On various kinds of integrability*

Leonhard Euler

§1 If the variable quantity p is considered without any restrictions and one asks of what nature the quantity V has to be such that the formula Vdp becomes integrable, then there is no doubt that the quantity V has to be a certain function of p. For, I assume the term *integrability* to be understood in the broadest sense that, whatever function of p the letter V was, I say that the formula Vdp is always integrable, and it is of no interest, whether its integral is expressed algebraically or by logarithms or circular arcs or any higher transcendental quantities, since it is always possible to exhibit the integral formula $\int Vdp$ by means of quadrature of a certain curve.

§2 But matters behave quite differently, whenever the quantity p is referred to other variable quantities in a certain way; for, then aside from the functions of p, it is also possible to attribute other values to the quantity V, which render the formula Vdp integrable. For the sake of an example, if p is related to the two coordinates x and y in such a way that

$$dy = pdx \quad \text{or} \quad p = \frac{dy}{dx},$$

then one can also take x instead of V, since the formula xdp is indeed integrable, since obviously

$$\int xdp = px - \int pdx, \quad \text{because of} \quad \int pdx = y,$$

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it will be

$$\int xdp = px - y,$$

of course.

§3 Yes, the same will also happen in the primitive differentials dx and dy; namely, that the formula Vdx is integrable is not only the case, if V was an arbitrary function of x, but also in the case $V = p$, since

$$\int pdx = y,$$

in like manner, for the the function $\int Vdy$ to become integrable, one will not only be able to take an arbitrary function of y for V, but also in the case

$$V = \frac{1}{p} \quad \text{it is} \quad \int \frac{dy}{p} = x.$$

§4 To prosecute this subject in more generality, let us call the quantity V a multiplier, by which a certain differential formula is rendered integrable, whence from the things mentioned in advance it is clear, if the differential formula was

$$\text{either} \quad dp \quad \text{or} \quad dx \quad \text{or} \quad dy,$$

that the multiplier then is

$$\text{either} \quad V = x \quad \text{or} \quad V = p \quad \text{or} \quad V = \frac{1}{p}.$$

§5 Although all this might seem trivial and obvious, nevertheless the investigation of multipliers of this kind is often detected to be highly difficult, and since it is so extremely useful, this way it is possible to solve differential equation so of second as of higher degrees sufficiently conveniently, which aside through these tools seem to be completely inaccessible. For, I have already noted more often that the main task in the integration of differential equations is reduced to finding appropriate multipliers, from which the investigation of multipliers of this kind is without a doubt to be considered of highest importance.

§6 The occasion to investigate this more accurately was given to me by the question in which the curve expressed by the coordinates x and y is asked for, whose radius of curvature is equal to the line

$$\sqrt{xx+yy};$$

for, since having put $dy = pdx$ the radius of curvature is

$$= \frac{dx}{dp}(1+pp)^{\frac{3}{2}},$$

one has to solve this equation

$$\frac{dp}{(1+pp)^{\frac{3}{2}}} = \frac{dx}{\sqrt{xx+yy}},$$

each of both sides of which I, not without admiration, discovered to become integrable by means of the multiplier $x + py$; for, then for the right-hand side

$$\frac{dx(x+py)}{\sqrt{xx+yy}} = \frac{xdx+ydy}{\sqrt{xx+yy}},$$

whose integral is

$$\sqrt{xx+yy};$$

for the left-hand side on the other hand matters are not so obvious; but having put

$$y = px + v$$

such that

$$dy = pdx = pdx + xdp + dv \quad \text{and hence} \quad dv = -xdp,$$

we will have

$$dp(x+py) = xdp + ypdp = x(1+pp)dp + vpdp = -dv(1+pp) + vpdp$$

such that the left-hand side becomes

$$\frac{-dv(1+pp)+vpdp}{(1+pp)^{\frac{3}{2}}},$$

whose integral obviously is

$$= -\frac{v}{\sqrt{1+pp}} = \frac{px-y}{\sqrt{1+pp}},$$

and so the whole integral equation will be

$$\sqrt{xx+yy} + C = \frac{px-y}{\sqrt{1+pp}}.$$

§7 Therefore, considering this I do not doubt any more that all differential formulas of this kind admit two multipliers such that, if such a formula is already integrbale per se, which we will call dv in general, aside from natural multiplier, which are functions of v, also multipliers of a different nature are given, which are not functions of v, as we saw it to happen in the mentioned examples.

§8 But as soon as one single multiplier was known, from it infinitely many other multipliers can be concluded; since the multiplier of the formula dx is p and $\int pdx = y$, then any arbitrary function of y, which we call Y, if multiplied by p, will also give a suitable multiplier; for, dx multiplied by Yp gives Ydy, which is obviously integrable. Further, if X denotes an arbitrary function of x, the formula dy will have the multiplier $\frac{X}{p}$; for, then

$$\frac{Xdy}{p} = Xdx$$

results. Since in like manner

$$\int xdp = px - y,$$

if V denotes an arbitrary function of the formula $px - y$, then Vx will be a multiplier of dp, since it will be

$$Vxdp = V \cdot d(px-y);$$

but multipliers of this kind, that are concluded in this way from a certain known multiplier, are all to be considered of the same nature, whence in each class I will call the most simple of them the primitive one, having known which, all the remaining ones become known.

§9 Therefore, if in general a differential formula of this kind is given:

$$Pdp + Qdx + Rdy,$$

where P, Q, R are arbitrary functions of x, y and p, which is rendered integrable by means of the multipliers M, all the remaining multipliers of the same kind can be found in the following way. Set

$$M(Pdp + Qdx + Rdy) = dv$$

such that dv is a true differential, and let V be an arbitrary function of v, and it will be obvious that also VM will be a multiplier, since then one will have

$$VM(Pdp + Qdx + Rdy) = Vdv,$$

which formula is integrable by assumption.

§10 In like manner, if for the same given formula

$$Pdp + Qdx + Rdy$$

still another primitive multiplier N had been found, then from it one will also be able to find infinitely many of the same kind such that this way one obtained two general formulas for the multipliers of the given differential formula. Thus, hence the question of greatest importance arises, which will be worth the effort to consider it separately.

