Analytical Exercises *

Leonhard Euler

§1 The following relation, I found some time ago, among the sums of these divergent series

$$1-2^m+3^m-4^m+5^m-\text{etc.}$$

and these convergent series

$$1+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\frac{1}{9^n}+\text{etc.}$$

is very remarkable and is expressed as follows

$$\begin{aligned}
1-2^0+3^0-4^0+\text{etc.} &= \frac{1}{2},\\
1-2^1+3^1-4^1+\text{etc.} &= \frac{1}{4} = +\frac{2\cdot 1}{\pi^2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\text{etc.}\right),\\
1-2^2+3^2-4^2+\text{etc.} &= \frac{0}{8},\\
1-2^3+3^3-4^3+\text{etc.} &= -\frac{2}{16} = -\frac{2\cdot 1\cdot 2\cdot 3}{\pi^4}\left(1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\text{etc.}\right),\\
1-2^4+3^4-4^4+\text{etc.} &= \frac{0}{32},\\
1-2^5+3^5-4^5+\text{etc.} &= \frac{16}{64} = +\frac{2\cdot 1\cdot 2\cdot 3\cdot 4\cdot 5}{\pi^6}\left(1+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\text{etc.}\right),
\end{aligned}$$

*Original title: "Exercitationes analyticae", first published in „*Novi commentarii academiae scientiarum Petropolitanae 17*, 1773, pp. 131-167", reprinted in „*Opera Omnia*: Series 1, Volume 15, pp. 131 - 167 ", Eneström number E432, translated by: Alexander Aycock"

$$
\begin{aligned}
&1-2^6+3^6-4^6+\text{etc.}= \frac{0}{128},\\
&1-2^7+3^7-4^7+\text{etc.}=-\frac{272}{256}=-\frac{2\cdot 1\cdot 2\cdot 3\cdots 7}{\pi^8}\left(1+\frac{1}{3^8}+\frac{1}{5^8}+\frac{1}{7^8}+\text{etc.}\right),\\
&1-2^8+3^8-4^8+\text{etc.}= \frac{0}{512},\\
&1-2^9+3^9-4^9+\text{etc.}= \frac{7936}{1024}=+\frac{2\cdot 1\cdot 2\cdot 3\cdots 9}{\pi^{10}}\left(1+\frac{1}{3^{10}}+\frac{1}{5^{10}}+\frac{1}{7^{10}}+\text{etc.}\right),
\end{aligned}
$$

where π denotes the circumference of the circle whose diameter is $=1$.

§2 From this it is possible to conclude that in general there is a relation among these infinite series

$$1-2^{n-1}+3^{n-1}-4^{n-1}+\text{etc.}$$

and these ones

$$1+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\text{etc.}$$

of such a kind that it is

$$1-2^{n-1}+3^{n-1}-4^{n-1}+\text{etc.}=\frac{2\cdot 1\cdot 2\cdot 3\cdots (n-1)}{\pi^n}N\left(1+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\text{etc.}\right),$$

where we indeed know, as often as n is an odd number, except for the case $n=1$ that N will be $=0$, but as often as n is an even number, N is either $+1$ or -1. Of course, if n is an even number of the form $4m+2$, N will be $+1$, but if n is an even number of the form $4m$, N will be -1. Hence it is easy to conclude, a function of what kind N is of n, because, if

$$n= \quad 2,\ 3, \quad 4,\ 5 \quad 6,\ 7 \quad 8,\ 9 \quad 10,\ 11 \quad 12,\ 13 \quad \text{etc.},$$

it is

$$N= \quad +1,\ 0, \quad -1,\ 0 \quad +1,\ 0 \quad -1,\ 0 \quad +1, \quad 0 \quad -1, \quad 0 \quad \text{etc.}$$

§3 And, if we consider this with more attention, we will understand that even case $n = 1$ does not violate this rule, according to which N has to become $= 0$ in this case; for, there is no obstruction that we allow this equality

$$1 - 2^0 + 3^0 - 4^0 + \text{etc.} = \frac{2}{\pi} \cdot 0 \cdot \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \text{etc.}\right),$$

since the sum of the series

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \text{etc.}$$

is infinite, whence it can certainly be

$$\frac{2}{\pi} \cdot 0 \cdot \infty = \frac{1}{2}$$

or equal to the sum of the series

$$1 - 1 + 1 - 1 + \text{etc.}$$

Therefore, without any exception, if it was

$$n = 1, \quad 2,\ 3, \quad 4,\ 5 \quad 6,\ 7 \quad 8,\ 9 \quad \text{etc.},$$

it will be

$$N = 0, \quad +1,\ 0, \quad -1,\ 0 \quad +1,\ 0 \quad -1,\ 0 \quad \text{etc.}$$

which rule is certainly correct for innumerable functions N of n. But it is very probable that the simplest and most natural choice is the right one here, which is

$$N = \cos\frac{n-2}{2}\pi,$$

while π denotes the angle equal to two right ones here, because the whole sine is assumed to be $= 1$, so that π is the half of the circumference of the circle.

§4 After having admitted this conjecture, we will therefore have in general

$$1-2^{n-1}+3^{n-1}-4^{n-1}+\text{etc.}$$
$$=2\cos\frac{n-2}{2}\pi\cdot\frac{1\cdot 2\cdot 3\cdots(n-1)}{\pi^n}\left(1+\frac{1}{3^n}+\frac{1}{5^n}+\text{etc.}\right)$$

or by manipulating this equation

$$1+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\text{etc.}$$
$$=\frac{1}{2\cos\frac{n-2}{2}\pi}\cdot\frac{\pi^n}{1\cdot 2\cdot 3\cdots(n-1)}(1-2^{n-1}+3^{n-1}-4^{n-1}+\text{etc.})$$

and from the preceding it is obvious that this equality indeed holds, as often as n was an even number, and does not deviate from the truth in the cases, in which n is an odd number. Hence if it is true for the cases, in which n is a fractional number, the values of the formula $1\cdot 2\cdot 3\cdots(n-1)$ have to be assigned by interpolation, which can indeed by done explicitly for fractions whose was denominator two; for, we find the following relations: For

$$\frac{1}{2},\quad \frac{3}{2},\quad \frac{5}{2},\quad \frac{7}{2},\quad \frac{9}{2}\quad \text{etc.}$$

the interpolated values are

$$\frac{1}{2}\sqrt{\pi},\quad \frac{1\cdot 3}{2\cdot 2}\sqrt{\pi},\quad \frac{1\cdot 3\cdot 5}{2\cdot 2\cdot 2}\sqrt{\pi},\quad \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 2\cdot 2\cdot 2}\sqrt{\pi},\quad \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 2\cdot 2\cdot 2\cdot 2}\sqrt{\pi}\quad \text{etc.}$$

and for $\cos\frac{(n-2)\pi}{2}=$

$$\frac{1}{\sqrt{2}},\quad \frac{1}{\sqrt{2}},\quad -\frac{1}{\sqrt{2}},\quad -\frac{1}{\sqrt{2}},\quad +\frac{1}{\sqrt{2}}\quad \text{etc.}$$

