy}\right) = x\left(\frac{ddV}{dxdy}\right) + f \cdot y,$$

that $\left(\frac{dV}{dx}\right)$ will be a function only of y.

COROLLARY 3

§25 In like manner it can be shown, if $Q = \left(\frac{dV}{dy}\right)$ was a function only of x, that it will be

$$V = Qy + X,$$

where X denotes an arbitrary function only of x; but then it can also be proved, if it was

$$\left(\frac{dV}{dx}\right) = y\left(\frac{ddV}{dxdy}\right) + f \cdot x,$$

that $\left(\frac{dV}{dy}\right)$ will be a function only of x.

Problem 5

§26 While $dV = Pdx + Qdy$, if P was a homogeneous function of x and y, whose number of dimensions is $= n$, to define the nature of the function V.

Solution

Since P is a homogeneous function of n dimensions, if the part Pdx of the differential is integrated having considered y as a constant, the integral will be a homogeneous function of $n+1$ dimensions, let Z be such an arbitrary homogeneous function, and it will be $V = Z + Y$, while Y is an arbitrary function of y only, which is the nature in question of the function V.

Corollary

§27 Therefore, in like manner it will be shown, if Q was a homogeneous function of n dimensions, that it will be $V = Z + X$, where, as before, Z denotes an arbitrary homogeneous function of $n+1$ dimensions and X one of x only.

Problem 6

§28 While $dV = Pdx + Qdy$, if it was $Q = nP$, to define the nature of the function V.

Solution

Because it is $Q = nP$, it will be $dV = P(dx + ndy)$; hence having put $x + ny = s$, it will be $dV = Pds$. From this V can only have a certain value, if P is a function of s, from which also V will be a function of s. As a logical consequence, if it was $Q = nP$, the nature of the quantity V will be that it is an arbitrary function of the formula $x + ny$; or if the character Φ is used to denote an arbitrary function of the quantity written behind it, it will be $V = \Phi : (x + ny)$.

Problem 7

§29 While $dV = Pdx + Qdy$, if it was

$$Py + Qx = 0,$$

to find the nature of the function V.

SOLUTION

Since it is $Py + Qx = 0$, it will be $Q = -\frac{Py}{x}$, and hence

$$dV = Pdx - \frac{Pydy}{x} = \frac{P}{x}(xdx - ydy).$$

Therefore, having put $xx - yy = s$, because of $xdy - ydy = \frac{1}{2}ds$, it is $dV = \frac{P}{2x}ds$. Since this formula is integrable by assumption, it is necessary that $\frac{P}{2x}$ is a function of s, whence also V will result as a function of $s = xx - yy$. Therefore, the nature in question will be that V is an arbitrary function of the quantity $xx - yy$.

PROBLEM 8

§30 While it is $dV = Pdx + Qdy$, if it was $Q = Pp$, while p expresses an arbitrary given function of x and y, to define the nature of the function V.

SOLUTION

Therefore, we will have $dV = Pdx + Ppdy = P(dx + pdy)$. Now, consider the formula $dx + pdy$, if which was not integrable per se, a multiplicator q will be given, which renders is integrable. Therefore, let $qdx + pqdy = ds$, and s will also be a given function of x and y, and because of $dx + pdy = \frac{ds}{q}$, one will have $dV = \frac{P}{q}ds$. Therefore, it is necessary that this formula is integrable and hence the nature in question will be that V is an arbitrary function of the quantity s, how which is composited of x and y, must be concluded from the condition of the given quantity p.

COROLLARY 1

§31 This problem extends very far, since in it an arbitrary ratio of the quantities P and Q or $\left(\frac{dV}{dx}\right)$ and $\left(\frac{dV}{dy}\right)$ is propounded. For, if it is $P : Q = 1 : p$, whatever function of x and y was given for p, it can be defined, what kind of function V is.

COROLLARY 2

§32 But although always a multiplicator q exists, which renders the formula $dx + pdy$ integrable, it can nevertheless happen that because of the missing Analysis this multiplicator cannot be assigned. And in these cases the solution of the problem is impeded.

COROLLARY 3

§33 But at another occasion I showed that multiplicators of this kind can always be exhibited, if the equation $dx + pdy = 0$ can be resolved. Hence if p was not a function of x and y of such kind that the equation $dx + pdy = 0$ can be resolved, this is to be attributed to the missing Analysis, if the problem cannot be resolved.

PROBLEM 9

§34 If X and Y are given functions, the first of x only, the other on the other hand of y only, but then p is also a given function of x and y, to define the nature of the function V, that having put $dV = Pdx + Qdy$ it is $Q = (P + X)p + Y$.

SOLUTION

Therefore, because it is $Q = (P + X)p + Y$, the differential equation will be $dV = Pdx + Ppdy + XPdy + Ydy$, which is reduced to this form:

$$dV = (P + X)(dx - dy) - Xdx + Ydy,$$

where the parts Xdx and Ydy are integrable per se. Therefore, again find the multiplicator q rendering the formula $dx + pdy$ integrable, and let $q = (dx + pdy) = ds$, and it will be

$$dV = \frac{P + X}{q} ds - Xdx + Ydy,$$

since which formula is integrable by assumption, it is necessary that $\frac{P+X}{q}$ is a function of s only, if which is put $= S$, $V = \int Sds - \int Xdx + \int Ydy$ will result, which is the nature in question of the function V.

COROLLARY 1

§35 Therefore, because from the given function p the function s is defined, if for Σ an arbitrary function of this quantity s is taken, the function V in question must be of such a nature that it is

$$V = \Sigma - \int Xdx + \int Ydy.$$

COROLLARY 2

§36 Therefore, this addition of the functions X and Y does not render the solution of the problem more difficult, if X is function of x only, and Y a function of y only. But the solution, as before, depends on the resolution of the differential equation $dx + pdy = 0$, if which exceeds our powers, we also cannot resolve the problem.