PROBLEM

If the formula

$$Pdp + Qdx + Rdy$$

becomes integrable so by the multiplier M as the other one N of different nature, to find the general expression that contains completely all possible multipliers of the same formula in it.

SOLUTION

Since M and N are multipliers, let us put

$$M(Pdp + Qdx + Rdy) = dv$$

and

$$N(Pdp + Qdx + Rdy) = du$$

and the quantities u and v will be known functions, now let z denote an arbitrary function of these two variables v and u, whose differential therefore will have a form of this kind:

$$dz = Sdv + Tdu,$$

where the functions S and T will be known from the function z; having already found this form, I say that the general expression, containing completely all multipliers in it, will be

$$= SM + TN;$$

for, then one will have

$$(SM + TN)(Pdp + Qdx + Rdy) = Sdv + Tdu = dz,$$

whose integral by assumption is z, where for z one can take any function of the two variables v and u as one desires.

§11 To illustrate this with an example, let the formula dp be given, whose multipliers are known

$$\begin{aligned} M &= 1 \qquad \text{and} \qquad N = x, \quad \text{thus, hence} \\ dp &= dv \qquad \text{and} \quad xdp = du, \quad \text{and hence} \\ v &= p \qquad \text{and} \qquad u = px - y, \end{aligned}$$

thus, if z denotes an arbitrary function of these two variables v and u and

$$dz = Sdv + Tdu,$$

the universal multiplier will be $S + Tx$.

§12 About this formula it is to be noted that it is absolutely not necessary that the values of the letters S and T are derived from a certain function z. For, as long as for the letters S and T functions of v and u of such a kind are taken that $\left(\frac{dS}{du}\right) = \left(\frac{dT}{dv}\right)$, then the formula $SM + TN$ will be a suitable multiplier of the differential formula

$$Pdp + Qdx + Rdy,$$

and the integral of the product will be that function z, which can be easily found from the letters S and T.

§13 But that this formula $SM + TN$ contains completely all multipliers of the given differential formula in it, is clear since one can always exhibit just two primitive multipliers M and N of this kind that do not depend on each other; for, if more multipliers of such a kind could find a place, then that form would certainly not be general, but another much more general one could be exhibited; but the reason why only two multipliers of this kind can take place is that just one relation between our three variables x, y and p is given, namely $p = \frac{dy}{dx}$; for, if we want to proceed further and introduce the letter q such that

$$dp = qdx \quad \text{or} \quad q = \frac{dp}{dx},$$

any arbitrary differential formula would even admit three multipliers, as it is evident from the most simple form dq, which first is integral, or the multiplier is $= 1$, the second multiplier is y, since

$$\int ydq = qy - \int qdy \quad \text{but} \quad \int qdy = \int pqdx = \int pdp = \frac{pp}{2},$$

whence

$$\int ydq = qy - \frac{pp}{2},$$

the third multiplier is x, since

$$\int xdq = xq - p,$$

from which it is sufficiently clear that three kinds of integrability occur here.

§14 But let us contemplate only the three variables x, y and p here, while $p = \frac{dy}{dx}$, and for it to become clearer that there are always two multipliers, let us focus on simpler cases, in which it was possible to find these multipliers, either by a divination or in any other way, which cases we want to add as follows.

$$\text{I.} \quad \alpha x dp + \beta p dx.$$

§15 This formula first is integrable per se, since its integral is

$$\alpha(px - y) + \beta y$$

such the first multiplier is $= 1$. The other multiplier will be $p^{\alpha-1}x^{\beta-1}$, for then the integral becomes $p^{\alpha}x^{\beta}$. Therefore, for finding the universal multiplier from § 10 it will be $M = 1$ and $N = p^{\alpha-1}x^{\beta-1}$, hence

$$v = \alpha(px - y) + \beta y = \alpha px + (\beta - \alpha)y \quad \text{and} \quad u = p^{\alpha}x^{\beta};$$

hence, if it was

$$dz = Sdv + Tdu,$$

the general multiplier will be

$$S \cdot 1 + Tp^{\alpha-1}x^{\beta-1}.$$

§16 But in the one case, in which $\alpha = 1$ and $\beta = 1$, this solution becomes incongruent, in which both multipliers are not different anymore; for, both of them would become $= 1$, and this inconvenience even occurs, if $\beta = \alpha$; for, then the first integral is αpx and the other multiplier $p^{\alpha-1}x^{\beta-1}$ would be its power, and would therefore not be different form the first multiplier, which is certainly obvious per se, since the whole task just depends on the ratio of α and β, whence here the new questions arises, whether the in the case $\beta = \alpha$ another multiplier can be exhibited, and how it will be expressed, which case we want to discuss separately.

$$\text{II.} \quad xdp + pdx.$$

§17 Concerning the first multiplier $= 1$, there is no difficulty, since the integral is pdx, the other multiplier on the the other hand does not reveal itself so easily; but having studied this more diligently the multiplier shows itself to be $= Lx$; for, it will be

$$\int (xdp + pdx)Lx = pxLx - y,$$

but in like manner another multiplier is calculated to be $\frac{y}{ppxx}$; for, the integral will become

$$= \frac{-y}{px} + \int \frac{dy}{px} = \frac{-y}{px} + Lx,$$

and this third multiplier does not differ from the first two, since from the first

$$M = 1, \quad v = px \quad \text{and} \quad N = Lx, \quad u = pxLx - y,$$

it is obvious that the third integral is a function of u and v, since

$$\frac{u}{v} = Lx - \frac{y}{px}.$$

Therefore, it is understood from this example that it can often seem that there are many different multipliers, although they can be reduced to two; to decide about this, from two multipliers just find the letters v and u, from which one will always find the remaining integrals, no matter how they were found, to be constructed.