§5 Therefore, we will have for these cases

$$1+\frac{1}{3\sqrt{3}}+\frac{1}{5\sqrt{5}}+\frac{1}{7\sqrt{7}}+\text{etc.}$$
$$=+\frac{\sqrt{2}}{1}\pi(1-\sqrt{2}+\sqrt{3}-\sqrt{4}+\sqrt{5}+\text{etc.})$$

$$1+\frac{1}{3^2\sqrt{3}}+\frac{1}{5^2\sqrt{5}}+\frac{1}{7^2\sqrt{7}}+\text{etc.}$$
$$=+\frac{2\sqrt{2}}{1\cdot 3}\pi^2(1-2\sqrt{2}+3\sqrt{3}-4\sqrt{4}+5\sqrt{5}+\text{etc.})$$
$$1+\frac{1}{3^3\sqrt{3}}+\frac{1}{5^3\sqrt{5}}+\frac{1}{7^3\sqrt{7}}+\text{etc.}$$
$$=-\frac{4\sqrt{2}}{1\cdot 3\cdot 5}\pi^3(1-2^2\sqrt{2}+3^2\sqrt{3}-4^2\sqrt{4}+5^2\sqrt{5}+\text{etc.})$$
$$1+\frac{1}{3^4\sqrt{3}}+\frac{1}{5^4\sqrt{5}}+\frac{1}{7^4\sqrt{7}}+\text{etc.}$$
$$=-\frac{8\sqrt{2}}{1\cdot 3\cdot 5\cdot 7}\pi^4(1-2^3\sqrt{2}+3^3\sqrt{3}-4^3\sqrt{4}+5^3\sqrt{5}+\text{etc.})$$
$$1+\frac{1}{3^5\sqrt{3}}+\frac{1}{5^5\sqrt{5}}+\frac{1}{7^5\sqrt{7}}+\text{etc.}$$
$$=+\frac{16\sqrt{2}}{1\cdot 3\cdot 5\cdot 7\cdot 9}\pi^5(1-2^4\sqrt{2}+3^4\sqrt{3}-4^4\sqrt{4}+5^4\sqrt{5}+\text{etc.});$$

But I do not dare to confirm with absolute certainty that these equations are absolutely true. Therefore, it is convenient to investigate, whether these equations are true at least approximately; and for the first we indeed calculate

$$1-\sqrt{2}+\sqrt{3}-\sqrt{4}+\sqrt{5}-\text{etc.}=0.380317$$

approximately, which number multiplied by $\pi\sqrt{2}$ gives 1.689655; and the sum of the series

$$1+\frac{1}{3\sqrt{3}}+\frac{1}{5\sqrt{5}}+\frac{1}{7\sqrt{7}}+\text{etc.}$$

is indeed detected to be approximately equal to that value.

§6 But because it seems that by taking odd numbers for n nothing can be concluded from this case, since the one side of our equation tends to $\frac{0}{0}$; in order to find even these values, let us take a number exceeding an integer infinitely less for n or let us write $n+\omega$ in instead of n, ω denoting an infinitely small fraction; and we will have

$$1+\frac{1}{3^{n+\omega}}+\frac{1}{5^{n+\omega}}+\frac{1}{7^{n+\omega}}+\text{etc.}$$

$$= \frac{1}{2\cos\frac{n-2+\omega}{2}\pi} \cdot \frac{\pi^{n+\omega}}{1\cdot 2\cdots(n-1+\omega)}(1 - 2^{n-1+\omega} + 3^{n-1+\omega} - 4^{n-1+\omega} + \text{etc.})$$

Therefore, I observe here at first that

$$\frac{1}{a^{n+\omega}} = a^{-n-\omega} = a^{-n}(1-\omega\ln a),$$

where the logarithms are to be understood to be natural or hyperbolic ones, so that

$$\frac{1}{a^{n+\omega}} = \frac{1}{a^n} - \frac{\omega\ln a}{a^n}.$$

In like manner it will be

$$a^{n-1+\omega} = a^{n-1} + a^{n-1}\omega\ln a$$

and

$$\pi^{n+\omega} = \pi^n(1+\omega\ln\pi);$$

then it indeed is

$$\cos\frac{n-2+\omega}{2}\pi = \cos\frac{n-2}{2}\pi - \frac{1}{2}\omega\pi\sin\frac{n-2}{2}\pi.$$

Finally, since I once proved that the value of the formula $1\cdot 2\cdots(n-1+\omega)$ is $= 1 - 0.57721566\omega$ in the case $n = 1$, if, for the sake of brevity, we write

$$\lambda = 0.5572156649015328$$

by taking

$n =$	1,	2,	3,	4,	5	etc.
it is						
$1\cdot 2\cdots(n-1+\omega) =$	$1-\lambda\omega$,	$1+(1-\lambda)\omega$,	$2+(3-2\lambda)\omega$,	$6+(11-6\lambda)\omega$,	$24+(50-24\lambda)\omega$	etc.

§7 Let us hence mainly consider the case $n = 3$, because this series

$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \text{etc.}$$

is of such a nature that all the efforts to investigate its sum up to now were without success. But because it is

$$\cos\frac{n-2}{2}\pi = 0 \quad \text{and} \quad \sin\frac{n-2}{2}\pi = 1,$$

our equation will take this form

$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \text{etc.} - \omega\left(\ln 1 + \frac{\ln 3}{3^3} + \frac{\ln 5}{5^3} + \frac{\ln 7}{7^3} + \text{etc.}\right)$$
$$= \frac{-1}{\pi\omega} \cdot \frac{\pi^3(1+\omega\ln\pi)}{2+(3-2\lambda)\omega}\left(1 - 2^2 + 3^2 - 4^2 + \text{etc.} - \omega(2^2\ln 2 - 3^2\ln 3 + 4^2\ln 4 - \text{etc.})\right).$$

But since

$$1 - 2^2 + 3^2 - 4^2 + \text{etc.} = 0,$$

we will have for $\omega = 0$

$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \text{etc.} = \frac{1}{2}\pi^2(2^2\ln 2 - 3^2\ln 3 + 4^2\ln 4 - 5^2\ln 5 + \text{etc.})$$

and so we would achieve the aim, if it would be possible to assign the sum of this logarithmic series

$$2^2\ln 2 - 3^2\ln 3 + 4^2\ln 4 - 5^2\ln 5 + \text{etc.}$$

But in like manner one finds for the remaining powers

$$1 + \frac{1}{3^5} + \frac{1}{5^5} + \frac{1}{7^5} + \text{etc.} = \frac{-\pi^4}{1\cdot 2\cdot 3\cdot 4}(2^4\ln 2 - 3^4\ln 3 + 4^4\ln 4 - 5^4\ln 5 + \text{etc.}),$$
$$1 + \frac{1}{3^7} + \frac{1}{5^7} + \frac{1}{7^7} + \text{etc.} = \frac{+\pi^6}{1\cdot 2\cdots 6}(2^6\ln 2 - 3^6\ln 3 + 4^6\ln 4 - 5^6\ln 5 + \text{etc.}),$$
$$1 + \frac{1}{3^9} + \frac{1}{5^9} + \frac{1}{7^9} + \text{etc.} = \frac{-\pi^8}{1\cdot 2\cdots 8}(2^8\ln 2 - 3^8\ln 3 + 4^8\ln 4 - 5^8\ln 5 + \text{etc.})$$
$$\text{etc.}$$

§8 So let this infinite series be propounded

$$2^2\ln 2 - 3^2\ln 3 + 4^2\ln 4 - 5^2\ln 5 + 6^2\ln 6 - 7^2\ln 7 + \text{etc.} = Z,$$

that it is

$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \text{etc.} = \frac{1}{2}\pi\pi Z,$$

and to be less intimidated by this equation note that it is

$$\ln 2 - \ln 3 + \ln 4 - \ln 5 + \ln 6 - \text{etc.} = \frac{1}{2}\ln\frac{\pi}{2}.$$