COROLLARY 3

§37 One can also assume other functions of the variables x and y, say M and N, instead of X and Y, as long as the formula $Mdx + Ndy$ admits an integration. For, if this condition is propounded that it is $Q - N = (P - M)p$, the propounded function V will result expressed as this:

$$V = \Sigma + \int (Mdx + Ndy).$$

PROBLEM 10

§38 While $dV = Pdx + Qdy$, to find, what kind of function V of x and y must be that it is

$$Q = \frac{Py}{x} + nx.$$

SOLUTION

Because it is

$$Q = \frac{Py}{x} + nx,$$

it will be

$$dV = \frac{P}{x}\,(xdx + ydy) + nxdy.$$

Set $xx + yy = ss$ that it is $x = \sqrt{ss - yy}$ and it will be

$$dV = \frac{Ps}{x}ds + nxdy = \frac{Ps}{x}ds + ndy\sqrt{ss - yy},$$

whence it is plain that V is function of y and s, and of course such a one that having put s to be constant it, having differentiated it, yields $ndy\sqrt{ss - yy}$. Hence vice versa the function V will be found, if the formula $ndy\sqrt{ss - yy}$, having considered s as a constant, is integrated, and additionally any function of s is added. Therefore, because it is

$$\int ndy\sqrt{ss - yy} = \frac{1}{2}ny\sqrt{ss - yy} + \frac{1}{2}nss \arcsin\frac{y}{s},$$

by using the letter Φ to denote an arbitrary function of the expression following it, it will be

$$V = \frac{1}{2}nxy + \frac{1}{2}n(xx + yy)\arctan\frac{y}{x} + \Phi : (xx + yy),$$

in which form the function V in question is always contained.

Scholium

§39 This example, no matter how particular, is nevertheless not contained in the preceding problem, and also not in its generalization made in the last corollary, since in the reduced formula the term

$$nxdy = ndy\sqrt{ss - yy}$$

is not integrable. Hence, carefully note the artifice I used here and which is that I reduced the value of the differential dV to the two other variables s and y, of course I found $dV = Rds + Tdy$, whose first term Tdy is given absolutely, whence the problem belongs to the first class, in which the one of the two quantities P and Q is known. And by means of this artifice the following problem extending a lot further can be solved.

PROBLEM 11

§40 If p and t are given arbitrary functions of the two variables x and y, to define the nature of the function V that having put $dV = Pdx + Qdy$ it is $Q = Pp + t$.

SOLUTION

Therefore, having substituted that value for Q, one will find

$$dV = P(dx + pdy) + tdy, \tag{1}$$

where for the differential formula $dx + pdy$ again a suitable multiplicator q is to be found, which renders it integrable. Therefore, let $q(dx + pdy) = ds$, and now introduce the quantity s as a new variable, by means of which and y the other variable x is defined. This way x will become equal to a certain given function of s and y, which is to be understood to be be written in t instead of x everywhere, and so t will also be a given function of s and y. Therefore, since because of $dx + pdy = \frac{ds}{q}$ it is $dV = \frac{P}{q}ds + tdy$, V will be a function of s and y of such a kind which having considered s as a constant after a differentiation yields tdy, whence vice versa in order to find the function V integrate the differential formula tdy, having considered s as a constant this time; let the integral resulting this way be $\int tdy = T$, which is therefore also given, then, since the quantity P is not given, it will be $V = T + \Phi : s$. Finally, here and in T substitute the value of s in terms of x and y again, and it will become plain, how the function V is composited of x and y.

EXAMPLE

§41 *Let the nature of the function V be in question that having put $dV = Pdx + Qdy$ it is $Px + Qy = n(xx + yy)$.*

Therefore, because it is

$$Q = -\frac{Px}{y} + \frac{n(xx+yy)}{y}, \quad \text{it will be} \quad p = -\frac{x}{y} \quad \text{and} \quad t = \frac{n(xx+yy)}{y},$$

whence

$$dx + pdy = dx - \frac{xdy}{y}.$$

Take $q = \frac{1}{y}$, it will be

$$ds = \frac{dx}{y} - \frac{xdy}{yy} \quad \text{and} \quad s = \frac{x}{y}$$

and hence

$$x = sy \quad \text{and} \quad t = ny(ss+1).$$

Hence having considered s as a constant one will have

$$\int tdy = \frac{1}{2}nyy(ss+1) = \frac{1}{2}n(xx+yy) = T;$$

and so finally this equation results

$$V = \frac{1}{2}n(xx+yy) + \Phi : \frac{x}{y},$$

where $\Phi : \frac{x}{y}$ expresses an arbitrary function of no dimension of the variables x and y.

SCHOLIUM

§42 Hence, we successfully covered the case, in which the relation among P and Q is expressed by means of an arbitrary equation of first order, in which the quantities P and Q do not rise higher than the first dimension. For, from such an equation Q will always be defined in such a way that it is $Q = Pp + t$, while p and t are arbitrary given functions of x and y. But here we cannot proceed any further, if the equation $dx + pdy = 0$ cannot be resolved, since then the quantity s cannot be found. But then, even though this quantity s was found, since it is highly transcendental, from it it will nevertheless in most cases be very difficult to define the variable x such that only the two s and y remain in the calculation. One will be able to think of auxiliary tools and theorems applying which, although the variable x is not eliminated from the given function t, one can nevertheless find the integral T of the formula t, which has to result, having considered s as a constant. But no matter how large these difficulties are, they are not to be attributed to this method, which I started to explain. Therefore, let us see how far we can proceed, if a relation among P and Q either of second or higher order is given.

PROBLEM 12

§43 While it is $dV = Pdx + Qdy$, to define the nature of the function V that it is $PQ = \alpha$.