III. $\alpha ydp + \beta pdy$.

§18 Here again one multiplier reveals itself immediately, namely $p^{\alpha-1}y^{\beta-1}$, to which the integral $p^\alpha y^\beta$ corresponds or, what goes back to the same, having taken the multiplier $\frac{1}{py}$, the integral will be

$$= \alpha Lp + \beta Ly = Lp^\alpha y^\beta,$$

since which is the logarithm of the latter, it is also to be considered to not differ from the first; but having studied this more diligently the other multiplier is calculated to be $\frac{1}{\alpha}xy^{\frac{\beta-\alpha}{\alpha}}$; for, the integral will be

$$xpy^{\frac{\beta}{\alpha}} - \left(\frac{\alpha}{\beta+\alpha}\right) y^{\frac{\beta+\alpha}{\alpha}},$$

then the multiplier $\frac{1}{pp}$ is seen immediately; for, it will be

$$\int \frac{\alpha y dp}{pp} = -\frac{\alpha y}{p} + \int \frac{\alpha dy}{p} = -\frac{\alpha y}{p} + \alpha x \quad \text{and} \quad \int \frac{pdy}{pp} = \int \frac{dy}{p} = x,$$

whence the whole integral will be

$$= \frac{-\alpha y}{p} + (\alpha + \beta)x,$$

but this is already contained in the two preceding ones; for, it will be

$$M = p^{\alpha - 1}y^{\beta - 1} \quad \text{and} \quad v = p^{\alpha}y^{\beta},$$
$$N = xyp^{\frac{\beta}{\alpha}} \quad \text{and} \quad u = xpy^{\frac{\beta}{\alpha}} - \left(\frac{\alpha}{\beta + \alpha}\right) y^{\frac{\beta + \alpha}{\alpha}},$$

whence dividing u by

$$\frac{v^{\frac{1}{\alpha}}}{\alpha + \beta} = \frac{py^{\frac{\beta}{\alpha}}}{\alpha + \beta}$$

the integral is $(\alpha + \beta)x - \frac{\alpha y}{p}$.
But this reduction does not succeed in the case $\beta = -\alpha$, which case we want to discuss separately.

IV. $ydp - pdy$.

§19 Since in this case $\alpha = 1$ and $\beta = -1$, the first multiplier will be $\frac{1}{yy}$; furthermore, one calculates the multiplier $\frac{x}{yy}$; for, then it will be

$$\int \frac{x(ydp - pdy)}{yy} = \frac{px}{y} - Ly,$$

for, this formula, if differentiated, yields

$$\frac{x(ydp - pdy)}{yy} = \frac{pdx + xdp}{y} - \frac{dypx}{yy} - \frac{dy}{y} \quad \text{because of} \quad dy = pdx;$$

since now we have two multipliers, the one $M = \frac{1}{yy}$ and the other $N = \frac{x}{yy}$, whence

$$v = \frac{p}{y} \quad \text{and} \quad u = \frac{px}{y} - Ly,$$

if z denotes an arbitrary function of the two quantities v and u, the general multiplier will be

$$= M\left(\frac{dz}{dv}\right) + N\left(\frac{dz}{du}\right) = \frac{1}{yy}\left(\frac{dz}{dv}\right) + \frac{x}{yy}\left(\frac{dz}{du}\right).$$

V. $pdp + xdx$.

§20 First, this formula is integrable per se such that

$$M = 1 \quad \text{and} \quad v = \frac{1}{2}(pp + xx),$$

but then another multiplier is detected as the arc, whose tangent is $\frac{x}{p} = N$; for, then it will be

$$\int (pdp + xdx) \arctan\frac{x}{p} = \frac{1}{2}(pp + xx) \arctan\frac{x}{p} - \int \frac{1}{2}(pp + xx) d \cdot \arctan\frac{x}{p}$$

$$= \frac{1}{2}(pp + xx) \arctan\frac{x}{p} - \int \frac{1}{2}(pdx - xdp);$$

but

$$\int \frac{1}{2}(pdx - xdp) = y - \frac{px}{2},$$

whence the integral in question will be

$$\frac{1}{2}(pp + xx) \arctan\frac{x}{p} - y + \frac{px}{2}.$$

§21 From these examples it is abundantly clear that finding of multipliers of this kind is not obvious at all, in most cases it is actually so difficult that it can even seem that it exceeds the powers of analysis. Nevertheless, I will present a method here, accommodated to the task at hand, by means of which it is possible to find such multipliers in most cases.

§22 Since the reason of two multipliers lies in the fact that the differential formulas of this kind are of second degree, whence it happens that each of both multipliers just involves one single integration, and two integrations also require two multipliers, hence vice versa it will be possible to find both multipliers, if we perform each of both integrations. Therefore, we will teach in the following examples, how this method has to be used.

EXAMPLE 1

§23 Given the differential formula $xdp + pdx$, to find each of its two multipliers. Since this formula is integrable per se, and hence $M = 1$, put

$$xdp + pdx = dv \quad \text{and it will be} \quad px = v,$$

and so one integration has been done, for the other, since $p = \frac{v}{x}$, multiplying by dx because of $pdx = dy$ we will have $dy = \frac{vdx}{x}$, whence by integration we find

$$y = vLx - \int dv \cdot Lx \quad \text{and hence}$$

$$\int dvLx = vLx - y = pxLx - y,$$

whence we understand the formula $dvLx$ to be integrable, since its integral is $pxLx - y$; since dv denotes our given formula, $xdp + pdx$, it is clear that its multiplier will be Lx.

§24 In the same way one can even find other multipliers; for, since

$$dy = \frac{vdx}{x}, \quad \text{it will also be} \quad \frac{dy}{v} = \frac{dx}{x},$$

hence by integrating

$$\frac{y}{v} + \int \frac{ydv}{vv} = Lx, \quad \text{thus,} \quad \int \frac{ydv}{vv} = Lx - \frac{y}{v} = Lx - \frac{y}{px},$$

hence the integrable formula is

$$\frac{ydv}{vv} \quad \text{or} \quad dv \cdot \frac{y}{vv},$$

and so the multiplier will be

$$\frac{y}{vv} = \frac{y}{ppxx},$$

which can therefore be taken instead of N, whence, since

$$v = px \quad \text{and} \quad u = Lx - \frac{y}{px},$$

if z denotes an arbitrary function of v and u, the general multiplier will be

$$\left(\frac{dz}{dv}\right) + \frac{y}{ppxx}\left(\frac{dz}{du}\right);$$

for example, if it was $z = vu$, it will be

$$\left(\frac{dz}{dv}\right) = u \quad \text{and} \quad \left(\frac{dz}{du}\right) = v,$$

whence this multiplier arises:

$$u + \frac{y}{ppxx}v = Lx - \frac{y}{px} + \frac{y}{px} = Lx,$$

which is the multiplier found first.