But that series Z can be transformed into several others, as for example

$$Z = \ln 2 - 3\ln\frac{3}{2} + 6\ln\frac{4}{3} - 10\ln\frac{5}{4} + 15\ln\frac{6}{5} - 21\ln\frac{7}{6} + \text{etc.}$$

and

$$\begin{aligned}Z = \ln\frac{2\cdot 2}{1\cdot 3} + 4\ln\frac{4\cdot 4}{3\cdot 5} + 9\ln\frac{6\cdot 6}{7\cdot 9} + 16\ln\frac{8\cdot 8}{7\cdot 9} + 25\ln\frac{10\cdot 10}{9\cdot 11} + \text{etc.}\\ -2\ln\frac{3\cdot 3}{2\cdot 4} - 6\ln\frac{5\cdot 5}{4\cdot 6} - 12\ln\frac{7\cdot 7}{6\cdot 8} - 20\ln\frac{9\cdot 9}{8\cdot 10} - \text{etc.}\end{aligned}$$

Since if we put in general

$$Z = \alpha\ln\frac{2\cdot 2}{1\cdot 3} - \beta\ln\frac{3\cdot 3}{2\cdot 4} + \gamma\ln\frac{4\cdot 4}{3\cdot 5} - \delta\ln\frac{5\cdot 5}{4\cdot 6} + \varepsilon\ln\frac{6\cdot 6}{5\cdot 7} - \zeta\ln\frac{7\cdot 7}{6\cdot 8} + \text{etc.},$$

it has to be

$$\begin{array}{rcl}
+2\alpha + \beta = 4 & \text{and hence} & \beta = 4 - 2\alpha,\\
\alpha + 2\beta + \gamma = 9 & & \gamma = 1 + 3\alpha,\\
\beta + 2\gamma + \delta = 16 & & \delta = 10 - 4\alpha,\\
\gamma + 2\delta + \varepsilon = 25 & & \varepsilon = 4 + 5\alpha,\\
\delta + 2\varepsilon + \zeta = 36 & & \zeta = 18 - 6\alpha,\\
\varepsilon + 2\zeta + \eta = 49 & & \eta = 9 + 7\alpha,\\
\zeta + 2\eta + \theta = 64 & & \theta = 28 - 8\alpha,\\
\eta + 2\theta + \iota = 81 & & \iota = 16 + 9\alpha,\\
\text{etc.} & & \text{etc.}
\end{array}$$

Here we indeed took $\alpha = 1$ that the progression becomes as regular as possible.

§9 This last formula seems to be the most appropriate for our purpose, because the logarithms are resolved into convergent series. For this aim I will use this resolution for the positive terms: Because any of them is contained in this form

$$xx \ln \frac{4xx}{4xx-1} = -xx \ln \left(1 - \frac{1}{4xx}\right),$$

hence this infinite series results

$$xx\left(\frac{1}{4xx} + \frac{1}{2\cdot 2^4x^4} + \frac{1}{3\cdot 2^6x^6} + \frac{1}{4\cdot 2^8x^8} + \text{etc.}\right)$$

or this one

$$\frac{1}{2^2} + \frac{1}{2\cdot 2^4}\cdot\frac{1}{xx} + \frac{1}{3\cdot 2^6}\cdot\frac{1}{x^4} + \frac{1}{4\cdot 2^8}\cdot\frac{1}{x^6} + \frac{1}{5\cdot 2^{10}}\cdot\frac{1}{x^8} + \text{etc.}$$

But for the negative terms the general form is

$$-x(x+1)\ln\frac{(2x+1)^2}{4x(x+1)} = -x(x+1)\ln\left(1+\frac{1}{4x(x+1)}\right),$$

which is resolved into this series

$$-\frac{1}{2^2} + \frac{1}{2\cdot 2^4}\cdot\frac{1}{x(x+1)} - \frac{1}{3\cdot 2^6}\cdot\frac{1}{x^2(x+1)^2} + \frac{1}{4\cdot 2^8}\frac{1}{x^3(x+1)^3} - \text{etc.},$$

whence the value of Z is transformed into these series

$$\begin{aligned} Z = \frac{1}{2^2}&\left(1 - 1 + 1 - 1 + \text{etc.}\right) + \frac{1}{2\cdot 2^4}\left(1 + \frac{1}{1\cdot 2} + \frac{1}{2^2} + \frac{1}{2\cdot 3} + \frac{1}{3^2} + \text{etc.}\right) \\ + \frac{1}{3\cdot 2^6}&\left(1 - \frac{1}{1^2\cdot 2^2} + \frac{1}{2^4} - \frac{1}{2^2\cdot 3^2} + \text{etc.}\right) + \frac{1}{4\cdot 2^8}\left(1 + \frac{1}{1^3\cdot 2^3} + \frac{1}{2^6} + \frac{1}{2^3\cdot 3^3} + \frac{1}{3^6} + \text{etc.}\right) \\ + \frac{1}{5\cdot 2^{10}}&\left(1 - \frac{1}{1^4\cdot 2^4} + \frac{1}{2^8} - \frac{1}{2^4\cdot 3^4} + \text{etc.}\right) + \frac{1}{6\cdot 2^{12}}\left(1 + \frac{1}{1^5\cdot 2^5} + \frac{1}{2^{10}} + \frac{1}{2^5\cdot 3^5} + \frac{1}{3^{10}} + \text{etc.}\right) \\ &\text{etc.} \end{aligned}$$

§10 Hence, if, for the sake of brevity, we set

$$\begin{aligned}
1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\text{etc.} &= \alpha\pi^2,\\
1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\text{etc.} &= \beta\pi^4,\\
1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\text{etc.} &= \gamma\pi^6,\\
1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\text{etc.} &= \delta\pi^8\\
&\text{etc.},
\end{aligned}$$

where the numbers α, β, γ, δ etc. are known, and because it is

$$1-1+1-1+\text{etc.} = \frac{1}{2},$$

it will be

$$\begin{aligned}
Z = \frac{1}{2^2}\cdot\frac{1}{2} \qquad\qquad\qquad\qquad\qquad\qquad & + \frac{1}{2\cdot 2^4}\left(\alpha\pi^2+\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\text{etc.}\right)\\
+\frac{1}{3\cdot 2^6}\left(\beta\pi^4-\frac{1}{2^2}-\frac{1}{6^2}-\frac{1}{12^2}-\text{etc.}\right) & + \frac{1}{4\cdot 2^8}\left(\gamma\pi^4+\frac{1}{2^3}+\frac{1}{6^3}+\frac{1}{12^3}+\text{etc.}\right)\\
+\frac{1}{3\cdot 2^{10}}\left(\delta\pi^8-\frac{1}{2^4}-\frac{1}{6^4}-\frac{1}{12^4}-\text{etc.}\right) & + \frac{1}{6\cdot 2^{12}}\left(\varepsilon\pi^{10}+\frac{1}{2^5}+\frac{1}{6^5}+\frac{1}{12^5}+\text{etc.}\right)\\
&\text{etc.},
\end{aligned}$$

where the whole task is now already reduced to the summation of these series

$$\frac{1}{2^n}+\frac{1}{6^n}+\frac{1}{12^n}+\frac{1}{20^n}+\text{etc.};$$

the numbers 2, 6, 12, 20 etc. are the pronic numbers[1].

