Integration of the Equation $\frac{dx}{\sqrt{A+Bx+Cx^2+Dx^3+Ex^4}} = \frac{dy}{\sqrt{A+By+Cy^2+Dy^3+Ey^4}}$ *

Leonhard Euler

§1 Using a very singular and non-intuitive method I had once got to the integration of this equation whose integral, and even the complete integral, I detected to be contained in an algebraic equation among x and y. That seems to be even more remarkable, since the integral of each formula separately can not even not be expressed algebraically, but also not in terms of the quadrature of the circle or the hyperbola. But then it especially occurred that there is no direct way to find that integral algebraically. But no occasion seems more fitting to extend the limits of Analysis than if, what was found by a non-intuitive method in non straight-forward fashion, we try to investigate the same by a direct method. Therefore, since I recently defined curves which a body attracted to fixed centres of forces describes, and reduced them to a similar equation, hence it will vice versa be possible to complete the integration of this equation; how this is to be done, I decided to explain here.

§2 And first I observe that the propounded equation can always be transformed into a form of such a kind in which the coefficients B and D vanish, which is certainly known about each one separately from elementary results. But that both at the same time can be reduced to zero, is a special property of

*Original title: "Integratio Aequationis $\frac{dx}{\sqrt{A+Bx+Cx^2+Dx^3+Ex^4}} = \frac{dy}{\sqrt{A+By+Cy^2+Dy^3+Ey^4}}$", first published in: Novi Commentarii academiae scientiarum Petropolitanae 12, 1768, pp. 3-16, reprint in: Opera Omnia: Series 1, Volume 20, pp. 302 - 317, Eneström-Number E345, translated by: Alexander Aycock for the project „Euler-Kreis Mainz".

such a form; for, having put $x = \frac{mz+a}{nz+b}$, the first form, which is the same as the other, goes over into this one

$$\frac{(mb-na)dz}{\sqrt{A(nz+b)^4+B(nz+b)^3(mz+a)+C(nz+b)^2(mz+a)^2+D(nz+b)(mz+a)^3+E(m+a)^4}},$$

in whose denominator it is possible to cancel the terms so affected with the quantity z as with its cube z^3. The first condition yields this equation

$$4Anb^3+Bmb^3+3Bnabb+2Cmabb+2Cnaab+3Dmaab+Dna^3+4Ema^3=0,$$

the second on the other this one

$$4An^3b+Bn^3a+3Bmnnb+2Cmnna+2Cmmnb+3Dmmna+Dm^3b+4Em^3a=0,$$

whence so the ratio $a:b$ as the ratio $m:n$ can be found.

§3 For, let us put $a = bp$ and $m = nq$ that we have these equations

$$\begin{aligned} 4A+Bq+3Bp+2Cpq+2Cpp+3Dppq+Dp^3+4Ep^3q&=0,\\ 4A+Bp+3Bq+2Cpq+2Cqq+3Dpqq+Dq^3+4Epq^3&=0, \end{aligned}$$

whose difference divided by $p-q$ yields

$$2B+2C(p+q)+D(pp+4pq+qq)+4Epq(p+q)=0.$$

But then the first multiplied by q subtracted from the second multiplied by p, after having divided by $p-q$, gives

$$-4A-B(p+q)+Dpq(p+q)+4Eppqq=0;$$

now let us set $p+q=r$ and $pq=s$ and from the equations

$$2B+2Cr+Drr+2Ds+4Ers=0,$$

$$-4A-Br+Drs+4Ess=0,$$

by eliminating $r = \frac{4(A-Ess)}{Ds-B}$ we obtain this cubic equation

$$\left.\begin{array}{l}+D^3\\-4CDE\\+8BEE\end{array}\right\}s^3\left.\begin{array}{l}-BDD\\+4BCE\\-8ADE\end{array}\right\}s^2\left.\begin{array}{l}-BBD\\+4ACD\\-8ABE\end{array}\right\}s\left.\begin{array}{l}+B^3\\-4ABC\\+8AAD\end{array}\right\}=0,$$

§4 Therefore, since without loss of generality the coefficients B and D can be assumed equal to zero, our question is about finding the integral of this equation

$$\frac{dx}{\sqrt{A+Cxx+Dx^4}}=\frac{dy}{\sqrt{A+Cyy+Dy^4}},$$

which we want to represent this way

$$\frac{dx}{dy}=\sqrt{\frac{A+Cxx+Dx^4}{A+Cyy+Dy^4}},$$

whence the relation among the variables x and y must be found in general, what I will try to achieve as follows.