EXAMPLE 2

§25 Given the differential formula

$$\alpha x dp + \beta p dx,$$

to find both of its multipliers. Since this formula is already integrable per se, it will be $M = 1$, and having put

$$\alpha x dp + \beta p dx = dv \quad \text{it will be} \quad v = \alpha p x + (\beta - \alpha) y,$$

whence one calculates

$$p = \frac{v}{\alpha x} + \frac{(\alpha - \beta) y}{\alpha x},$$

which, if multiplied by dx, yields

$$dy = p dx = \frac{v dx}{\alpha x} + \frac{\alpha - \beta}{\alpha} \cdot \frac{y dx}{x},$$

and hence

$$\frac{dx}{x} = \frac{\alpha dy}{v + (\alpha - \beta)y};$$

thus, by integrating we will obtain

$$Lx = \frac{\alpha}{\alpha - \beta}L[v + (\alpha - \beta)y] - \int \frac{\alpha}{\alpha - \beta} \cdot \frac{dv}{v + (\alpha - \beta)y},$$

and so it will be

$$\alpha \int \frac{dv}{v + (\alpha - \beta)y} = \alpha L[v + (\alpha - \beta)y] - (\alpha - \beta)Lx,$$

whence it is clear that multiplier of our formula dv will be

$$\frac{\alpha}{v + (\alpha - \beta)y} = \frac{1}{px},$$

as it is obvious per se; for, then the integral will be

$$\alpha Lp + \beta Lx.$$

EXAMPLE 3

§26 Given the formula $pdp + xdx$, to find both of its multipliers. Here, again the first multiplier is $M = 1$ and having put

$$pdp + xdx = dv \quad \text{it will be} \quad pp + xx = 2v, \quad \text{whence} \quad p = \sqrt{2v - xx}$$

and by multiplying by dx

$$dy = pdx = dx\sqrt{2v - xx},$$

put $2v = ss$, and it will be

$$y = \frac{1}{2}x \cdot \sqrt{ss - xx} + \frac{ss}{2} \arcsin \frac{x}{s} - \int \left(\frac{xsds}{2\sqrt{ss - xx}} + sds \arcsin \frac{x}{s} - \frac{sxds}{2\sqrt{ss - xx}} \right)$$

$$= \frac{1}{2}x \cdot \sqrt{ss - xx} + \frac{ss}{2} \arcsin \frac{x}{s} - \int sds \arcsin \frac{x}{s}$$

and hence

$$\int sds \cdot \arcsin\frac{x}{s} = \frac{1}{2}x\sqrt{ss-xx} + \frac{ss}{2}\arcsin\frac{x}{s} - y$$
$$= \frac{px}{2} + \frac{1}{2}(pp+xx)\arcsin\frac{x}{\sqrt{pp+xx}} - y,$$

but

$$sds = pdp + xdx,$$

whence it is clear that the multiplier of our formula is

$$\arcsin\frac{x}{\sqrt{pp+xx}} \quad \text{or} \quad \arctan\frac{x}{p}.$$

§27 But this operation is too tedious for us to be able to use more complicated formulas in it; thus, let us try to simplify it as follows. Since we found

$$dy = dx\sqrt{ss-xx},$$

let us set $x = sz$ here and it will be

$$dy = ssdz\sqrt{1-zz} + zsds\sqrt{1-zz},$$

which divided by s gives

$$\frac{dy}{ss} = dz\sqrt{1-zz} + \frac{zds}{s}\sqrt{1-zz}$$

and by integrating

$$\frac{y}{ss} + 2\int\frac{yds}{s^3} = \int dz\sqrt{1-zz} + \int\frac{zds}{s}\sqrt{1-zz},$$

where the penultimate term

$$\int dz\sqrt{1-zz}$$

is given absolutely, since it is a certain function of $z = \frac{x}{s}$, but the last term, having resubstituted the value $\frac{x}{s}$ for z, goes over into

$$\int\frac{xds}{s^3}\sqrt{ss-xx},$$

whence by combining the two integrals affected by $\frac{ds}{s^3}$ we obtain

$$\int \frac{ds}{s^3}(2y - x\sqrt{ss - xx}) = \int dz\sqrt{1 - zz} - \frac{y}{ss},$$

from where it is clear that the multiplier of our formula

$$pdp + xdx = sds$$

is

$$\frac{1}{s^4}(2y - x\sqrt{ss - xx}),$$

which because of $ss = pp + xx$ is transformed into this form:

$$\frac{2y - px}{(pp + xx)^2},$$

if which is put $= N$, it will be

$$u = \int dz\sqrt{1 - zz} - \frac{y}{pp + xx}, \quad \text{while} \quad z = \frac{x}{\sqrt{pp + xx}},$$

and hence it is possible to find the general multiplier easily.

§28 Hence it is clear, if the formula $\alpha pdp + \beta xdx$ was given, in which case again

$$M = 1 \quad \text{and} \quad v = \frac{\alpha pp}{2} + \frac{\beta xx}{2},$$

that the other multiplier will be found to be

$$N = \frac{2y - xp}{(\alpha pp + \beta xx)^2};$$

for, then the integral arising from this will be

$$u = \int \frac{sds}{s^4}(2y - px) = \int dz\sqrt{\frac{1 - \beta zz}{\alpha}} - \frac{y}{\alpha pp + \beta xx},$$

while

$$z = \frac{x}{s} = \frac{x}{\sqrt{\alpha pp + \beta xx}}.$$

EXAMPLE 4

§29 Given the differential formula

$$p^{n-1}dp + \beta x^{n-1}dx,$$

to find its multiplier. Since the one multiplier $= 1$ is known again, having put our formula $= dv$, it will be

$$p^n + \beta x^n = nv,$$

whence

$$p = (nv - \beta x^n)^{\frac{1}{n}},$$

and hence

$$dy = pdx = dx(nv - \beta x^n)^{\frac{1}{n}},$$

now set

$$nv = s^n, \quad \text{let} \quad x = sz,$$

we will have

$$dy = (sdz + zds)s(1 - \beta z^n)^{\frac{1}{n}},$$

which divided by ss gives

$$\frac{dy}{ss} = dz(1 - \beta z^n)^{\frac{1}{n}} + \frac{zds}{s}(1 - \beta z^n)^{\frac{1}{n}},$$