§11 But the single terms of this series, whose form is $\frac{1}{x^n(x+1)^n}$, can be resolved into parts consisting of the simple powers; these parts are the following:

[1] By this Euler means numbers which are the product of two consecutive integers.

$$
\begin{aligned}
\frac{1}{x(x+1)} &= \frac{1}{x} - \frac{1}{x+1},\\
\frac{1}{x^2(x+1)^2} &= \frac{1}{x^2} + \frac{1}{(x+1)^2} - 2\left(\frac{1}{x} - \frac{1}{x+1}\right),\\
\frac{1}{x^3(x+1)^3} &= \frac{1}{x^2} - \frac{1}{(x+1)^3} - 3\left(\frac{1}{x^2} - \frac{1}{(x+1)^2}\right) + \frac{3\cdot 4}{1\cdot 2}\left(\frac{1}{x} - \frac{1}{x+1}\right),\\
\frac{1}{x^4(x+1)^4} &= \frac{1}{x^4} + \frac{1}{(x+1)^4} - 4\left(\frac{1}{x^3} - \frac{1}{(x+1)^3}\right) + \frac{4\cdot 5}{1\cdot 2}\left(\frac{1}{x^2} - \frac{1}{(x+1)^2}\right),\\
&\qquad - \frac{4\cdot 5\cdot 6}{1\cdot 2\cdot 3}\left(\frac{1}{x} - \frac{1}{x+1}\right)
\end{aligned}
$$

etc.

Because, indicating the sums with the prefixed sign $\int$, it is

$$\int \frac{1}{(x+1)^n} = \int \frac{1}{x^n} - 1,$$

it will be

$$
\begin{aligned}
\int \frac{1}{x(x+1)} &= 1,\\
\int \frac{1}{x^2(1+x)^2} &= 2\int \frac{1}{x^2} - 1 - 2,\\
\int \frac{1}{x^3(1+x)^3} &= 1 - 3\left(\int \frac{1}{x^2} - 1\right) + \frac{3\cdot 4}{1\cdot 2},\\
\int \frac{1}{x^4(1+x)^4} &= 2\int \frac{1}{x^4} - 1 - 4 + \frac{4\cdot 5}{1\cdot 2}\left(2\int \frac{1}{x^2} - 1\right) - \frac{4\cdot 5\cdot 6}{1\cdot 2\cdot 3}
\end{aligned}
$$

etc.

§12 In these single expressions it is possible to collect the absolute terms into one in a convenient way, and furthermore, because it is

$$\int \frac{1}{x^2} = \alpha\pi^2, \quad \int \frac{1}{x^4} = \beta\pi^4, \quad \int \frac{1}{x^6} = \gamma\pi^6, \quad \int \frac{1}{x^8} = \delta\pi^8 \quad \text{etc.},$$

we will have

$$\int \frac{1}{x(x+1)} = 1,$$
$$\int \frac{1}{x^2(x+1)^2} = 2\alpha\pi^2 - \frac{2\cdot 3}{1\cdot 2},$$
$$\int \frac{1}{x^3(1+x)^3} = -3\cdot 2\alpha\pi^2 + \frac{3\cdot 4\cdot 5}{1\cdot 2\cdot 3},$$
$$\int \frac{1}{x^4(1+x)^4} = 2\beta\pi^4 + \frac{4\cdot 5}{1\cdot 2}2\alpha\pi^2 - \frac{4\cdot 5\cdot 6\cdot 7}{1\cdot 2\cdot 3\cdot 4},$$
$$\int \frac{1}{x^4(1+x)^4} = -5\cdot 2\beta\pi^4 - \frac{5\cdot 6\cdot 7}{1\cdot 2\cdot 3}2\alpha\pi^2 + \frac{5\cdot 6\cdot 7\cdot 8\cdot 9}{1\cdot 2\cdot 3\cdot 4\cdot 5},$$
$$\int \frac{1}{x^6(1+x)^6} = 2\gamma\pi^6 + \frac{6\cdot 7}{1\cdot 2}2\beta\pi^4 + \frac{6\cdot 7\cdot 8\cdot 9}{1\cdot 2\cdot 3\cdot 4}2\alpha\pi^2 - \frac{6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6},$$
etc.,

where this remarkable reduction has to be observed

$$1 + \frac{n}{1} + \frac{n(n+1)}{1\cdot 2} + \cdots + \frac{n(n+1)\cdots(2n-2)}{1\cdot 2\cdots(n-1)} = \frac{n(n+1)(n+2)\cdots(2n-1)}{1\cdot 2\cdot 3\cdots n};$$

for, from a known rule this is

$$= \frac{(n+1)(n+2)(n+3)\cdots(2n-1)}{1\cdot 2\cdot 3\cdots(n-1)}.$$

§13 After having substituted these values we will obtain

$$Z = \frac{1}{2^2}\cdot\frac{1}{2} + \frac{1}{2\cdot 2^4}(\alpha\pi^2+1) + \frac{1}{3\cdot 2^6}\left(\beta\pi^4 + \frac{2\cdot 3}{1\cdot 2} - 2\alpha\pi^2\right)$$
$$+\frac{1}{4\cdot 2^8}\left(\gamma\pi^6 + \frac{3\cdot 4\cdot 5}{1\cdot 2\cdot 3} - \frac{3}{1}\,2\alpha\pi^2\right) + \frac{1}{5\cdot 2^{10}}\left(\delta\pi^8 + \frac{4\cdot 5\cdot 6\cdot 7}{1\cdot 2\cdot 3\cdot 4} - \frac{4\cdot 5}{1\cdot 2}\,2\alpha\pi^2 - 2\beta\pi^4\right)$$
$$+\frac{1}{6\cdot 2^{12}}\left(\varepsilon\pi^{10} + \frac{5\cdot 6\cdot 7\cdot 8\cdot 9}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{5\cdot 6\cdot 7}{1\cdot 2\cdot 3}\,2\alpha\pi^2 - \frac{5}{1}\,2\beta\pi^4\right)$$
$$+\frac{1}{7\cdot 2^{14}}\left(\zeta\pi^{12} + \frac{6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} - \frac{6\cdot 7\cdot 8\cdot 9}{1\cdot 2\cdot 3\cdot 4}\,2\alpha\pi^2 - \frac{6\cdot 7}{1\cdot 2}\,2\beta\pi^4 - 2\gamma\pi^6\right) + \text{etc.},$$

which expression is resolved into these series

$$Z = \frac{1}{2^2\cdot 2} + \frac{1}{2\cdot 2^4} + \frac{2\cdot 3}{2\cdot 3\cdot 2^6} + \frac{3\cdot 4\cdot 5}{2\cdot 3\cdot 4\cdot 2^8} + \frac{4\cdot 5\cdot 6\cdot 7}{2\cdot 3\cdot 4\cdot 5\cdot 2^{10}} + \frac{5\cdot 6\cdot 7\cdot 8\cdot 9}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 2^{12}} + \text{etc.}$$

$$+\frac{\alpha\pi^2}{2\cdot 2^4} + \frac{\beta\pi^4}{3\cdot 2^6} + \frac{\gamma\pi^6}{4\cdot 2^8} + \frac{\delta\pi^8}{5\cdot 2^{10}} + \frac{\varepsilon\pi^{10}}{6\cdot 2^{12}} + \text{etc.}$$

$$-2\alpha\pi^2\left(\frac{1}{3\cdot 2^6} + \frac{3}{1\cdot 4\cdot 2^8} + \frac{4\cdot 5}{1\cdot 2\cdot 5\cdot 2^{10}} + \frac{5\cdot 6\cdot 7}{1\cdot 2\cdot 3\cdot 6\cdot 2^{12}} + \frac{6\cdot 7\cdot 8\cdot 9}{1\cdot 2\cdot 3\cdot 4\cdot 7\cdot 2^{14}} + \text{etc.}\right)$$
$$-2\beta\pi^4\left(\frac{1}{5\cdot 2^{10}} + \frac{5}{1\cdot 6\cdot 2^{12}} + \frac{6\cdot 7}{1\cdot 2\cdot 7\cdot 2^{14}} + \frac{7\cdot 8\cdot 9}{1\cdot 2\cdot 3\cdot 8\cdot 2^{16}} + \frac{8\cdot 9\cdot 10\cdot 11}{1\cdot 2\cdot 3\cdot 4\cdot 9\cdot 2^{18}} + \text{etc.}\right)$$
$$-2\gamma\pi^6\left(\frac{1}{7\cdot 2^{14}} + \frac{7}{1\cdot 8\cdot 2^{16}} + \frac{8\cdot 9}{1\cdot 2\cdot 9\cdot 2^{18}} + \frac{9\cdot 10\cdot 11}{1\cdot 2\cdot 3\cdot 10\cdot 2^{20}} + \frac{10\cdot 11\cdot 12\cdot 13}{1\cdot 2\cdot 3\cdot 4\cdot 11\cdot 2^{22}} + \text{etc.}\right)$$
$$\text{etc.}$$