§5 First let us put $x=\sqrt{npq}$ and $y=\sqrt{n\frac{p}{q}}$; it will be

$$dx=\frac{\sqrt{n}(qdp+pdq)}{2\sqrt{pq}}\quad\text{and}\quad dy=\frac{\sqrt{n}(qdp-pdq)}{2q\sqrt{pq}}$$

and hence

$$\frac{dx}{dy}=\frac{q(qdp+pdq)}{qdp-pdq}.$$

But further

$$\frac{A+Cxx+Dx^4}{A+Cyy+Dy^4}=\frac{qq(A+nCpq+nnDppqq)}{Aqq+nCpq+nnDpp},$$

whence

$$\frac{qdp+pdq}{qdp-pdq}=\sqrt{\frac{A+nCpq+nnDppqq}{Aqq+nCpq+nnDpp}},$$

where now the number n can be assumed as we please.

§6 For the sake of brevity, let

$$\frac{A + nCpq + nnDppqq}{Aqq + nCpq + nnDpp} = \frac{P+Q}{P-Q};$$

it will be

$$\frac{P}{Q} = \frac{A(1+qq) + 2nCpq + nnDpp(1+qq)}{A(1-qq) - nnDpp(1-qq)} = \frac{(A+nnDpp)(1+qq) + 2nCpq}{(A-nnDpp)(1-qq)}.$$

But then, because of

$$\frac{qdp + pdq}{qdp - pdq} = \sqrt{\frac{P+Q}{P-Q}},$$

we will obtain

$$\frac{qdp}{pdq} = \frac{\sqrt{P+Q} + \sqrt{P-Q}}{\sqrt{P+Q} - \sqrt{P-Q}} = \frac{P + \sqrt{PP-QQ}}{Q}$$

and

$$\frac{pdq}{qdp} = \frac{p - \sqrt{PP-QQ}}{Q}.$$

§7 Now all focus in on a suitable substitution; and I observed that this one is to be used

$$q = u + \sqrt{uu-1}, \quad \text{whence} \quad \frac{dq}{q} = \frac{du}{\sqrt{uu-1}}$$

and further

$$1 + qq = 2qu, \quad 1 - qq = -2q\sqrt{uu-1},$$

from which one calculates

$$\frac{P}{Q} = \frac{(A+nnDpp)u + nCp}{(nnDpp - A)\sqrt{uu-1}},$$

and now it is certainly evident that the most convenient choice for n is 1. Therefore, since

$$\frac{P}{Q} = \frac{(A + Dpp)u + Cp}{(Dpp - A)\sqrt{uu - 1}},$$

it will be

$$\frac{\sqrt{PP - QQ}}{Q} = \frac{\sqrt{4ADppuu + 2Cp(A + Dpp)u + CCpp + (Dpp - A)^2}}{(Dpp - A)\sqrt{uu - 1}},$$

such that our equation which is to be integrated is

$$\frac{pdu}{dp} = \frac{(A + Dpp)u + Cp - \sqrt{4ADppuu + 2Cpu(A + Dpp) + CCpp + (Dpp - A)^2}}{Dpp - A}.$$