SOLUTION

Because it is $Q = \frac{\alpha}{P}$, it will be $dV = Pdx + \frac{\alpha dy}{P}$, and it is in question, what kind of function P must be so that this differential formula $Pdx + \frac{\alpha dy}{P}$ becomes integrable. But let us apply an obvious transformation of integrals here, according to which it is $\int zdu = zu - \int udz$, and one will find:

$$V = Px + \frac{\alpha y}{P} - \int xdP + \int \frac{\alpha ydP}{PP}.$$

Hence it is necessary that this differential formula $dP\left(\frac{\alpha y}{PP} - x\right)$ is integrable; this can only happen, if $\frac{\alpha y}{PP} - x$ is a function of P; in this case also the integral $\int dP\left(\frac{\alpha y}{PP} - x\right)$ will be a function of P. Therefore, let Π denote an arbitrary function of P, and put $\frac{\alpha y}{PP} - x = \Pi$, from the resolution of which equation the quantity P is understood to be defined by x and y. But having found this function P we will have

$$V = Px + \frac{\alpha y}{P} + \int \Pi dP.$$

COROLLARY 1

§44 Therefore, the simplest case solving this problem is, if it is $\Pi = 0$, in which case it is

$$P = \sqrt{\frac{\alpha y}{x}} \quad \text{and} \quad \int \Pi dP = \text{Const.}$$

Therefore, we will have

$$V = 2\sqrt{\alpha xy} + \text{Const.},$$

for because of

$$dV = \frac{dx\sqrt{\alpha y}}{\sqrt{x}} + \frac{dy\sqrt{\alpha x}}{\sqrt{y}}$$

it will be $PQ = \alpha$, of course.

COROLLARY 2

§45 Then having taken $\Pi = \beta$, it will be

$$P = \sqrt{\frac{\alpha y}{x+\beta}} \quad \text{and} \quad \int \Pi dP = \beta P.$$

Therefore, in this case we will obtain the following satisfying function:

$$V = x\sqrt{\frac{\alpha}{x+\beta}} + \sqrt{\alpha y(x+\beta)} + \beta\sqrt{\frac{\alpha y}{x+\beta}} = 2\sqrt{\alpha y(x+\beta)},$$

and more general this function is obvious to satisfy:

$$V = 2\sqrt{\alpha(x+\beta)(y+\gamma)}.$$

COROLLARY 3

§46 If we want higher composite functions, which can nevertheless be exhibited, let

$$\Pi = \beta PP \quad \text{and hence} \quad \int \Pi dP = \frac{1}{3}\beta P^3.$$

But because we have

$$\frac{\alpha y}{PP} - x = \beta PP \quad \text{and} \quad P^4 = \frac{-xPP + \alpha y}{\beta},$$

it will be

$$PP = \frac{-x + \sqrt{xx + 4\alpha\beta y}}{2\beta},$$

whence because of

$$V = \frac{PPy + \alpha y + \frac{1}{3}\beta P^4}{P} = \frac{2PPx + 4\alpha y}{3P}$$

and after the substitution

$$V = \sqrt{\frac{2}{9\beta}\left((xx + 4\alpha\beta y)^{\frac{3}{2}} + x(12\alpha\beta y - x^2)\right)}.$$

Scholium 1

§47 It is plain that one can resolve the problem the same way, if Q must be an arbitrary function of P. For, put $dQ = RdP$, and it will be

$$V = Px + Qy - \int (x + Ry)dP.$$

Therefore, it is necessary that $x + Ry$ is a function of P, of which also R is a given function. Hence, if one puts, as before, $x + Ry = \Pi$, from this equation P will be defined by means of x and y; having afterwards substituted this value in Q, R and Π one will obtain a function $V = Px + Qy - \int \Pi dP$ expressed by the two variables x and y only.

Example

§48 *Let the function V be in question that having put $dV = Pdx + Qdy$ it is $PP + QQ = aa$ or $Q = \sqrt{aa - PP}$.*

And hence it is

$$R = \frac{-P}{\sqrt{aa - PP}} \quad \text{and} \quad x - \frac{Py}{\sqrt{aa - PP}} = \Pi,$$

whence P must be defined. But now take $\Pi = 0$, it will be

$$P = \frac{ax}{\sqrt{xx + yy}}, \quad Q = \frac{ay}{\sqrt{xx + yy}} \quad \text{and} \quad V = a\sqrt{xx + yy}.$$

Scholium 2

§49 But if Q is not only given by means of P but also the variables x and y somehow enter the expression for it, then this way the task cannot be completed. But in these cases it is to be taken into account, how P and Q are treated as functions of x and y; for, any two of the four P, Q, x and y variables can be considered as functions of the two remaining ones. Hence in each case we are restricted to this formula $Pdx + Qdy$, which must be rendered integrable, but we will complete the task in like manner, if we render either this one $-ddP + Qdy$ or this one $Pdx - ydQ$ or this one $-xdP - yqQ$ integrable; one can even introduce new variables by means of substitutions, how the method of solving is extended a lot; it will be helpful to have given several examples of this.

PROBLEM 13

§50 While it is $dV = Pdx + Qdy$, if Q is given somehow by x and P, to find the nature of the function V.

SOLUTION

Since Q is put to be given by x and P, one will have an equation between the three quantities x, P and Q, from which one will also be able to define P by x and Q such that P becomes equal to a certain function of x and Q. Therefore, take x and Q for the two variables from which all remaining ones are to be determined, and because it is

$$V = Qy + \int (Pdx - ydQ),$$

it is necessary that the differential formula $Pdx - ydQ$ is integrable, whose integral is to be considered as function of x and Q. Therefore, because P is given by x and Q, but the first variable y is left undetermined, this integral $\int (Pdx - ydQ)$ will be found, if having considered Q as constant the formula Pdx is integrated and to the integral any arbitrary function of Q is added. Therefore, let the integral taken this way be $\int Pdx = R$, and R will be a given function of x and Q, whence it is $dR = Pdx + SdQ$, having assumed both quantities x and Q as variable, of course. Having constituted this we will have $\int (Pdx - ydQ) = R + \Phi : Q$ and $V = Qy + R + \Phi : Q$. Now denote the differential of $\Phi : Q$ by $dQ. \Phi' : Q$, and it will be

$$Pdx - ydQ = Pdx + SdQ + dQ. \Phi' : Q,$$

whence it is $y = -S - \Phi' : Q$. Finally, from this equation $y = -S - \Phi' : Q$ together with the relation among Q, P and x by means of the variables x and y define the other two P and Q, and their resubstituted values will show, what kind of function V must be of x and y; this is the nature in question of the function V.