§14 Hence we are led to this general infinite series comprising all those numerical series:

$$\frac{1}{n\cdot 2^{2n}} + \frac{n}{(n+1)2^{2n+2}} + \frac{(n+1)(n+2)}{2(n+2)2^{2n+4}} + \frac{(n+2)(n+3)(n+4)}{2\cdot 3(n+3)2^{2n+6}}$$
$$+\frac{(n+3)(n+4)(n+5)(n+6)}{2\cdot 3\cdot 4(n+4)2^{2n+8}} + \text{etc.},$$

whose sum is therefore to be investigated. So, if we indicate the sum of this series in general with this sign $S(n)$, we will have

$$Z = -\frac{1}{8} + S(1) + 2\alpha\pi^2\left(\frac{1}{4\cdot 2^4} - S(3)\right) + \beta\pi^4\left(\frac{1}{6\cdot 2^6} - S(5)\right)$$
$$+\gamma\pi^6\left(\frac{1}{8\cdot 2^8} - S(7)\right) + \delta\pi^8\left(\frac{1}{10\cdot 2^{10}} - S(9)\right) + \text{etc.}$$

But our general series can be exhibited in a more convenient form as follows:

$$n(n+1)S(n) = \frac{n+1}{2^{2n}} + \frac{nn}{2^{2n+2}} + \frac{n(n+1)(n+1)}{2\cdot 2^{2n+4}} + \frac{n(n+1)(n+2)(n+4)}{2\cdot 3\cdot 2^{2n+6}}$$
$$\frac{n(n+1)(n+3)(n+5)(n+6)}{2\cdot 3\cdot\cdot 2^{2n+8}} + \frac{n(n+1)(n+4)(n+6)(n+7)(n+8)}{2\cdot 3\cdot 4\cdot 5\cdot 2^{2n+10}} + \text{etc.},$$

where the denominators do not involve the number n. The single terms can also be represented by products in such a way that it is

$$S(n) = A + AB + ABC + ABCD + ABCDE + \text{etc.},$$

and the letters will be

$$A = \frac{1}{n \cdot 2^{2n}}, \quad B = \frac{nn}{4(n+1)}, \quad C = \frac{(n+1)(n+1)}{4 \cdot 2n}, \quad D = \frac{(n+2)(n+4)}{4 \cdot 3(n+1)},$$
$$E = \frac{(n+3)(n+5)(n+6)}{4 \cdot 4(n+2)(n+4)}, \quad F = \frac{(n+4)((n+7)(n+8)}{4 \cdot 5(n+3)(n+5)} \quad \text{etc.},$$

where one factor in general has this form

$$\frac{(n+\lambda-1)(n+2\lambda-3)(n+2\lambda-2)}{4\lambda(n+\lambda-2)(n+\lambda)}.$$

§15 Let us start from the simplest case $n = 1$, and because the factor in general is

$$= \frac{\lambda(2\lambda-2)(2\lambda-1)}{4\lambda(\lambda-1)(\lambda+1)} = \frac{2\lambda-1}{2\lambda+2},$$

it will be

$$A = \frac{1}{4}, \quad B = \frac{1}{8}, \quad C = \frac{3}{6}, \quad D = \frac{5}{8}, \quad E = \frac{7}{10}, \quad F = \frac{9}{12} \quad \text{etc.}$$

whence it is

$$S(1) = \frac{1}{4} + \frac{1}{4 \cdot 8}\left(1 + \frac{3}{6} + \frac{3 \cdot 5}{6 \cdot 8} + \frac{3 \cdot 5 \cdot 7}{6 \cdot 8 \cdot 10} + \frac{3 \cdot 5 \cdot 7 \cdot 9}{6 \cdot 8 \cdot 10 \cdot 12} + \text{etc.}\right).$$

But because it is

$$\sqrt{1-1} = 1 - \frac{1}{2} - \frac{1 \cdot 1}{2 \cdot 4} - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} - \frac{1 \cdot 1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} - \text{etc.} = 0,$$

it will be

$$1 + \frac{3}{6} + \frac{3 \cdot 5}{6 \cdot 8} + \frac{3 \cdot 5 \cdot 7}{6 \cdot 8 \cdot 10} + \text{etc.} = \frac{2 \cdot 4}{1 \cdot 1}\left(1 - \frac{1}{2}\right) = 4$$

and hence

$$S(1) = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$$

and

$$-\frac{1}{8} + S(1) = \frac{1}{4}.$$

§16 To to be able to determine the sums of the remaining series more easily, let us write x for of $\frac{1}{2^2}$ that $x = \frac{1}{4}$, and because we have

$$S(n) = \frac{1}{n}x^n + \frac{n}{n+1}x^{n+1} + \frac{(n+1)(n+2)}{2(n+2)}x^{n+2} + \frac{(n+2)(n+3)(n+4)}{2\cdot 3(n+3)}x^{n+3} + \text{etc.},$$

which in the case $n = 1$ becomes

$$S(1) = x + \frac{1}{2}xx + \frac{2\cdot 3}{2\cdot 3}x^3 + \frac{3\cdot 4\cdot 5}{2\cdot 3\cdot 4}x^4 + \frac{4\cdot 5\cdot 6\cdot 7}{2\cdot 3\cdot 4\cdot 5}x^5 + \text{etc.}$$

or

$$S(1) = x + \frac{1}{2}xx + x^3 + \frac{5}{2}x^4 + \frac{6\cdot 7}{2\cdot 3}x^5 + \frac{7\cdot 8\cdot 9}{2\cdot 3\cdot 4}x^6 + \frac{8\cdot 9\cdot 10\cdot 11}{2\cdot 3\cdot 4\cdot 5}x^7 + \text{etc.}$$

or

$$S(1) = x + \frac{1}{2}xx\left(1 + \frac{2}{1}x + \frac{2\cdot 5}{1\cdot 2}xx + \frac{2\cdot 5\cdot 14}{1\cdot 2\cdot 5}x^3 + \frac{2\cdot 5\cdot 14\cdot 9}{1\cdot 2\cdot 3\cdot 5}x^4 + \text{etc.}\right)$$

or

$$S(1) = x + \frac{1}{2}xx\left(1 + \frac{3}{6}4x + \frac{3\cdot 5}{6\cdot 8}4^2xx + \frac{3\cdot 5\cdot 7}{6\cdot 8\cdot 10}4^3x^3 + \frac{3\cdot 5\cdot 7\cdot 9}{6\cdot 8\cdot 10\cdot 12}4^4x^4 + \text{etc.}\right);$$

but it is

$$\sqrt{1-4x} = 1 - \frac{1}{2}\,4x - \frac{1\cdot 1}{2\cdot 4}\,4^2x^2 - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\,4^3x^3 - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\,4^4x^4 - \text{etc.},$$

whence

$$\frac{1\cdot 1}{2\cdot 4}4^2x^2\left(1 + \frac{3}{6}\,4x + \frac{3\cdot 5}{6\cdot 8}4^2x^2 + \text{etc.}\right) = 1 - 2x - \sqrt{1-4x},$$

therefore

$$S(1) = x + \frac{1-2x-\sqrt{1-4x}}{4} = \frac{1+2x-\sqrt{1-4x}}{4},$$

and hence for $x = \frac{1}{4}$ it is

$$S(1) = \frac{1}{4}\left(1+\frac{1}{2}\right) = \frac{3}{8},$$

as above.