§8 Represent this irrational formula this way

$$\sqrt{\left(2pu\sqrt{AD} + \frac{C(A + Dpp)}{2\sqrt{AD}}\right)^2 + \frac{(4AD - CC)(Dpp - A)^2}{4AD}}$$

and put

$$2pu\sqrt{AD} + \frac{C(A + Dpp)}{2\sqrt{AD}} = \frac{(Dpp - A)s\sqrt{4AD - CC}}{2\sqrt{AD}},$$

whence the surdic form becomes

$$= \frac{(Dpp - A)\sqrt{(4AD - CC)(1 + ss)}}{2\sqrt{AD}}$$

and

$$u = -\frac{C(A + Dpp)}{4ADp} + \frac{(Dpp - A)s\sqrt{4AD - CC}}{4ADp}$$

and hence

$$(A + Dpp)u + Cp = \frac{-C(Dpp - A)^2 + (A + Dpp)(Dpp - A)s\sqrt{4AD - CC}}{4ADp},$$

such that our equation now is

$$\frac{pdu}{dp} = \frac{-C(Dpp - A) + (A + Dpp)s\sqrt{4AD - CC}}{4ADp} - \frac{\sqrt{(4AD - CC)(1 + ss)}}{2\sqrt{AD}}.$$

§9 Hence we conclude

$$du = -\frac{Cdp(Dpp-A)}{4ADpp} + \frac{sdp(A+Dpp)\sqrt{4AD-CC}}{4ADpp} + \frac{ds(Dpp-A)\sqrt{4AD-CC}}{4ADp},$$

such that we obtain

$$\frac{pdu}{dp} = \frac{-C(Dpp-A)}{4ADp} + \frac{s(A+Dpp)\sqrt{4AD-CC}}{4ADp} + \frac{ds(Dpp-A)\sqrt{4AD-CC}}{4ADdp},$$

having equated which formula to the preceding ones it most conveniently happens that the most terms cancel and hence this equation results

$$\frac{ds(Dpp-A)\sqrt{4AD-CC}}{4ADdp} = \frac{-\sqrt{(4AD-CC)(1+ss)}}{2\sqrt{AD}},$$

whence it results

$$\frac{ds}{\sqrt{1+ss}} = \frac{-2dp\sqrt{AD}}{Dpp-A} = \frac{2dp\sqrt{AD}}{A-Dpp},$$

whose integral in logarithms is

$$\log\left(s+\sqrt{1+ss}\right) = \log\frac{\sqrt{A}+p\sqrt{D}}{\sqrt{A}-p\sqrt{D}} + \log\alpha$$

such that we have

$$s+\sqrt{1+ss} = \frac{\alpha\sqrt{A}+\alpha p\sqrt{D}}{\sqrt{A}-p\sqrt{D}}$$

and hence

$$s = \frac{\alpha\alpha(\sqrt{A}+p\sqrt{D})^2-(\sqrt{A}-p\sqrt{D})^2}{2\alpha(A-Dpp)}.$$

§10 If we go back from this, we will find

$$u = \frac{-C(A+Dpp)}{4ADp} + \frac{(\sqrt{A}-p\sqrt{D})^2-\alpha\alpha(\sqrt{A}+p\sqrt{D})^2}{8\alpha ADp}\sqrt{4AD-CC},$$

whence one has to define $q = u + \sqrt{uu-1}$. But since hence $u = \frac{1+qq}{2q}$, substituting $p = xy$ and $q = \frac{x}{y}$ again, our complete integral is

$$\frac{xx+yy}{2xy} = \frac{-C(A+Dxxyy)}{4ADxy} + \frac{(\sqrt{A}-xy\sqrt{D})^2 - \alpha\alpha(\sqrt{A}+xy\sqrt{D})^2}{8\alpha ADxy}\sqrt{4AD-CC}$$

or

$$4AD(xx+yy) + 2C(A+Dxxyy)$$
$$= \frac{\sqrt{4AD-CC}}{\alpha}\left((\sqrt{A}-xy\sqrt{D})^2 - \alpha\alpha(\sqrt{A}+xy\sqrt{D})^2\right),$$

which is expanded into this one

$$\frac{4AD(xx+yy)+2C(A+Dxxyy)}{\sqrt{4AD-CC}} = \frac{(1-\alpha\alpha)A - 2(1+\alpha\alpha)xy\sqrt{AD} + (1-\alpha\alpha)Dxxyy}{\alpha}$$

and putting

$$\alpha = \frac{\sqrt{4AD-CC}}{mC}$$

it results

$$4AD(xx+yy) + 2C(A+Dxxyy)$$
$$= \frac{((1+mm)CC-4AD)(A+Dxxyy) - 2((mm-1)CC+4AD)xy\sqrt{AD}}{mC}.$$