COROLLARY 1

§51 In like manner, if Q is given by y and P such that x does not enter this relation, one will have to use this form $V = Px + \int (Qdy - xdP)$, which, since Q must be considered as a given function of x and P, by the same operations will be rendered integrable.

COROLLARY 2

§52 But if either P or Q is determined by x and y, the question has no difficulty. For, if P is a given function of x and y, let the integral $\int Pdx$ be in question, having considered y as a constant, and having put $\int Pdx = R$, it will be $V = R + \Phi : y$.

COROLLARY 3

§53 Therefore, if a relation among the quantities P, Q, x and y is given by an equation of this kind, which only three of these four quantities enter, the nature of the function V can be defined by the problems already treated .

SCHOLIUM

§54 Therefore, from this class the cases remain, in which the given relation contains all four letters P, Q, x and y. But for this we expanded the case, in which it was $Q = Pp + t$, while p and t are arbitrary functions of x and y, whose solution was given in Problem 11. But instead of the two variables x and y the following pairs of variables can be treated:

I. If it is $Q = xM + N$,
while M and N are arbitrary functions of P and y.

II. If it is $P = yM + N$,
while M and N are arbitrary functions of Q and x.

III. If it is $y = xM + N$,
while M and N are arbitrary functions of P and Q. In these cases the solution can of course also be found by means of the prescription given in § 40.

EXAMPLE

§55 *While it is $dV = Pdx + Qdy$, the found function V is to be defined that it is $xyPQ = \alpha$.*

Therefore, because it is $Q = \frac{\alpha}{Pxy}$, it will be

$$dV = Pdx + \frac{\alpha dy}{Pdx},$$

which case is contained in none of the ones we treated. But having put $\log y = u$, because it is $dV = Pdx + \frac{\alpha du}{Pdx}$, if we consider u instead of y and compare this form to $Pdx + Qdy$, it will be $Q = \frac{\alpha}{Pdx}$, and hence given only by x and P, such that this example is reduced to present problem. Therefore, to not confound this Q with the principal one, because it is $P = \frac{\alpha}{Qx}$, we will have by writing y instead of u

$$V = Qy + \int \left(\frac{\alpha dx}{Qx} - ydQ \right),$$

therefore, having assumed Q to be constant it will be

$$\int \frac{\alpha dx}{Qx} = R = \frac{\alpha}{Q} \log x,$$

and hence $S = \frac{-\alpha \log x}{QQ}$, whence it is

$$u = \log y = \frac{\alpha \log x}{QQ} - \Phi' : Q$$

and

$$V = Q \log y + \frac{\alpha \log x}{Q} + \Phi : Q,$$

while $\Phi : Q = \int dQ \cdot \Phi' : Q$, where for $\Phi' : Q$ any function of Q can be taken. Therefore, for the simplest case let it be $\Phi' : Q = 0$, and it will be

$$Q = \sqrt{\frac{\alpha \log x}{\log y}} \quad \text{and} \quad V = 2\sqrt{\alpha \log x \cdot \log y} + \text{Const.}$$

And if one takes $\Phi' : Q = n - \frac{\alpha m}{QQ}$, it will be

$$\Phi : Q = nQ + \frac{\alpha m}{Q} + C \quad \text{and} \quad \log y + n = \frac{\alpha(\log x + m)}{QQ},$$

and hence

$$Q = \sqrt{\alpha \frac{\log x + m}{\log y + n}}$$

and

$$V = 2\sqrt{\alpha(\log x + m)(\log y + n)} + \text{Const.}$$

PROBLEM 14

§56 While it is $dV = Pdx + Qdy$, to define the nature of the function V that it is $PP + QQ = xx + yy$.

SOLUTION

This problem is contained in none of the cases treated up to this point; but nevertheless by a suitable transformation it can be reduced to a very simple case. For, put $PP + QQ = xx + yy = tt$, and by introducing two indefinite angles Φ and θ let it be:

$$P = t\sin\Phi, \quad Q = t\cos\Phi, \quad x = t\sin\theta \quad \text{and} \quad y = t\cos\theta,$$

because of

$$dx = dt\sin\theta + td\theta\cos\theta \quad \text{and} \quad dy = dt\cos\theta - td\theta\sin\theta$$

it will be

$$dV = tdt(\sin\Phi\sin\theta + \cos\Phi\cos\theta) - ttd\theta(\cos\Phi\sin\theta - \sin\Phi\cos\theta)$$

or

$$dV = tdt\cos(\theta - \Phi) - ttd\theta\sin(\theta - \Phi).$$

But it is

$$\int tdt\cos(\theta - \Phi) = \frac{1}{2}tt\cos(\theta - \Phi) + \frac{1}{2}\int tt(d\theta - d\Phi)\sin(\theta - \Phi),$$

whence it is

$$V = \frac{1}{2}tt\cos(\theta - \Phi) - \frac{1}{2}\int tt(d\theta + d\Phi)\sin(\theta - \Phi).$$

Therefore, because this formula must be integrable, it is necessary that it is $tt\sin(\theta - \Phi) = \text{funct.}(\theta + \Phi)$. Hence, because it is $tt = xx + yy$ and $\tan\theta = \frac{x}{y}$, the angle Φ will be determined, whose value, having substituted it, will give the function V expressed in terms of x and y. In order to find simpler algebraic functions, let it be

$$tt\sin(\theta-\Phi)=\alpha\sin(\theta+\Phi)+\beta\cos(\theta+\Phi),$$

and it will be

$$V=\frac{1}{2}tt\cos(\theta-\Phi)+\frac{1}{2}\alpha\cos(\theta+\Phi)-\frac{1}{2}\beta\sin(\theta+\Phi)$$

whence, if tt is eliminated, it results

$$2V\sin(\theta-\Phi)=\alpha\sin 2\alpha+\beta\cos 2\theta=\frac{2\alpha xy-\beta(xx-yy)}{xx+yy}.$$