§17 Now let us put $n = 3$ and let $S(3)$ be $= Q$, while

$$S(1) = P = \frac{1+2x-\sqrt{1-4x}}{4},$$

so that it is

$$\begin{aligned} P &= x + \frac{1}{2}xx \quad + \frac{2\cdot 3}{2\cdot 3}x^3 + \frac{3\cdot 4\cdot 5}{2\cdot 3\cdot 4}x^4 + \frac{4\cdot 5\cdot 6\cdot 7}{2\cdot 3\cdot 4\cdot 5}x^5 + \text{etc.}, \\ Q &= \frac{1}{3}x^3 + \frac{3}{4}x^4 + \frac{4\cdot 5}{2\cdot 5}x^5 + \frac{5\cdot 6\cdot 7}{2\cdot 3\cdot 6}x^6 + \frac{6\cdot 7\cdot 8\cdot 9}{2\cdot 3\cdot 4\cdot 7}x^7 + \text{etc.} \end{aligned}$$

Hence one calculates

$$Pxx - Q = \frac{2}{3}x^3 - \frac{1}{4}x^4 - \frac{2\cdot 3\cdot 5}{2\cdot 3\cdot 5}x^5 - \frac{3\cdot 4\cdot 5\cdot 8}{2\cdot 3\cdot 4\cdot 6}x^6 - \frac{4\cdot 5\cdot 6\cdot 7\cdot 11}{2\cdot 3\cdot 4\cdot 5\cdot 7}x^7$$
$$-\frac{5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 14}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 8}x^8 - \text{etc.}$$

and hence by differentiating with respect to x

$$2Px + \frac{xxdP}{dx} - \frac{dQ}{dx} = 2xx - x^3 - \frac{2\cdot 3}{2\cdot 3}\,5x^4 - \frac{3\cdot 4\cdot 5}{2\cdot 3\cdot 4}\,8x^5 - \frac{4\cdot 5\cdot 6\cdot 7}{2\cdot 3\cdot 4\cdot 5}\,11x^6$$
$$-\frac{5\cdot 6\cdot 7\cdot 8\cdot 9}{2\cdot 3\cdot 4\cdot 5\cdot 6}x^8 - \text{etc.},$$

whose triple added to the first yields

$$2Px + \frac{4xxdP}{dx} - \frac{dQ}{dx} = 5xx + 2x^3 + 4\cdot\frac{2\cdot 3}{2\cdot 3}x^4 + 4\cdot\frac{3\cdot 4\cdot 5}{2\cdot 3\cdot 4}x^5 + 4\cdot\frac{4\cdot 5\cdot 6\cdot 7}{2\cdot 3\cdot 4\cdot 5}x^6 + \text{etc.},$$

and

$$4Px = 4xx + 2x^3 + 4\cdot\frac{2\cdot 3}{2\cdot 3}x^4 + 4\cdot\frac{3\cdot 4\cdot 5}{2\cdot 3\cdot 4}x^5 + 4\cdot\frac{4\cdot 5\cdot 6\cdot 7}{2\cdot 3\cdot 4\cdot 5}x^6 + \text{etc.},$$

hence

$$-2px + \frac{4xxdP}{dx} - \frac{dQ}{dx} = xx$$

and

$$dQ = 4xxdP - 2Pxdx - xxdx,$$

whence because of

$$dP = \frac{1}{2}dx + \frac{dx}{2\sqrt{1-4x}}$$

one calculates

$$dQ = -\frac{1}{2}xdx + \frac{1}{2}xdx\sqrt{1-4x} + \frac{2xxdx}{\sqrt{1-4x}} = -\frac{1}{2}xdx + \frac{xdx}{2\sqrt{1-4x}}$$

and by integrating

$$Q = -\frac{1}{4}xx - \frac{1+2x}{24}\sqrt{1-4x} + \frac{1}{24}.$$

Let x be $= \frac{1}{4}$, in which case it is $P = \frac{3}{8}$; Q will be $= \frac{5}{192}$ so that it is

$$S(1) = \frac{3}{8}, \quad S(3) = \frac{5}{192}$$

and

$$-\frac{1}{8} + S(1) = \frac{1}{4}, \quad \frac{1}{4\cdot 2^4} - S(3) = -\frac{1}{96}.$$

§18 Now let us in general put $S(n) = P$ and the following sum $S(n+2) = Q$; it will be

$$P = \frac{1}{n}x^n + \frac{n}{n+1}x^{n+1} + \frac{n+1}{2}x^{n+2} + \frac{(n+2)(n+4)}{2 \cdot 3}x^{n+3}$$
$$+ \frac{(n+3)(n+5)(n+6)}{2 \cdot 3 \cdot 4}x^{n+4} + \text{etc.},$$
$$Q = \frac{1}{n+2}x^{n+2} + \frac{n+2}{n+3}x^{n+3} + \frac{n+3}{2}x^{n+4} + \frac{(n+4)(n+6)}{2 \cdot 3}x^{n+5}$$
$$+ \frac{(n+5)(n+7)(n+8)}{2 \cdot 3 \cdot 4}x^{n+6},$$

whence one concludes that it will be

$$Q = Px - \frac{1}{2}(n+1)\int Pdx - \frac{1}{2}(n-1)xx\int \frac{Pdx}{xx}.$$

Hence one can define the value $S(n+2)$ from the value $S(n)$, except in the case $n = 1$, since then in $\int \frac{Pdx}{xx}$ $\int \frac{dx}{x}$ will appear.

But since the case $n = 3$ is already known

$$S(3) = \frac{1 - 6xx - (1+2x)\sqrt{1-4x}}{24},$$

if one takes this for P, it will be

$$S(5) = Px - 2\int Pdx - xx\int \frac{Pdx}{xx},$$

which, having expanded the integrals, gives

$$S(5) = \frac{1}{60}(1 - 15xx + 10x^3 - (1 + 2x - 9xx)\sqrt{1-4x}).$$

Hence for $x = \frac{1}{4}$

$$S(5) = \frac{7}{32 \cdot 60} = \frac{7}{1920}$$

and therefore

$$\frac{1}{6 \cdot 2^6} - S(5) = -\frac{1}{960}.$$

§19 Further, let n be $=5$ and

$$P = \frac{1}{60}(1 - 15xx + 10x^3 - (1 + 2x - 9xx)\sqrt{1-4x});$$

it will be

$$S(7) = Px - 3\int P dx - 2xx \int \frac{P dx}{xx};$$

after having solved these integrals one finally finds

$$S(7) = \frac{1}{112}(1 - 28xx + 56x^3 - 14x^4 - (1 + 2x - 22xx + 20x^3)\sqrt{1-4x}).$$

For $x = \frac{1}{4}$ this equation becomes

$$S(7) = \frac{1}{112}\left(1 - \frac{7}{4} + \frac{7}{8} - \frac{7}{128}\right) = \frac{9}{2^{11} \cdot 7}$$

and hence

$$\frac{1}{8 \cdot 2^8} - S(7) = -\frac{1}{2^{10} \cdot 7}.$$

So if we collect everything we found up to now, we can also easily predict the following values:

$$\begin{aligned}
S(1) - \frac{1}{2 \cdot 2^2} &= \frac{1}{4} &&= \frac{1}{1 \cdot 2 \cdot 2^1}, \\
S(3) - \frac{1}{4 \cdot 4^2} &= \frac{1}{96} &&= \frac{1}{3 \cdot 4 \cdot 2^3}, \\
S(5) - \frac{1}{6 \cdot 2^6} &= \frac{1}{960} &&= \frac{1}{5 \cdot 6 \cdot 2^5}, \\
S(7) - \frac{1}{8 \cdot 2^8} &= \frac{1}{2^{10} \cdot 7} &&= \frac{1}{7 \cdot 8 \cdot 2^7}.
\end{aligned}$$