§11 For the case in which $\sqrt{AD}$ becomes an imaginary quantity not to cause any trouble, it will be helpful to investigate the integration in another way which is based on the cancellation of the terms observed in § 9. Of course, having propounded the equation

$$\frac{dx}{dy} = \sqrt{\frac{A+Cxx+Ex^4}{A+Cyy+Ey^4}}$$

let $x = \sqrt{pq}$ and $y = \sqrt{\frac{p}{q}}$, that hence one obtains

$$\frac{pdq}{qdp} = \frac{P-\sqrt{PP-QQ}}{Q}$$

while

$$\frac{P}{Q} = \frac{(A+Epp)(1+qq)+2Cpq}{(A-Epp)(1-qq)}.$$

Now put $q = u + \sqrt{uu-1}$ that

$$1+qq = 2qu, \quad 1-qq = 2qu - 2qq = -2q\sqrt{uu-1};$$

it will be

$$\frac{dq}{q} = \frac{du}{\sqrt{uu-1}} \quad \text{and} \quad \frac{P}{Q} = \frac{u(A+Epp)+Cp}{(Epp-A)\sqrt{uu-1}},$$

whence this transformed equation results

$$\frac{pdu}{dp} = \frac{u(A+Epp)+CP-\sqrt{4AEppuu+2Cpu(A+Epp)+CCpp+(Epp-A)^2}}{Epp-A}.$$

§12 Having massaged this equation and, for the sake of brevity, having put the irrational part $= \sqrt{M}$, it will be

$$udp(A+Epp)+Cpdp-pdu(Epp-A) = dp\sqrt{M}$$

and having first thrown out this irrational term one finds the integral

$$\frac{C+2EpU}{Epp-A} = \text{Const.};$$

but take the variable quantity s instead of the constant that

$$2Epu+C = s(Epp-A) \quad \text{and} \quad u = \frac{s(Epp-A)-C}{2Ep},$$

and hence the rational part becomes

$$\frac{-ds(Epp-A)^2}{2E}$$

and the irrational formula

$$(E-ppA)\sqrt{\frac{Ass+Cs+E}{E}},$$

such that now

$$\frac{ds}{2}(Epp - A) = dp\sqrt{E(Ass + Cs + E)}$$

or

$$\frac{ds}{\sqrt{E(Ass + Cs + E)}} + \frac{2dp}{Epp - A} = 0,$$

whose integral is

$$\frac{1}{\sqrt{AE}}\log\frac{p\sqrt{E} - \sqrt{A}}{p\sqrt{E} + \sqrt{A}} + \frac{1}{\sqrt{AE}}\log\left(As + \frac{1}{2}C + \sqrt{A(Ass + Cs + E)}\right) = \text{Const.}$$

§13 Therefore, this equation reduces to this form

$$As + \frac{1}{2}C + \sqrt{A(Ass + Cs + E)} = \alpha\frac{p\sqrt{E} + \sqrt{A}}{p\sqrt{E} - \sqrt{A}} = T,$$

whence one finds

$$AE = TT - T(2As + C) + \frac{1}{4}CC$$

or

$$2As + C = \frac{TT + \frac{1}{4}CC - AE}{T} = \frac{\alpha\alpha(p\sqrt{E} + \sqrt{A})^2 + \left(\frac{1}{4}CC - AE\right)(p\sqrt{E} - \sqrt{A})^2}{\alpha(Epp - A)}.$$