But having expanded those angles it is

$$ttx\cos\Phi-tty\sin\Phi=\alpha x\cos\Phi+\alpha y\sin\Phi+\beta y\cos\Phi-\beta x\sin\Phi,$$

and hence

$$\tan\Phi=\frac{ttx-\alpha x-\beta y}{tty+\alpha y-\beta x}$$

and

$$\sec\Phi=\frac{\sqrt{t^6-2\alpha tt(xx-yy)-4\beta ttxy+\alpha\alpha tt+\beta\beta tt}}{tty+\alpha y-\beta x},$$

let

$$T=t\sqrt{t^4-2\alpha(xx-yy)-4\beta xy+\alpha\alpha+\beta\beta},$$

it will be

$$\sin\Phi=\frac{ttx-\alpha x-\beta y}{T}\quad\text{and}\quad\cos\Phi=\frac{tty+\alpha y-\beta x}{T},$$

and hence

$$\sin(\theta-\Phi)=\frac{2\alpha xy-\beta(xx-yy)}{Tt},$$

having substituted which value $V=\frac{T}{2t}$ will result and hence

$$V=\frac{1}{2}\sqrt{(xx+yy)^2-2\alpha(xx-yy)-4\beta xy+\alpha\alpha+\beta\beta},$$

which function can also be represented this way:

$$V = \frac{1}{2}\sqrt{(\alpha - xx + yy)^2 + (\beta - 2xy)^2}.$$

The simplest case results by taking $\alpha = 0$ and $\beta = 0$, in which it is

$$V = \frac{1}{2}(xx + yy), \quad \text{and} \quad dV = xdx + ydy.$$

SCHOLIUM

§57 From this problem it is understood, how questions of this kind, which, since all four letters enter the prescribed relation, seem to have a most difficult solution, nevertheless by means of a suitable substitution sometimes can be reduced to cases already treated. But on the other hand I still do not see a way, by which in general, no matter of what nature the relation among the four quantities P, Q, x and y was, the solution can be obtained; I could give a lot of other examples of this kind, in which the reduction to cases already treated is possible; but since I am certain not to have exhausted this subject by any means, I proceed to the following chapters, in which the prescribed relation except for the quantities P, Q, x and y also contains the function in question V itself; here, it is perspicuous per se, if only a relation among V, x and y would be propounded, that it would not even be a question, since the function V would be given immediately by x and y. Hence I will begin with problems of such a kind, where the prescribed relation except for the function V contains the one of the two quantities P and Q or even both, while the variables x and y either enter at the same time or not. But it is easily understood that these problems are a lot more difficult than the preceding ones.

PROBLEM 15

§58 While it is $dV = Pdx + Qdy$, to define the nature of the function V that it is $P = nV$.

SOLUTION

Because it is $P = nV$, it will be $dV = nVdx + Qdy$ or $dV - nVdx = Qdy$. Multiply the left-hand side by e^{-nx} that it becomes integrable, and its integral $e^{-nx}V = \int e^{-nx}Q$ will have to become equal to an arbitrary function of y, which we want to put $= Y$. Hence the function in question will be $V = e^{nx}Y$.

Another Solution

Since V must be a function of x and y so that its differential is $dV = nVdx + Qdy$, it is perspicuous, if the function V is differentiated having put y to be constant, that $dV = nVdx$ will result. Hence vice versa from the integration of the formula $dV = nVdx$ the function V will be found, if y is considered as a constant; but then the constant introduced by integration can involve the quantity y somehow. But the equation $dV = nVdx$ integrated yields

$$\log V = nx + \log Y \quad \text{or} \quad V = e^{nx}Y,$$

as before.

Corollary 1

§59 The same way one will be able to resolve the further extending question, if P must be an arbitrary function of V. For, consider, having treated y as a constant, this differential equation $dV = Pdx$, which, because it contains only the two variables V and x, is to be integrated and then somehow involves the quantity y in the constant introduced by integration.

Corollary 2

§60 Since the two variables x and y are interchangeable, the problem is resolved the same way, if Q must be an arbitrary function of V.

Problem 16

§61 While it is $dV = Pdx + Qdy$, to define the nature of the function V that P becomes an arbitrary function of V and x.

Solution

Therefore, because P is given by means of V and x, if we consider y as a constant, we will have this equation $dV = Pdx$ between the two variables x and V. Therefore, integrate it, and instead of the constant introduce an arbitrary function of y into the integral equation; this way one will obtain an equation between V, x and y, by which the nature of the function V will be defined by means of x and y.

COROLLARY 1

§62 Therefore, whatever relation among the three quantities V, P and x is propounded, whether from it V is defined by x and P or P by V and x or x by V and P, the solution will always be easy.

COROLLARY 2

§63 Because of the interchangeability of the variables x and y, the problem will be solved in the same way, if an arbitrary relation among Q, V and y is propounded, and it is not necessary that we expand this case separately.

EXAMPLE 1

§64 *Having put $dV = Pdx + Qdy$ is shall have to be*

$$P = \frac{mV}{x} + nx.$$

Therefore, having considered y as a constant, it will be

$$dV = \frac{mVdx}{x} + nxdx \quad \text{or} \quad dV - \frac{mVdx}{x} = nxdx,$$

whose integral is

$$\frac{V}{x^m} = \frac{nx^{2-m}}{2-m} + Y$$

where Y is an arbitrary function of y. Hence it will be

$$V = x^m Y + \frac{n}{2-m} xx,$$

for $m = 2$, it would be $V = xxY + nxx \log x$.

EXAMPLE 2

§65 *Having put $dV = Pdx + Qdy$ is shall have to be $aV = P(aa - xx)$.*

Therefore, because it is

$$P = \frac{aV}{aa - xx},$$

having assumed y to be a constant, it will be

$$dV = \frac{aVdx}{aa - xx} \quad \text{or} \quad \frac{dV}{V} = \frac{adx}{aa - xx},$$

whose integral is

$$\log V = \frac{1}{2} \log \frac{a + x}{a - x} + \log Y,$$

whence one will have

$$V = Y\sqrt{\frac{a + x}{a - x}},$$

where Y denotes an arbitrary function of y.