§20 Hence we finally obtain the following value for Z:

$$Z = \frac{1}{4} - \frac{\alpha\pi^2}{3 \cdot 4 \cdot 2^2} - \frac{\beta\pi^4}{5 \cdot 6 \cdot 2^4} - \frac{\gamma\pi^6}{7 \cdot 8 \cdot 2^6} - \frac{\delta\pi^8}{9 \cdot 10 \cdot 2^8} - \frac{\varepsilon\pi^{10}}{11 \cdot 12 \cdot 2^{10}} - \text{etc.},$$

so that

$$1+\frac{1}{3^3}+\frac{1}{5^3}+\frac{1}{7^3}+\text{etc.}=\frac{1}{2}\pi\pi Z$$

or

$$1+\frac{1}{3^3}+\frac{1}{5^3}+\frac{1}{7^3}+\text{etc.}=\frac{1}{8}\pi\pi-\frac{2\alpha\pi^2}{3\cdot 4\cdot 2^2}-\frac{2\beta\pi^4}{5\cdot 6\cdot 2^4}-\frac{2\gamma\pi^6}{7\cdot 8\cdot 2^6}-\text{etc.}$$

To investigate the sum of this series, let us consider π as a variable quantity and having set $\frac{\pi}{2}=\varphi$ let us put

$$\frac{\alpha\varphi^4}{3\cdot 4}+\frac{\beta\varphi^6}{5\cdot 6}+\frac{\gamma\varphi^8}{7\cdot 8}+\frac{\delta\varphi^{10}}{9\cdot 10}+\text{etc.}=s;$$

it will be

$$\frac{dds}{d\varphi^2}=\alpha\varphi^2+\beta\varphi^4+\gamma\varphi^6+\delta\varphi^8+\text{etc.}=z,$$

whence we can form this equation

$$\begin{aligned}2zz=2\alpha\alpha\varphi^4+4\alpha\beta\varphi^6&+4\alpha\gamma\varphi^8+4\alpha\delta\varphi^{10}+\text{etc.}\\&+2\beta\beta\quad\;\;+4\beta\gamma.\end{aligned}$$

Now because it is

$$\beta=\frac{2\alpha\alpha}{5},\quad\gamma=\frac{4\alpha\beta}{7},\quad\delta=\frac{4\alpha\gamma+2\beta\beta}{9}\quad\text{etc.},$$

it will be

$$2\int zzd\varphi=\beta\varphi^5+\gamma\varphi^7+\delta\varphi^9+\text{etc.}=z\varphi-\alpha\varphi^3$$

and hence

$$2zzd\varphi=zd\varphi+\varphi dz-3\alpha\varphi\varphi d\varphi.$$

§21 Because now it is $\alpha = \frac{1}{6}$, by integration one finds

$$z = \frac{1}{2} - \frac{\varphi}{2\tan\varphi},$$

as it will be seen clearly, if the calculation is actually done. Hence, because it is

$$dds = zd\varphi^2,$$

one concludes this relation

$$\frac{ds}{d\varphi} = \int zd\varphi = \frac{1}{2}\varphi - \frac{1}{2}\int \frac{\varphi d\varphi}{\tan\varphi}$$

and

$$s = \frac{1}{4}\varphi^2 - \frac{1}{2}\int d\varphi \int \frac{\varphi d\varphi}{\tan\varphi}$$

and hence

$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \text{etc.} = \frac{1}{8}\pi\pi - \frac{1}{2}\varphi\varphi + \int d\varphi \int \frac{\varphi d\varphi}{\tan\varphi}$$

and because of $\varphi = \frac{\pi}{2}$ or $\pi = 2\varphi$ it will be

$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \text{etc.} = \int d\varphi \int \frac{\varphi d\varphi}{\tan\varphi} = \frac{\pi}{2}\int \frac{\varphi d\varphi}{\tan\varphi} - \int \frac{\varphi\varphi d\varphi}{\tan\varphi}$$

$$= 2\int \varphi d\varphi \ln\sin\varphi - \frac{\pi}{2}\int d\varphi \ln\sin\varphi;$$

but it is

$$\int d\varphi \ln\sin\varphi = -\frac{\pi\ln 2}{2},$$

hence

$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \text{etc.} = \frac{\pi\pi}{4}\ln 2 + 2\int \varphi d\varphi \ln\sin\varphi;$$

here, having taken the integrals in such a way that they vanish for $\varphi = 0$, one has to put $\varphi = \frac{\pi}{2}$ in order to obtain the sum of the propounded series. But even if the integration cannot be done explicitly, its value can nevertheless be defined by quadratures. But the series found before by means of Z itself is very appropriate to determine the sum approximately.

§22 Hence I take the opportunity to consider this series more accurately

$$P=\frac{1}{n}x^n+\frac{n}{n+1}x^{n+1}+\frac{n+1}{2}x^{n+2}+\frac{(n+2)(n+4)}{2\cdot 3}x^{n+3}$$
$$\frac{(n+3)(n+5)(n+6)}{2\cdot 3\cdot 4}x^{n+4}+\text{etc.},$$

whose values we determined by means of a special method for the cases, in which n is an odd integer: These values are the following:

$$\begin{aligned}
\text{if}\quad n=1,\quad P&=\frac{1}{4}\left(1+2x\quad-\sqrt{(1-4x)}\right),\\
n=3,\quad P&=\frac{1}{24}\left(1-6xx\quad-(1+2x)\sqrt{1-4x}\right),\\
n=5,\quad P&=\frac{1}{60}\left(1-15xx+10x^3-(1+2x-9xx)\sqrt{1-4x}\right),\\
n=7,\quad P&=\frac{1}{112}\left(1-28xx+56x^3-14x^4-(1+2x-22xx+20x^3)\sqrt{1-4x}\right).
\end{aligned}$$

Therefore, because the summation in general can be reduced to an differential equation, it seems worth one's while to examine, how these values satisfy it in these cases. But it will be more convenient to consider the differentials, which are:

$$\begin{aligned}
\text{if}\quad n=1,\quad \frac{dP}{dx}&=\frac{1}{2}\left(1+\frac{1}{\sqrt{1-4x}}\right),\\
n=3,\quad \frac{dP}{dx}&=\frac{1}{2}\left(-x+\frac{x}{\sqrt{1-4x}}\right),\\
n=5,\quad \frac{dP}{dx}&=\frac{1}{2}\left(-x+xx+\frac{x-3xx}{\sqrt{1-4x}}\right),\\
n=7,\quad \frac{dP}{dx}&=\frac{1}{2}\left(-x+3xx-x^3\frac{x-5xx+5x^3}{\sqrt{1-4x}}\right).
\end{aligned}$$

§23 But in general by differentiating it is

$$\frac{dP}{dx}=x^{n-1}+nx^n+\frac{(n+1)(n+2)}{1\cdot 2}x^{n+1}+\frac{(n+2)(n+3)(n+4)}{1\cdot 2\cdot 3}x^{n+2}+\text{etc.}$$