Now since $p = xy$ and $q = \frac{x}{y}$, it will be

$$u = \frac{xx + yy}{2xy} \quad \text{and} \quad s = \frac{E(xx + yy) + C}{Exxyy - A},$$

from which one obtains

$$\frac{2AE(xx + yy) + CExxyy + AC}{Exxyy - A} = T + \frac{CC - 4AE}{4T}$$

while

$$T = \alpha \cdot \frac{xy\sqrt{E} + \sqrt{A}}{xy\sqrt{E} - \sqrt{A}} = \alpha \cdot \frac{Exxyy + A + 2xy\sqrt{AE}}{Exxyy - A}$$

and

$$\frac{1}{T} = \frac{1}{\alpha} \cdot \frac{Exxyy + A - 2xy\sqrt{AE}}{Exxyy - A}$$

and hence

$$2AE(xx+yy) + CExxyy + AC = \alpha(Exxyy + A) + 2\alpha xy\sqrt{AE}$$
$$+\frac{CC-4AE}{4\alpha}(Exxyy + A) - \frac{2(CC-4AE)}{4\alpha}xy\sqrt{AE}.$$

§14 For this expression not to involve imaginary quantities, let us change the form of the constant α in such a way that

$$\alpha + \frac{CC-4AE}{4\alpha} = F \quad \text{or} \quad 4\alpha\alpha = 4\alpha F - CC + 4AE$$

and hence

$$2\alpha = F + \sqrt{FF + 4AE - CC} \quad \text{and} \quad \frac{1}{2\alpha} = \frac{F - \sqrt{FF+4AE-CC}}{CC-4AE},$$

whence

$$2\alpha - \frac{CC-4AE}{2\alpha} = 2\sqrt{FF + 4AE - CC}$$

and

$$2AE(xx+yy) = (F-C)(Exxyy + A) + 2xy\sqrt{AE(AE + CG + GG)},$$

which is the complete integral equation of this differential equation

$$\frac{dx}{\sqrt{A + Cxx + Ex^4}} = \frac{dy}{\sqrt{A + Cyy + Ey^4}},$$

where the constant G must be taken in such a way that the irrational formula

$$\sqrt{AE(AE + CG + GG)}$$

does not become imaginary.

§15 This integral formula can still be simplified even more by putting $G = Eff$ and so the integral equation will become

$$A(xx+yy) = ff(A+Exxyy) + 2xy\sqrt{A(A+Cff+Ef^4)},$$

where f is an arbitrary constant. But hence one finds

$$y = \frac{x\sqrt{A(A+Cff+Ef^4)} \pm f\sqrt{A(A+Cxx+Ex^4)}}{A-Effxx}$$

and in like manner

$$x = \frac{y\sqrt{A(A+Cff+Ef^4)} \pm f\sqrt{A(A+Cyy+Ey^4)}}{A-Effyy}.$$

These formulas agree perfectly with those I had once given.

§16 Here I certainly obtained the integral of the propounded differential equation by a direct method, but nevertheless I cannot deny that this was achieved in a very non straight-forward fashion, so that it is hardly to be expected that anyone could have had these operations in mind. From this the method I used here seems to promise much to be discovered about it and there is no doubt that by investigating it more diligently a way to many other beautiful result is opened and maybe another new method to achieve the same is detected, whence many auxiliary tools to perfect Analysis can be obtained.

§17 The operations here can be varied a bit, to have noted which will be useful. Of course, I represent the propounded differential equation this way

$$\frac{ydx}{xdy} = \sqrt{\frac{Ayy+Cxxyy+Ex^4yy}{Axx+Cxxyy+Exxy^4}} = \sqrt{\frac{P+Q}{P-Q}}$$

that

$$\frac{P}{Q} = \sqrt{\frac{(A+Exxyy)(xx+yy)+2Cxxyy}{(A-Exxyy)(yy-xx)}},$$

and it will be

$$\frac{ydx + xdy}{ydx - xdy} = \frac{\sqrt{P+Q} + \sqrt{P-Q}}{\sqrt{P+Q} - \sqrt{P-Q}} = \frac{P + \sqrt{PP - QQ}}{Q},$$

then also

$$\frac{ydx - xdy}{ydx + xdy} = \frac{P - \sqrt{PP - QQ}}{Q}.$$

Now let us make this substitution

$$x = p\left(\sqrt{\frac{q+1}{2}} - \sqrt{\frac{q-1}{2}}\right) \quad \text{and} \quad y = p\left(\sqrt{\frac{q+1}{2}} + \sqrt{\frac{q-1}{2}}\right);$$