Problem 17

§66 While $dV = Pdx + Qdy$, to define the nature of the function V, if P is an arbitrary given function of x, y and V.

Solution

Here, I assume that the propounded relation is contained in a certain equation between the quantities x, y, V and P; therefore, from this P can be defined by x, y and V. Therefore, consider y as a constant quantity, and because it is $dV = Pdx$, this equation will already involve only the two variables x and V. Therefore, integrate it, and instead of the constant introduce an arbitrary function of y, and this way an equation will result showing the nature of the function V.

Corollary

§67 Therefore, the problem will be solved in like manner, if an arbitrary relation among the four quantities x, y, Q and V is propounded, in which case only this difference is to be observed, that at first the quantity x is considered as a constant.

Example

§68 *Having put $dV = Pdx + Qdy$ it shall have to be $V = \frac{Px}{y}$.*

Therefore, because it is $P = \frac{Vy}{x}$, having assumed y to be constant it will be

$$dV = \frac{Vydx}{x} \quad \text{and hence} \quad \log V = y \log x + \log Y,$$

whence the function is question result as $V = x^y V$.

SCHOLIUM

§69 Therefore, if only one of the quantities P and Q enter the propounded relation, the problems are easily solvable. But if both quantities P and Q are in it, a major difficulty occurs, which is always so great that it cannot be overcome. Therefore, since in this case a general solution cannot be expected, let us run through several examples, which already extend sufficiently far.

PROBLEM 18

§70 While $dV = Pdx + Qdy$ to find the nature of the function V that it is $V = mPx + nQy$.

SOLUTION

Since hence it is

$$Q = \frac{V - mPx}{ny},$$

it will be

$$dV - \frac{Vdy}{ny} = Pdx - \frac{mPxdy}{ny} = \frac{P}{ny}(nydx - mxdy).$$

Therefore, find the multiplicator which renders the formula $nydx - mxdy$ integrable, since which is $\frac{1}{xy}$ and hence

$$dV - \frac{Vdy}{ny} = \frac{Px}{n}\left(\frac{ndx}{x} - \frac{mdy}{y}\right),$$

put

$$n \log x - m \log y = \log z \quad \text{or} \quad z = \frac{x^n}{y^m};$$

hence it is $x = y^{\frac{m}{n}} z^{\frac{1}{n}}$ which value is to be assumed to be substituted for x. Hence because it is

$$dV = \frac{Vdy}{ny} + \frac{Pxdz}{nz},$$

the quantity V can be considered as function of the two quantities y and z, which must therefore be such a one that having assumed z to be constant it is $dV = \frac{Vdy}{ny}$. Therefore, hence by integration it will result:

$$\log V = \frac{1}{n} \log y + \log Z \quad \text{or} \quad V = y^{\frac{1}{n}} Z$$

having taken an arbitrary function of $z = \frac{x^n}{y^m}$ for Z; and so one will have

$$V = y^{\frac{1}{n}} \Phi : \frac{x^n}{y^m}.$$

COROLLARY 1

§71 Because $\frac{x^{\frac{1}{m}}}{y^{\frac{1}{n}}}$ is a function of $\frac{x^n}{y^m}$, it will also be

$$V = x^{\frac{1}{m}} \Phi : \frac{x^n}{y^m}.$$

But then it can also exhibited this way

$$V = x^{\frac{1}{m}} \Phi : \frac{x^{\lambda n}}{y^{\lambda m}} \quad \text{or} \quad V = y^{\frac{1}{n}} \Phi : \frac{x^{\lambda n}}{y^{\lambda m}},$$

having assumed an arbitrary number for λ.

COROLLARY 2

§72 If it is $m = n$, one will have the case of homogeneous functions covered above. For, having taken $\lambda = \frac{1}{n}$, $\Phi : \frac{x}{y}$ will denote an arbitrary function of no dimension of x and y; and V will be a homogeneous function of the same, the number of dimension of which function is $= \frac{1}{n}$.

COROLLARY 3

§73 If we put in general

$$x^{\frac{1}{m}} = t \quad \text{and} \quad y^{\frac{1}{n}} = u,$$

but then take $\lambda = \frac{1}{mn}$, we will have

$$V = t\Phi : \frac{t}{u},$$

or V will be a homogeneous function of one dimension of the two quantities t and u.

SCHOLIUM

§74 If it is desired that it is $V = mP + nQ$ the solution will equally have hardy any difficulty. For, because of

$$Q = \frac{V}{n} - \frac{mP}{n}$$

it will be

$$dV - \frac{Vdy}{n} = P\left(dx - \frac{mdy}{n}\right).$$

Set $x - \frac{my}{n} = z$ that it is

$$dV = \frac{Vdy}{n} + Pdz;$$

now having considered z as a constant it will be

$$\log V = \frac{y}{n} + \Phi : z,$$

and hence

$$V = e^{\frac{y}{n}}\Phi : (nx - my).$$

But if it must be $V = mPy + nQx$, because of

$$Q = \frac{V - mPy}{nx}$$

it will be

$$dV = P\left(dx - \frac{mydy}{nx}\right) + \frac{Vdy}{nx}.$$

Now set $nxx - myy = zz$ that it is

$$x = \sqrt{\frac{zz + myy}{n}},$$

and because it is

$$dV = \frac{Vdy}{\sqrt{n(zz + myy)}} + \frac{P}{nx}dz,$$

consider the quantity z as constant and because of

$$\frac{dV}{V} = \frac{dy}{\sqrt{nzz + mnyy}},$$

it will be

$$\log V = \frac{1}{\sqrt{mn}} \log(y\sqrt{mn} + \sqrt{n(zz + myy)}) + \log Z,$$

and hence because of $\sqrt{n(zz + myy)} = nx$, it will result

$$V = (y\sqrt{m} + x\sqrt{n})^{\frac{1}{\sqrt{mn}}} \Phi : (nxx - myy).$$

Hence, if it must be $V = Py + Qx$, it will be

$$V = (x + y)\Phi : (xx - yy).$$

But the following problem contains all cases of this kind in it.