hence by integrating

$$\frac{y}{ss} + \int \frac{2yds}{s^3} = \int dz(1 - \beta z^n)^{\frac{1}{n}} + \int \frac{zds}{s}(1 - \beta z^n)^{\frac{1}{n}},$$

where the penultimate term is determined, a certain function of

$$z = \frac{x}{s} = \frac{x}{(nv)^{\frac{1}{n}}},$$

the last term on the other hand, if one sets $z = \frac{x}{s}$ in it again, goes over into

$$\int \frac{xds}{s^3}(s^n - \beta x^n)^{\frac{1}{n}},$$

and because of

$$(s^n - \beta x^n)^{\frac{1}{n}} = p$$

it will be

$$\int \frac{xds}{s^3}(s^n - \beta x^n)^{\frac{1}{n}} = \int \frac{pxds}{s^3},$$

having substituted which it will be

$$\int \frac{ds}{s^3}(2y - x) = \int dz(1 - \beta z^n)^{\frac{1}{n}} - \frac{y}{ss}.$$

But since $s = (nv)^{\frac{1}{n}}$, it will be

$$ds = (nv)^{\frac{1}{n}-1}dv \quad \text{and} \quad \frac{ds}{s^3} = \frac{dv}{(nv)^{\frac{2+n}{n}}},$$

from which the first term will become

$$\int \frac{dv}{(nv)^{\frac{2+n}{n}}}(2y - px),$$

since whose integral has already been found, it is plain that the multiplier of the given formula dv is

$$= \frac{2y - px}{(nv)^{\frac{2+n}{n}}} = \frac{2y - px}{(p^n + \beta x^n)^{\frac{2+n}{n}}}.$$

EXAMPLE 5

§30 Having given the differential formula:

$$p^{n-1}dp + \beta x^{n-1}dx = dv,$$

to finds its our multiplier. Since hence

$$v = \frac{1}{m}p^m + \frac{\beta}{n}x^n, \quad \text{it will be}$$

$$p = (mv - \frac{m\beta}{n} \cdot x^n)^{\frac{1}{m}} \quad \text{and} \quad dy = pdx = dx(mv - \frac{m\beta}{n} \cdot x^n)^{\frac{1}{m}};$$

here, again set

$$mv = s^n \quad \text{and} \quad x = sz, \quad \text{it will be}$$
$$dy = (sdz + zds)s^{\frac{n}{m}} \left(1 - \frac{m\beta}{n} \cdot z^n\right)^{\frac{1}{m}},$$

which divided by $s^{\frac{m+n}{m}}$ yields

$$\frac{dy}{s^{\frac{m+n}{m}}} = dz\left(1 - \frac{m\beta}{n} \cdot z^n\right)^{\frac{1}{m}} + \frac{zds}{s}\left(1 - \frac{m\beta}{n} \cdot z^n\right)^{\frac{1}{m}},$$

hence by integrating

$$\frac{y}{s^{\frac{m+n}{m}}} + \int \frac{m+n}{m} \cdot \frac{yds}{s^{\frac{2m+n}{m}}} = \int dz\left(1 - \frac{m\beta}{n} \cdot z^n\right)^{\frac{1}{m}} + \int \frac{zds}{s}\left(1 - \frac{m\beta}{n} \cdot z^n\right)^{\frac{1}{m}},$$

where the penultimate term is a known function of z, but the last term, having substituted the value $\frac{x}{s}$ instead of z, goes over into

$$\int \frac{xds}{s^{\frac{2m+n}{m}}}\left(s^n - \frac{m\beta}{n} \cdot x^n\right)^{\frac{1}{m}} = \int \frac{pxds}{s^{\frac{2m+n}{m}}},$$

whence we calculate

$$\int \frac{ds}{s^{\frac{2m+n}{m}}}\left(\frac{m+n}{m}y - px\right) = \int dz\left(1 - \frac{m\beta}{n} \cdot z^n\right)^{\frac{1}{m}} - \frac{y}{s^{\frac{m+n}{m}}},$$

but since

$$s = (mv)^{\frac{1}{n}}, \quad \text{it will be} \quad ds = \frac{1}{n} \cdot m^{\frac{1}{n}} v^{\frac{1}{n}-1} dv$$

and

$$\frac{ds}{s^{\frac{2m+n}{m}}} = \frac{1}{n} \cdot \frac{dv}{m^{\frac{m+n}{mn}} v^{\frac{m+n}{mn}+1}},$$

from which one concluded that the multiplier of our given formula is

$$\frac{\frac{m+n}{m}y - px}{v^{\frac{m+n}{mn}+1}} = \frac{\frac{m+n}{m}y - px}{\left(\frac{1}{m}p^m + \frac{\beta}{n}x^n\right)^{\frac{m+n}{mn}+1}}.$$

§31 Up to this point we have considered formulas of such kind that are both integrable per se and contain just the two variables p and x; but in like manner it will be possible to treat those formulas that just involve these two variables p and y, where we certainly assume that they are integrable per se.

EXAMPLE 6

Given the formula

$$p^{m-1}dp + \beta y^{n-1}dy = dv,$$

to investigate its multiplier. First, by integrating we find

$$v = \frac{1}{m}p^m + \frac{\beta}{n}y^n,$$

whence

$$p = \left(mv - \frac{m\beta}{n} \cdot y^n\right)^{\frac{1}{m}}$$

and

$$dy = pdx = dx(mv - \frac{m\beta}{n} \cdot y^n)^{\frac{1}{m}}$$

and hence

$$dx = \frac{dy}{(mv - \frac{m\beta}{n} \cdot y^n)^{\frac{1}{m}}},$$

which equation divided by $s^{\frac{m-n}{m}}$ gives

$$\frac{dx}{s^{\frac{m-n}{m}}} = \frac{dz}{\left(1 - \frac{m\beta}{n} \cdot z^n\right)^{\frac{1}{m}}} + \frac{zds}{s\left(1 - \frac{m\beta}{n} \cdot z^n\right)^{\frac{1}{m}}};$$

this equation is integrated in the same way as above, and from the penultimate term again a certain function of z arises; if we resubstitute the value $\frac{y}{s}$ instead of z in the last term, we obtain