it will be

$$xy = pp, \quad xx + yy = 2ppq, \quad yy - xx = 2pp\sqrt{qq-1},$$

further,

$$\frac{dx}{x} = \frac{dp}{p} - \frac{dq}{2\sqrt{qq-1}} \quad \text{and} \quad \frac{dy}{y} = \frac{dp}{p} + \frac{dq}{2\sqrt{qq-1}};$$

hence

$$\frac{ydx}{xdy} = \frac{\frac{dp}{p} - \frac{dq}{2\sqrt{qq-1}}}{\frac{dp}{p} + \frac{dq}{2\sqrt{qq-1}}} \quad \text{and} \quad \frac{ydx - xdy}{ydx + xdy} = \frac{-pdq}{2dp\sqrt{qq-1}}$$

and

$$\frac{P}{Q} = \frac{2(A + Ep^4)ppq + 2Cp^4}{2(A - Ep^4)pp\sqrt{qq-1}} = \frac{(A + Ep^4)q + Cpp}{(A - Ep^4)\sqrt{qq-1}},$$

whence

$$\frac{\sqrt{PP - QQ}}{Q} = \frac{\sqrt{4AEp^4qq + 2Cppq(A + Ep^4) + CCp^4 + (A - Ep^4)^2}}{(A - Ep^4)\sqrt{qq-1}}.$$

§18 Let $pp = r$ and, because of $\frac{dp}{p} = \frac{dr}{2r}$, it will be

$$0 = \frac{rdq}{dr} + \frac{(A-Err)q + Cr - \sqrt{4AErrqq + 2Crq(A+Err) + CCrr + (A-Err)^2}}{A-Err}$$

or

$$\begin{aligned} &rdq(A-Err) + qdr(A+Err) + Crdr \\ &= dr\sqrt{4AErrqq + 2Crq(A+Err) + CCrr + (A-Err)^2}. \end{aligned}$$

Exhibit the quantity under the square root sign this way

$$\begin{aligned} &\frac{1}{4AE}(16AAEErrqq + 8ACErq(A+Err) + 4ACCErr + 4AE(A-Err)^2) \\ &= \frac{1}{4AE}\left((4AErq + C(A+Err))^2 + (4AE-CC)(A-Err)^2\right). \end{aligned}$$

Therefore, let us put

$$4AErqC(A+Err) = s(A-Err)\sqrt{4AE-CC}$$

and the surdic formula will be

$$= \frac{(A-Err)\sqrt{(4AE-CC)(1+ss)}}{2\sqrt{AE}}$$

and because of

$$s\sqrt{4AE-CC} = \frac{4AErq + C(A+Err)}{A-Err}$$

and by differentiating

$$ds\sqrt{4AE-CC} = \frac{4AAE(rdq+qdr) - 4AEEr^3dq + 4AEErrqdr + 4ACErdr}{(A-Err)^2}$$

and hence

$$rdq(A-Err) + qdr(A+Err) + Crdr = \frac{ds(A-Err)^2\sqrt{4AE-CC}}{4AE};$$

since this is the first

$$\frac{dr(A - Err)\sqrt{(4AE - CC)(1 + ss)}}{2\sqrt{AE}},$$

we will have

$$\frac{ds(A - Err)}{2\sqrt{AE}} = dr\sqrt{1 + ss} \quad \text{and} \quad \frac{2dr\sqrt{AE}}{A - Err} = \frac{ds}{\sqrt{1 + ss}},$$

whose integral is

$$s + \sqrt{1 + ss} = \alpha \cdot \frac{\sqrt{A} + r\sqrt{E}}{\sqrt{A} - r\sqrt{E}},$$

whence

$$1 = \alpha\alpha \left(\frac{\sqrt{A} + r\sqrt{E}}{\sqrt{A} - r\sqrt{E}} \right)^2 - 2\alpha s \cdot \frac{\sqrt{A} + r\sqrt{E}}{\sqrt{A} - r\sqrt{E}}.$$