PROBLEM 19

§75 If p is an arbitrary given function of x and y, but M even an arbitrary given function of x, y and V, to define the nature of the function V that having put $dV = Pdx + Qdy$ it is $Q = Pp + M$.

SOLUTION

Having substituted this value for Q we have

$$dV = Mdy + P(dx + pdy).$$

Find the multiplicator q rendering the formula $dx + pdy$ integrable and let $\int q(dx + pdy) = z$; hence define the value of x by y and z, and substitute it in M, if x is in it, for x, having done which it will be

$$dV = Mdy + \frac{Pdz}{q},$$

and so V can be considered as a function of y and z. Now consider z as a constant quantity, and because it is $dV = Mdy$, where only the two variables y and V are to understood to be contained in it, integrate this equation and instead of the constant introduce an arbitrary function of z; if in it for z the value expressed in terms of x and y, of course $\int q(dx + pdy)$, is resubstituted, that equation $dV = Mdy$ integrated will exhibit the nature of the function V, how is must depend on the two variables x and y.

EXAMPLE

§76 *Having put $dV = Pdx + Qdy$ it shall have to be $V = pyy + Qxx$.*

Therefore, it is

$$Q = -\frac{Pyy}{xx} + \frac{V}{xx},$$

whence it is

$$dV = \frac{Vdy}{xx} + P\left(dx - \frac{yydy}{xx}\right).$$

Now take $q = xx$, it will be

$$\int(xxdx - yydy) = z = \frac{1}{3}x^3 - \frac{1}{3}y^3$$

or $x^3 = y^3 + 3z$ and hence $xx = (y^3 + 3z)^{\frac{2}{3}}$. Therefore, having assumed z to be constant, one has

$$\frac{dV}{V} = \frac{dy}{(y^3 + 3z)^{\frac{2}{3}}}.$$

Therefore, let S be the integral of the formula $\frac{dy}{(y^3+3z)^{\frac{2}{3}}}$, while z is assumed to be constant, and one will obtain

$$V = e^S \Phi : z = e^S \Phi : (x^3 - y^3),$$

of course in S one has to substitute its value $\frac{1}{3}x^3 - \frac{1}{3}y^3$ for z everywhere.

Problem 20

§77 While it is $dV = Pdx + Qdy$, to define the nature of the function V that it is $V = nPQ$.

Solution

Therefore, because of $Q = \frac{V}{nP}$ it will be

$$dV = Pdx + \frac{Vdy}{nP}.$$

That now V can be separated from the second term, put $P = R\sqrt{V}$ and this equation will result

$$\frac{dV}{\sqrt{V}} = Rdx + \frac{dy}{nR},$$

whence by integrating one obtains

$$2\sqrt{V} = Rx + \frac{y}{nR} - \int dR \left(x - \frac{y}{nRR} \right).$$

Hence it is necessary that $x - \frac{y}{nRR}$ is a function of R only; and having assumed such a function one will be able to define R by x and y, whence also the function in question V will be found expressed in terms of x and y.

Another Solution

Since it is $V = nPQ$, eliminate V that one has this equation

$$nPdQ + nQdP = Pdx + Qdy,$$

from which it is

$$dy = -\frac{Pdx}{Q} + \frac{nPdQ}{Q} + ndP,$$

and hence

$$y = nP + \int \frac{P}{Q}(ndQ - dx).$$

Therefore, it is necessary that $\frac{P}{Q}$ is a function of the quantity $nQ - x$. Put $nQ - x = z$; and let $\int \frac{P}{Q}dz = \Phi : z$, it will be $\frac{P}{Q} = \Phi' : z$ and

$$y = nP + \Phi : z = nQ\Phi' : z + \Phi : z.$$

But it is $V = nQQ\Phi' : z$, whence $Q = \sqrt{\frac{V}{n\Phi' : z}}$; and so one will have these equations:

$$\sqrt{\frac{nV}{\Phi' : z}} = x + \quad \text{and} \quad y = \Phi : z + \sqrt{nV\Phi' : z},$$

from which one concludes $nV = (x+z)(y - \Phi : z)$, and if the quantity z is eliminated, the function V will result expressed in terms of x and y.

COROLLARY 1

§78 Assume z as a constant or $ndQ - dx = 0$, it will be

$$Q = \frac{x+a}{n} \quad \text{and} \quad y = nP - B \quad \text{or} \quad P = \frac{y+b}{n},$$

whence this equation arises

$$V = \frac{(x+a)(y+b)}{n},$$

which is the simplest case.

COROLLARY 2

§79 If one sets $\Phi' : z = a$, it will be $\phi : z = az + b$, whence it is

$$\sqrt{\frac{nV}{a}} = x + z \quad \text{and} \quad nV = (y - az - b)\sqrt{\frac{nV}{a}} \quad \text{and} \quad \sqrt{naV} = y - b - az,$$

from which having combined them we obtain $2\sqrt{naV} = ax + y - b$, and hence

$$V = \frac{(ax + y - b)^2}{4na},$$

which is another very simple case.

COROLLARY 3

§80 Let $\Phi' : z = \frac{1}{(az+b)^2}$ that it is $\Phi : z = -\frac{1}{a(az+b)} + c$, and it will be

$$(az + b)\sqrt{nV} = x + z \quad \text{and} \quad y - c + \frac{1}{a(az + b)} = \frac{\sqrt{nV}}{az + b}$$

or

$$a(az + b)(y - c) = a\sqrt{nV} - 1;$$

but hence it is

$$z = \frac{x - b\sqrt{nV}}{a\sqrt{nV} - 1} \quad \text{and hence} \quad az + b = \frac{ax - b}{a\sqrt{nV} - 1};$$

and having substituted this value

$$a(ax - b)(y - c) = (a\sqrt{nV} - 1)^2,$$

which expansion yields

$$V = \frac{1 + a(ax - b)(y - c) \pm 2m\sqrt{a(ax - b)(y - c)}}{naa}.$$

PROBLEM 21

§81 While it is $dV = Pdx + Qdy$, if V is given somehow by P and Q, to define the nature of the function V or how V is determined by x and y.