$$\frac{x}{s^{\frac{m-n}{m}}} + \int \frac{m-n}{m} \cdot \frac{xds}{s^{\frac{2m-n}{m}}} = \int \frac{dz}{\left(1 - \frac{m\beta}{n} \cdot z^n\right)^{\frac{1}{m}}} + \int \frac{yds}{p \cdot s^{\frac{2m-n}{m}}};$$

thus, from all this we conclude f

$$\int \frac{ds}{s^{2-\frac{n}{m}}}\left(\left(1-\frac{n}{m}\right)x-\frac{y}{p}\right)=\int \frac{dz}{\left(1-\frac{m\beta}{n}\cdot z^n\right)^{\frac{1}{m}}}-\frac{x}{s^{\frac{m-n}{m}}},$$

since now $s=(mv)^{\frac{1}{n}}$, it will be

$$ds=\frac{1}{n}\cdot m^{\frac{1}{n}}v^{\frac{1}{n}-1},$$

hence

$$\frac{ds}{s^{2-\frac{n}{m}}}=\frac{1}{n}dv\cdot\frac{1}{m^{\frac{1}{n}-\frac{1}{m}}v^{\frac{1}{n}-\frac{1}{m}+1}},$$

whence we conclude that the multiplier of our given formula will be

$$\frac{\left(1-\frac{n}{m}\right)x-\frac{y}{p}}{v^{\frac{1}{n}-\frac{1}{m}+1}}=\frac{\left(1-\frac{n}{m}\right)x-\frac{y}{p}}{\left(\frac{1}{m}p^m+\frac{\beta}{n}y^n\right)^{\frac{1}{n}-\frac{1}{m}+1}}.$$

§32 Even though these examples seem to extend sufficiently far, nevertheless, if the matter itself is concerned, they are even still extremely particular ones, since for v a binomial formula arises for each of both cases, involving the letters p and x in the penultimate example, but p and y in the last one, and hence it is hardly clear how these operations have to be executed, if more terms occur in the value of v. Nevertheless, that investigation can be generalised quite a bit as follows.

PROBLEM

If Ω was a function of the quantities p and x that it, having put $p=x^\lambda q$, takes this form $x^n Q$ such that Q is a function of q only, then, given the differential formula $d\Omega$, to find its multiplier.

SOLUTION

Having put $d\Omega=dv$ as before such that $v=\Omega$, set $p=x^\lambda q$ and by assumption it will be

$$v = x^n Q, \quad \text{hence} \quad Q = \frac{v}{x^n},$$

now further set $x^n = \frac{v}{z}$ that $Q = z$; now, no matter how many dimensions of q are contained in the function Q, even if the resolution of this equation exceeds the possibilities of analysis, nevertheless it is certain that the value of the root q is expressed by a certain function of z, which we will call Z, such that $q = Z$; hence

$$p = x^\lambda Z \quad \text{and} \quad dy = pdx = x^\lambda Zdx,$$

since now

$$x^n = \frac{v}{z}, \quad \text{it will be} \quad x^{\lambda+1} = \left(\frac{v}{z}\right)^{\frac{\lambda+1}{n}}$$

and

$$x^\lambda dx = \frac{1}{n} \cdot \frac{zdv - vdz}{zz} \left(\frac{v}{z}\right)^{\frac{\lambda+1}{n}-1} = \frac{1}{n}\left(\frac{dv \cdot v^{\frac{\lambda+1}{n}-1}}{z^{\frac{\lambda+1}{n}}} - \frac{dz \cdot v^{\frac{\lambda+1}{n}}}{z^{\frac{\lambda+1}{n}+1}}\right)$$

and multiplying by $\frac{n}{v^{\frac{\lambda+1}{n}}}$ we will have

$$\frac{ndy}{v^{\frac{\lambda+1}{n}}} = \frac{Zdv}{v \cdot z^{\frac{\lambda+1}{n}}} - \frac{Zdz}{z^{\frac{\lambda+1}{n}+1}},$$

where the last term containing the variable z will give a determined function of z, the penultimate on the other can, having resubstituted the value $\frac{v}{x^n}$ for z, because of $Z = \frac{p}{x^\lambda}$ goes over into $\frac{pxdv}{v^{\frac{\lambda+1}{n}+1}}$, whence by integrating we will have

$$\frac{ny}{v^{\frac{\lambda+1}{n}}} + \int (\lambda+1)\frac{ydv}{v^{\frac{\lambda+1}{n}+1}} = \int \frac{pxdv}{v^{\frac{\lambda+1}{n}+1}} - \int \frac{Zdz}{z^{\frac{\lambda+1}{n}+1}},$$

whence

$$\int \frac{dv}{v^{\frac{\lambda+1}{n}+1}}(px - (\lambda+1)y) = \int \frac{Zdz}{z^{\frac{\lambda+1}{n}+1}} + \frac{ny}{v^{\frac{\lambda+1}{n}}}.$$

Therefore, we conclude that the multiplier in question of our formula $dv = d\Omega$ is

$$\frac{px - (\lambda+1)y}{(\Omega)^{\frac{\lambda+1}{n}+1}}.$$

Problem

§33 If Ω was a function of p and y of such a kind that, having put $p = y^\lambda q$, it obtains this form $y^n Q$, while Q is function of q, then, given the differential formula $d\Omega$, to find its multiplier.

Solution

Having put $d\Omega = dv$ again, it will be $v = \Omega$, and having put $p = y^\lambda q$ it will be $v = y^n Q$ and hence $Q = \frac{v}{y^n}$; now set $y^n = \frac{v}{z}$ such that $Q = z$; since hence Q is a function of q, by the resolution of the equation q will become equal to a certain of z, which we will call Z, such that

$$q = Z \quad \text{hence} \quad p = y^\lambda Z$$

$$\text{and} \quad dy = pdx = y^\lambda Z dx, \quad \text{whence} \quad dx = \frac{dy}{y^\lambda Z},$$

but since $y^n = \frac{v}{z}$, it will be

$$y = \frac{v^{\frac{1}{n}}}{z^{\frac{1}{n}}} \quad \text{and} \quad dy = \frac{1}{n}\left(\frac{dv \cdot v^{\frac{1}{n}-1}}{z^{\frac{1}{n}}} - \frac{v^{\frac{1}{n}}dz}{z^{\frac{1}{n}+1}}\right),$$