But on the other hand

$$s = \frac{4AEqr + C(A + Err)}{(A - Err)\sqrt{4AE - CC}}$$

and

$$r = pp = xy \quad \text{and} \quad q = \frac{xx + yy}{2xy}$$

and hence

$$s = \frac{2AE(xx + yy) + C(A + Exxyy)}{(A - Exxyy)\sqrt{4AE - CC}}.$$

§19 We can achieve the same without a new substitution; for, the moment we arrive at this equation

$$rdq(A - Err) + qdr(A + Err) + Crdr$$
$$= dr\sqrt{\frac{(4AErq + C(A + Err))^2 + (4AE - CC)(A - Err)^2}{4AE}},$$

note that the first term

$$= \frac{(A - Err)^2}{4AE} d. \frac{4AErq + C(A + Err)}{A - Err},$$

the second on the other hand can be expressed this way

$$\frac{dr(A - Err)}{2\sqrt{AE}} \sqrt{4AE - CC + \left(\frac{4AErq + C(A + Err)}{A - Err} \right)^2};$$

hence, for the sake of brevity, having set

$$\frac{4AErq + C(A + Err)}{A - Err} = v,$$

it will be

$$\frac{(A - Err)^2}{4AE} dv = \frac{dr(A - Err)}{2\sqrt{AE}} \sqrt{4AE - CC + vv}$$

and hence

$$\frac{dv}{\sqrt{4AE - CC + vv}} = \frac{2dr\sqrt{AE}}{A - Err}.$$

§20 Intending to give another specimen of this reduction, I will consider this equation

$$\frac{dx}{\sqrt{Bx + Cxx + Dx^3}} = \frac{dy}{\sqrt{By + Cyy + Dy^3}},$$

which I represent this way

$$\frac{ydx}{xdy} = \sqrt{\frac{Bxyy + Cxxyy + Dx^3yy}{Bxxy + Cxxyy + Dxxy^3}} = \sqrt{\frac{P + Q}{P - Q}},$$

that

$$\frac{P}{Q} = \frac{Bxy(y + x) + 2Cxxyy + Dxxyy(x + y)}{Bxy(y - x) + Dxxyy(x - y)}$$

or

$$\frac{P}{Q} = \frac{(B + Dxy)(x + y) + 2Cxy}{(B - Dxy)(y - x)},$$

and it will be

$$\frac{ydx - xdy}{ydx + xdy} = \frac{P + \sqrt{PP - QQ}}{Q}.$$

§21 Now set

$$x = p\left(u + \sqrt{uu - 1}\right) \quad \text{and} \quad y = p\left(u - \sqrt{uu - 1}\right);$$

it will be

$$\frac{dx}{x} = \frac{dp}{p} + \frac{du}{\sqrt{uu - 1}} \quad \text{and} \quad \frac{dy}{y} = \frac{dp}{p} - \frac{du}{\sqrt{uu - 1}}$$

and hence

$$\frac{ydx - xdy}{ydx + xdy} = \frac{pdu}{dp\sqrt{uu - 1}}.$$

Further, because of

$$xy = pp \quad \text{and} \quad x + y = 2pu, \quad y - x = -2p\sqrt{uu - 1},$$

it will be

$$\frac{P}{Q} = \frac{(B + Dpp)u + Cp}{-(B - Dpp)\sqrt{uu - 1}}$$

and hence

$$\frac{pdu}{dp} = \frac{(B + Dpp)u + Cp - \sqrt{4DBppuu + 2Cpu(B + Dpp) + CCpp + (B - Dpp)^2}}{Dpp - B},$$

whence

$$udp(B + Dpp) - pdu(Dpp - B) + Cpdp = dp\sqrt{\cdots}.$$

The first term is

$$(B - Dpp)^2 d.\frac{pu + \frac{C}{4BD}(B + Dpp)}{B - Dpp}$$

or

$$\frac{(B-Dpp)^2}{4BD}d.\frac{4BDpu+C(B+Dpp)}{B-Dpp},$$

but the quantity under the square root sign can be written this way

$$\frac{1}{4BD}(16BBDDppuu+8BCDpu(B+Dpp)+4BCCDpp+4BD(B-Dpp)^2)$$

$$=\frac{1}{4BD}((4BDpu+C(B+Dpp))^2+(4BD-CC)(B-Dpp)^2)),$$

whence the irrational term will be

$$\frac{B-Dpp}{2\sqrt{BD}}\sqrt{4BD-CC+\left(\frac{4BDpu+C(B+Dpp)}{B-Dpp}\right)^2}.$$