SOLUTION

Therefore, because V is a function of the two quantities P and Q, put its differential $dV = MdP + NdQ$, and also M and N will be given functions of P and Q. Hence, because it is

$$MdP + NdQ = Pdx + Qdy, \tag{2}$$

it will be

$$dy = -\frac{Pdx}{Q} + \frac{MdP + NdQ}{Q},$$

and hence

$$y = -\frac{Px}{Q} + \int \left(xd.\frac{P}{Q} + \frac{MdP}{Q} + \frac{NdQ}{Q} \right).$$

Put $P = QS$, which value is to be understood to be substituted for P in M and N, such that the variables Q and S are now to be considered, and it will be

$$y = -Sx + \int (dS(x + M) + \frac{dQ}{Q}(N + MS)).$$

Since here M and N are given functions of Q and S, assume S to be constant, and put the integral

$$\int \frac{dQ}{Q}(N + MS) = R + \Phi : S,$$

therefore, it will be

$$x + M = \left(\frac{dR}{dS}\right) + \Phi' : S,$$

with $\Phi : s = \int dS\Phi' : S$ and

$$y = MS - S\left(\frac{dR}{dS}\right) - S\Phi' : S + R + \Phi : S.$$

Since now R and M are given by Q and S and because of $P = QS$ also V is given by Q and S. If this relation is combined with these two

$$x = -M + \left(\frac{dR}{dS}\right) + \Phi' : S \quad \text{and} \quad y = -Sx + R + \Phi : S,$$

one will hence be able to eliminate the two quantities S and Q, having done which an equation will result, from which V will be determined by x and y.

EXAMPLE 1

§82 *While it is $dV = Pdx + Qdy$, it shall have to be $V = mPP + nQQ$.*

Therefore, because it is $dV = 2mPdP + 2nQdQ$, it will be $M = 2mP$ and $N = 2nQ$ or $M = 2mQS$ because of $P = QS$ such that it is $V = QQ(mSS + n)$. Therefore, we will have

$$N + MS = 2Q(mSS + n),$$

and hence having considered S as a constant

$$R = \int \frac{dQ}{Q}(N + MS) = 2Q(mSS + n),$$

and hence $\left(\frac{dR}{dS}\right) = 4mQS$.

Hence we obtain these three equations:

$$\begin{aligned} &\text{I.} \quad V = QQ(mSS+n), \\ &\text{II.} \quad x + 2mQS = 4mQS + \Phi' : S \quad \text{or} \quad x = 2mSQ + \Phi' : S, \\ &\text{III.} \quad y + Sx = 2Q(mSS+n) + \Phi : S \quad \text{or} \quad y = 2nQ + \Phi : S - S\Phi' : S. \end{aligned}$$

Hence, if Q is eliminated from II and III, it will be

$$\text{IV.} \quad nx - mSy = (mSS + n)\Phi' : S - mS\Phi : S,$$

but from the same combined it is $Q = \frac{Sx + y - \Phi : S}{2(mSS+n)}$, which with the first gives

$$\text{V.} \quad 2\sqrt{V(mSS+n)} = Sx + y - \Phi : S.$$

Hence it remains that S is eliminated from IV and V, and so the function V will result expressed by x and y.

Let $\Phi' : S = a$, it will be $\Phi = S = aS + B$ and

$$\begin{aligned} &\text{IV.} \quad nx - mSy = na - mbS \\ &\text{V.} \quad 2\sqrt{V(mSS+n)} = Sx + y - aS - b. \end{aligned}$$

Hence it is $S = \frac{n(x-a)}{m(y-b)}$, having substituted which value, it will be

$$2\sqrt{mnV} = \sqrt{n(x-a)^2 + m(y-b)^2},$$

and hence

$$V = \frac{n(x-a)^2 + m(y-b)^2}{4mn}.$$

EXAMPLE 2

§83 *While it is $dV = Pdx + Qdy$ it shall have to be $V = \frac{P}{Q}$.*

Therefore, it will be

$$M = \frac{1}{Q}, \quad N = -\frac{P}{QQ} = -\frac{S}{Q}$$

because of $P = QS$ and $V = S$ and $N + MS = 0$, whence it is $R = 0$. Hence this equation results

$$x + \frac{1}{Q} = \Phi' : S \quad \text{and} \quad y + Sx = \Phi : S,$$

and because it is $S = V$, the function V is determined by x and y that it is $y + Vx = \Phi : V$.

Put $\Phi : V = \frac{\alpha + 2\beta V + \gamma VV}{2\delta + 2\varepsilon V}$ that it is

$$2\delta y + 2\varepsilon Vy + 2\delta Vx + 2\varepsilon VVx = \alpha + 2\beta V + \gamma VV,$$

and hence

$$VV = \frac{2V(\delta x + \varepsilon y - \beta) + 2\delta y - \alpha}{\gamma - 2\varepsilon x}$$

and

$$V = \frac{\delta x + \varepsilon y - \beta \pm \sqrt{(\delta x - \varepsilon y)^2 + 2(\alpha\varepsilon - \beta\delta) + 2(\gamma\delta - \beta\varepsilon)y + \beta\beta - \alpha\gamma}}{\gamma - 2\varepsilon x};$$

if it is $\gamma = 0$ and $\varepsilon = 0$, it will be

$$V = \frac{2\delta y - \alpha}{2\beta - 2\delta x} \quad \text{or} \quad V = \frac{y - m}{n - x}.$$

SCHOLIUM

§84 Many other questions of this kind could be propounded and resolved, but since their solution is based on the same principles we used until now, I will not spend more time on multiplying them, since the ones mentioned already seem to suffice, to develop the elements of this new method. One could still add several things for the cases, in which even the formulas of this kind $\left(\frac{ddV}{dx^2}\right)$, $\left(\frac{ddV}{dxdy}\right)$, $\left(\frac{ddV}{dy^2}\right)$ etc. enter the propounded relation, and similarly for the cases, in which the function to be found must be defined by three or more variables; but that this treatise does not become too long, I will retain it for another occasion.