from where one will have

$$dx = \frac{1}{n}\frac{v^{\frac{1}{n}-\frac{\lambda}{n}-1}dv}{z^{\frac{1-\lambda}{n}}\cdot Z} - \frac{1}{n}\cdot\frac{v^{\frac{1-\lambda}{n}}dz}{Z\cdot z^{\frac{1-\lambda}{n}+1}},$$

which multiplied by $n \cdot v^{\frac{\lambda-1}{n}}$ will yield

$$ndx \cdot v^{\frac{\lambda-1}{n}} = \frac{dv}{v\cdot z^{\frac{1-\lambda}{n}}\cdot Z} - \frac{dz}{z^{\frac{1-\lambda}{n}+1}\cdot Z};$$

here the integral of the last term obviously is a certain power of z that can at least be exhibited by means of quadratures; the penultimate term on the other hand because of

$$Z = q = \frac{p}{y^\lambda} \quad \text{goes over into} \quad \frac{dvy^\lambda}{pv\cdot z^{\frac{1-\lambda}{n}}}$$

and having substituted its value $\frac{v}{y^n}$ for z is transformed into

$$\frac{ydv}{p\cdot v^{\frac{1-\lambda}{n}+1}},$$

whence, by integrating the first term by parts, we obtain

$$nx\cdot v^{\frac{\lambda-1}{n}}-\int(\lambda-1)x\cdot v^{\frac{\lambda-1}{n}-1}dv=\int\frac{ydv\cdot v^{\frac{\lambda-1}{n}-1}}{p}-\int\frac{dz\cdot z^{\frac{\lambda-1}{n}-1}}{Z},$$

and hence one concludes

$$\int dv\cdot v^{\frac{\lambda-1}{n}-1}\left((\lambda-1)x+\frac{y}{p}\right)=nx\cdot v^{\frac{\lambda-1}{n}}+\int\frac{dz\cdot z^{\frac{\lambda-1}{n}-1}}{Z},$$

since therefore $dv = d\Omega$, it is clear that the given formula $d\Omega$ is rendered integrable, if it is multiplied by

$$(\Omega)^{\frac{\lambda-1}{n}-1}\left((\lambda-1)x+\frac{y}{p}\right),$$

which therefore is the multiplier in question.

§34 Since in the penultimate problem the multiplier of the formula $d\Omega$ is

$$\frac{px-(\lambda+1)y}{(\Omega)^{\frac{\lambda-1}{n}+1}},$$

the multiplier of this formula

$$\frac{d\Omega}{(\Omega)^{\frac{\lambda+1}{n}+1}},$$

which is even a true differential, whose integral obviously is

$$\frac{n}{\lambda+1}\cdot\frac{1}{(\Omega)^{\frac{\lambda+1}{n}}},$$

will be

$$px-(\lambda+1)y;$$

since in like manner in the last problem the multiplier of the formula $d\Omega$ was found to be

$$(\Omega)^{\frac{\lambda-1}{n}-1}\left((\lambda-1)x+\frac{y}{p}\right),$$

the multiplier of this formula

$$d\Omega(\Omega)^{\frac{\lambda-1}{n}-1},$$

which also is a true differential, whose integral is

$$\frac{n}{\lambda-1}\cdot(\Omega)^{\frac{\lambda-1}{n}},$$

will be

$$(\lambda-1)x+\frac{y}{p};$$

because of the simplicity of the multipliers these two cases seem to be especially noteworthy such that it will be worth one's while to investigate all differential formulas in general that have such a multiplier, for which aim we mention this Lemma in advance.

LEMMA

§35 If the multiplier of the differential formula $d\Omega$ was V, then vice versa Ω will be the multiplier of the differential formula dV; for, since

$$\int \Omega dV = V\Omega - \int V d\Omega,$$

since the formula $Vd\Omega$ is integrable by assumption, it is necessary that the formula $\int \Omega dV$ is also integrable.

PROBLEM

§36 To find all differential formulas $d\Omega$ that have this multiplier

$$\alpha y + px,$$

while α denotes an arbitray number.

Solution

Since because of $dv = pdx$

$$d(\alpha y + px) = (\alpha + 1)pdx + xdp,$$

the multiplier of this formula has to be Ω, from which condition one can determine the quantity Ω. Therefore, let this differential formula be given

$$(\alpha + 1)pdx + xdp,$$

for which one has the multiplier

$$M = 1 \quad \text{and it will be} \quad v = \alpha y + px,$$

the other multiplier will be

$$N = x^{\alpha}, \quad \text{and then it will then be} \quad u = p \cdot x^{\alpha+1};$$

thus, if Z denotes an arbitrary function of the two variables

$$v = \alpha y + px \quad \text{and} \quad u = p \cdot x^{\alpha+1},$$

the multiplier of our formula in general will be

$$M\left(\frac{dZ}{dv}\right) + N\left(\frac{dZ}{du}\right)$$

such that, since

$$\Omega = \left(\frac{dZ}{dv}\right) + x^{\alpha}\left(\frac{dZ}{du}\right),$$

its differential $d\Omega$ will contain all differential formulas that have the multiplier $\alpha y + px$.

§37 If one takes an arbitrary function of u for Z, it will be

$$\left(\frac{dZ}{dv}\right) = 0$$

and $\left(\frac{dZ}{du}\right)$ will be a certain function of u, which we will call $f:u$; hence our Ω will be

$$x^\alpha \cdot f : u = x^\alpha f : p \cdot x^{\alpha+1},$$

which form agrees perfectly with the problem in § 32 [see also § 34], where the multiplier was $px - (\lambda+1)y$, such that $\alpha = -(\lambda+1)$. Further, since here Ω is what

$$\text{was} \quad \frac{dv}{v^{\frac{\lambda+1}{n}+1}} \quad \text{there,}$$

and so what is Ω here,

$$\text{was} \quad \frac{n}{\lambda+1} \cdot \frac{dv}{v^{\frac{\lambda+1}{n}}} \quad \text{there,}$$

but there it was

$$v = x^n Q = x^n \Phi : \frac{p}{x^\lambda}$$

such that from this

$$\frac{1}{v^{\frac{\lambda+1}{n}}} = x^{-(\lambda+1)} \cdot \Delta : \frac{p}{x^\lambda},$$

that which form is contained in that one is quite obvious. Thus, hence it is clear that that problem is a highly particular case of this problem and the solution of this one extends infinitely further.