Hence, for the sake of brevity, having set

$$\frac{4BDpu+C(B+Dpp)}{B-Dpp}=s,$$

it will be

$$\frac{(B-Dpp)^2}{4BD}ds=\frac{(B-Dpp)dp}{2\sqrt{BD}}\sqrt{4BD-CC+ss},$$

whence

$$\frac{ds}{\sqrt{4BD-CC+ss}}=\frac{2dp\sqrt{BD}}{B-Dpp}$$

and by integrating

$$s+\sqrt{4BD-CC+ss}=\alpha\cdot\frac{\sqrt{B}+p\sqrt{D}}{\sqrt{B}-p\sqrt{D}}$$

and hence

$$4BD-CC=\alpha\alpha\left(\frac{\sqrt{B}+p\sqrt{D}}{\sqrt{B}-p\sqrt{D}}\right)^2-2\alpha s\cdot\frac{\sqrt{B}+p\sqrt{D}}{\sqrt{B}-p\sqrt{D}}.$$

§22 Therefore, the foundation of this reduction consists in this that first one sets $x = pq$ and $y = \frac{p}{q}$, but then for q a formula of such a kind is taken, by which the parts $x \pm y$, $xx \pm yy$ etc., which are contained in the formula $\frac{P}{Q}$, are simplified as much as possible. As in the case in § 17 we took

$$q = \sqrt{\frac{u+1}{2}} + \sqrt{\frac{u-1}{2}}$$

or $qq = u + \sqrt{uu-1}$, in the last on the other hand $q = u + \sqrt{uu-1}$; for, there it had not been necessary that $x + y$ is expressed rationally, whence it sufficed to attribute the form $u + \sqrt{uu-1}$ to qq, but here it was necessary that $x + y$ obtains a rational value.

§23 Finally, I cannot omit the simpler case in which this equation is propounded

$$\frac{dx}{\sqrt{A+Cxx}} = \frac{dy}{\sqrt{A+Cyy}},$$

which I represent this way

$$\frac{ydx}{xdy} = \sqrt{\frac{Ayy + Cxxyy}{Axx + Cxxyy}} = \sqrt{\frac{P+Q}{P-Q}};$$

therefore, having put

$$x = p\left(\sqrt{\frac{q+1}{2}} - \sqrt{\frac{q-1}{2}}\right) \quad \text{and} \quad y = p\left(\sqrt{\frac{q+1}{2}} + \sqrt{\frac{q-1}{2}}\right),$$

it will be

$$\frac{-pdq}{2dp\sqrt{qq-1}} = \frac{P - \sqrt{PP-QQ}}{Q}$$

while

$$\frac{P}{Q} = \frac{Aq + Cpp}{A\sqrt{qq-1}} \quad \text{and} \quad \frac{\sqrt{PP-QQ}}{Q} = \frac{\sqrt{2ACppq + CCp^4 + AA}}{A\sqrt{qq-1}}$$

whence having taken $pp = r = xy$ it will be

$$0 = \frac{rdq}{dr} + \frac{Aq + Cr - \sqrt{2ACrq + CCrr + AA}}{A}$$

and hence

$$\frac{A(rdq + qdr) + Crdr}{\sqrt{2ACrq + CCrr + AA}} = dr,$$

whose integral is

$$Cr + F = \sqrt{2ACrq + CCrr + AA} \quad \text{or} \quad FF + 2CFr = 2ACrq + AA;$$

but on the other hand

$$r = xy \quad \text{and} \quad q = \frac{xx + yy}{2xy},$$

whence the integral equation is

$$FF + 2CFxy = AA + AC(xx + yy).$$

And so this comparison of x and y, which is usually shown using logarithms or circular arcs, was found algebraicially here.