Problem

§38 To find all differential formulas $d\Omega$ that have the multiplier $\alpha x + \frac{y}{p}$.

Solution

Since because of $dx = \frac{dy}{p}$

$$d \cdot \left(\alpha x + \frac{y}{p}\right) = (\alpha+1)\frac{dy}{p} - \frac{ydp}{pp},$$

let us consider this differential as the given formula, whose multiplier Ω is to be investigated, and since the first multiplier is $M = 1$, it will be $v = \alpha x + \frac{y}{p}$, but the other multiplier is calculated to be $N = y^\alpha$, from where $u = \frac{y^{\alpha+1}}{p}$; hence, if Z denotes an arbitrary function of the two variables

$$v = \alpha x + \frac{y}{p} \quad \text{and} \quad u = \frac{y^{\alpha+1}}{p},$$

the general expression for the multiplier in question will be

$$\Omega = \left(\frac{dZ}{dv}\right) + y^\alpha \left(\frac{dZ}{du}\right),$$

where one has to note, if a function of just the one variable u is taken for Z, that this solution then leads to the case of the problem in § 33.

§39 What has been taught about the investigation of multipliers up to this point is tremendously useful in the resolution of differential equations of second degree; for, since because of $dy = pdx$ all equations of this order can be reduced to this form:

$$Rdp + Qdx + Rdy = 0,$$

it is obvious, if one multiplier is known, that then immediately a one time integrated equation, which will be of first order, is obtained, which should thereafter be treated according to the known rules; but if two multipliers of the formula were known, then one will immediately be able to find a finite or two times integrated equation such that repeated integration is not necessary, which operation we will teach in the following.

Problem

§40 Given the differential equation

$$Pdp + Qdx + Rdy = 0,$$

if two of its multipliers M and N are known, to find its finite two times integrated equation.

SOLUTION

Since M and N are known multipliers, let us put

$$M(Pdp + Qdx + Rdy) = dv$$

and

$$N(Pdp + Qdx + Rdy) = du,$$

hence one will have the quantities v and u, no matter how they are conflated from the three variables p, x and y. Thus, because of the first multiplier it will be $v = a$ and because of the second $u = b$, where a and b are two constants, each of them entering through integration; therefore, since one has two finite equations between the three variables x, y and p, if p is eliminated from them, a finite equation between the two coordinates x and y will result or, what is essentially the same, one will be able to determined two of these letters in terms of the third from this.

§41 One could easily illustrate this method from the preceding with many examples, but the problem mentioned at the beginning of the dissertation will be prototypical for all of them, in which the curve that is to be expressed as an equation between the two coordinates x and y and whose radius of curvature is equal to the formula $\frac{1}{n}\sqrt{xx+yy}$ is in question.

EXAMPLE

Given the differential equation of second degree

$$\frac{dp}{(1+pp)^{\frac{3}{2}}} - \frac{ndx}{\sqrt{xx+yy}} = 0,$$

to find a finite equation between x and y by means of two multipliers. We already above [§6] that the first multiplier is $x + py$, whence

$$dv = \frac{dp(x+py)}{(1+pp)^{\frac{3}{2}}} - \frac{n(xdx+ydy)}{\sqrt{xx+yy}},$$

and hence it will be

$$v = \frac{px-y}{\sqrt{1+pp}} - n\sqrt{xx+yy} = a.$$

But for the other multiplier, since it is not seen that easily, let us investigate so it as the integral arising from it at the same time; first, let us put $y = xz$ and because of $dy = pdx$ it will be

$$pdx = zdx + xdz \quad \text{and hence} \quad \frac{dx}{x} = \frac{dz}{p-z},$$

having substituted which value our formula will be

$$\frac{dp}{(1+pp)^{\frac{3}{2}}} - \frac{ndz}{(p-z)\sqrt{1+zz}} = 0,$$

whence having further put $z = \frac{p+q}{1-pq}$, whence

$$p - z = -\frac{q(1+pp)}{1-pq}, \quad \sqrt{1+zz} = \frac{\sqrt{(1+pp)(1+qq)}}{1-pq}$$

and

$$dz = \frac{dp(1+qq) + dq(1+pp)}{(1-pq)^2},$$

our formula is transformed into this one:

$$\frac{dp}{(1+pp)^{\frac{3}{2}}} + \frac{ndp(1+qq) + ndq(1+pp)}{q(1+pp)^{\frac{3}{2}}\sqrt{1+qq}} = \frac{(q\sqrt{1+qq} + n(1+qq))dp + ndq(1+pp)}{q(1+pp)^{\frac{3}{2}}\sqrt{1+qq}}$$

or into

$$\frac{q+n\sqrt{1+qq}}{q\sqrt{1+pp}}\left(\frac{dp}{1+pp} + \frac{ndq}{q\sqrt{1+qq}+n(1+qq)}\right);$$

therefore, it is clear that for the other multiplier one has to take

$$N = \frac{q\sqrt{1+pp}}{q+n\sqrt{1+qq}},$$

and then

$$du = \frac{dp}{1+pp} + \frac{ndq}{q\sqrt{1+qq}+n(1+qq)},$$

which formula is integrable and from this it will be

$$u = \int \frac{dp}{1+pp} + \int \frac{ndq}{q\sqrt{1+qq}+n(1+qq)} = b,$$

from which either q in terms of p or p in terms of q will be determined; but then, since

$$z = \frac{p+q}{1-pq} = \frac{y}{x},$$

hence

$$y = \frac{(p+q)x}{1-pq},$$

which value, if substituted in the first equation, yields

$$\frac{-qx\sqrt{1+pp}}{1-pq} - nx\frac{\sqrt{(1+pp)(1+qq)}}{1-pq} = a;$$

therefore, one can now find x, whence

$$x = \frac{-a(1-pq)}{(q+n\sqrt{1+qq})\sqrt{1+pp}},$$

and since p is given in terms of q or q in terms of p, x will be defined in the same way; further, it will be

$$y = \frac{-a(p+q)}{(q+n\sqrt{1+qq})\sqrt{1+pp}},$$

which is the complete solution of the problem.