Let us put

$$x = yy \quad \text{and} \quad \frac{dP}{dx} = \frac{dP}{2ydy} = s,$$

that we have

$$s = y^{2n-2} + ny^{2n} + \frac{(n+1)(n+2)}{1 \cdot 2} y^{2n+2} + \frac{(n+2)(n+3)(n+4)}{1 \cdot 2 \cdot 3} y^{2n+4} + \text{etc.},$$

whence it will be

$$y^{2-n}s = y^n + ny^{n+2} + \frac{(n+1)(n+2)}{1 \cdot 2} y^{n+4} + \frac{(n+2)(n+3)(n+4)}{1 \cdot 2 \cdot 3} y^{n+6} + \text{etc.}$$

and hence further

$$\frac{dd(y^{2-n}s)}{dy^2} = n(n-1)y^{n-2} + \frac{n(n+1)(n+2)}{1} y^n$$
$$+ \frac{(n+1)(n+2)(n+3)(n+4)}{1 \cdot 2 \cdot 3} y^{2n+1} + \text{etc.},$$

which equation multiplied by y^{5-2n} and differentiated again produces

$$\frac{1}{2dy} d.\frac{s}{yy} = (n-2)y^{2n-5} + (n-1)y^{2n-3} + \frac{n(n+1)(n+2)}{1 \cdot 2} y^{2n-1}$$
$$+ \frac{(n+1)(n+2)(n+3)(n+4)}{1 \cdot 2} y^5 + \text{etc.},$$

which series by the one above also is

$$= \frac{y^{3-n} dd(y^{2-n}s)}{dy^2},$$

so that we have this equation between s and y

$$d.\left(y^{5-2n} d.\frac{s}{yy}\right) = 4y^{3-n} dd(y^{2-n}s).$$

§24 Having assumed the element dy to be constant this equation in expanded form gives

$$\begin{aligned} &y^{3-2n}dds+(1-2n)y^{2-2n}dyds-4(1-n)y^{1-2n}sdy^2\\ =&\,4y^{5-2n}dds+8(2-n)y^{4-2n}dyds+4(2-n)(1-n)y^{3-2n}sdy^2, \end{aligned}$$

which multiplied by y^{2n-1} becomes this one

$$\begin{aligned} &yy(1-4yy)dds+(1-2n)ydyds-4(1-n)sdy^2\\ &-8(2-n)y^3dyds-4(2-n)(1-n)yysdy^2=0, \end{aligned}$$

which for $yy=x$ and for constant dx is transformed into this one

$$\begin{aligned} &xx(1-4x)dds+(1-n)xdxds-(1-n)sdx^2\\ &-2(5-2n)xxdxds-(2-n)(1-n)sxdx^2=0, \end{aligned}$$

where it is

$$s=\frac{dP}{dx} \quad \text{and} \quad P=\int sdx.$$

But the integrals have to be taken in such a way that, while x is infinite small, it is

$$\frac{ds}{dx}=(n-1)x^{n-2}, \quad s=x^{n-1} \quad \text{and} \quad P=\frac{1}{n}x^n.$$

§25 If this equation is integrated by means of an infinite series whose first term is x^{n-1}, the propounded series itself is reproduced; but the initial term can also be the constant term x^0, whence also an integral is obtained; but that integral does not fulfill our requirements; but furthermore one can choose another integral that combined with that one solves the task. Hence let us assume this power series

$$s=O+Ax+Bx^2+Cx^3+Dx^4+Ex^5+Fx^6+\text{etc.},$$

it will be

$$\frac{ds}{dx}=A+2Bx+3Cxx+4Dx^3+5Ex^4+6Fx^5+\text{etc.}$$

and

$$\frac{dds}{dx^2} = 2B + 6Cx + 12Dxx + 20Ex^3 + 30Fx^4 + 42Gx^5 + \text{etc.};$$

after having substituted this series it has to be

$$\begin{array}{rrrrl}
 & & -(1-n)O & & \\
-(2-n)(1-n)Ox & -(2-n)(1-n)Ax^2 & -(2-n)(1-n)Bx^3 & -(2-n)(1-n)Cx^4 & -\text{etc.}=0, \\
+(1-n)A & +2(1-n)B & +3(1-n)C & +4(1-n)D & \\
-(1-n)A & -2(5-2n)A & -4(5-2n)B & -6(5-2n)C & \\
 & -(1-n)B & -(1-n)C & -(1-n)D & \\
 & +2B & +6C & +12D & \\
 & & -8B & -24C &
\end{array}$$

which equation is reduced to this form

$$\begin{array}{rrrrl}
 & -(1-n)O & -(2-n)(1-n)O\ x & & \\
+(3-n)Bx^2 & +2(4-n)Cx^3 & +3(5-n)Dx^4 & +4(6-n)Ex^5 & +\text{etc.} = 0. \\
+(3-n)(4-n)A & -(5-n)(6-n)B & -(7-n)(8-n)C & -(9-n)(10-n)D &
\end{array}$$

§26 Therefore, having set the single terms equal to zero, O has to be $= 0$, if n is not $= 1$, but for the remaining coefficients one will have

$$\begin{aligned}
B &= \frac{(3-n)(4-n)}{1(3-n)}\ A = \frac{4-n}{1}A, \\
C &= \frac{(5-n)(6-n)}{2(4-n)}\ B = \frac{(5-n)(6-n)}{1\cdot 2}A, \\
D &= \frac{(7-n)(8-n)}{3(5-n)}\ C = \frac{(6-n)(7-n)(8-n)}{1\cdot 2\cdot 3}A, \\
E &= \frac{(9-n)(10-n)}{4(6-n)}D = \frac{(7-n)(8-n)(9-n)(10-n)}{1\cdot 2\cdot 3\cdot 4}A \\
& \qquad\qquad \text{etc.},
\end{aligned}$$

whence the structure of the progression is obvious. But in the case $n = 1$ the quantity O remains undetermined, but then the equation is solved after having set all remaining coefficients equal to zero so that it is $s = O$, even though also finite values could be assumed for them from these determinations, as for example

$$B = \frac{3}{1}A, \quad C = \frac{4\cdot 5}{1\cdot 2}A, \quad D = \frac{5\cdot 6\cdot 7}{1\cdot 2\cdot 3}A \quad \text{etc.}$$

whence the complete integral will be

$$s = O + A\left(x + \frac{3}{1}x^2 + \frac{4\cdot 5}{1\cdot 2}x^3 + \frac{5\cdot 6\cdot 7}{1\cdot 2\cdot 3}x^4 + \frac{6\cdot 7\cdot 8\cdot 9}{1\cdot 2\cdot 3\cdot 4}x^5 + \text{etc.}\right).$$

§27 In the same way it is indeed $O = 0$ for the remaining cases, in which n is a whole number, but the number A is arbitrary; but furthermore another certain coefficient is also not defined, which can therefore assumed arbitrarily. Hence if one puts $= 0$, one will have an integral expressed in finite terms, which will be:

$$\begin{array}{llllllll}
\text{if} & n = 3, & O = 0 & A & \text{is undefined,} & B = 0, & C = 0 & \text{etc.;} \\
\text{if} & n = 4, & O = 0 & A & \text{is undefined,} & B = 0, & C = 0 & \text{etc.;} \\
\text{if} & n = 5, & O = 0 & A & \text{is undefined,} & B = -A, & C = 0, & D = 0 \quad \text{etc.;} \\
\text{if} & n = 6, & O = 0 & A & \text{is undefined,} & B = -2A, & C = 0, & D = 0 \quad \text{etc.;} \\
\text{if} & n = 7, & O = 0 & A & \text{is undefined,} & B = -3A, & C = A, & D = 0 \quad \text{etc.;} \\
\text{if} & n = 8, & O = 0 & A & \text{is undefined,} & B = -4A, & C = \frac{2\cdot 3}{1\cdot 2}A, & D = 0 \\
 & & & & & & & E = 0 \quad \text{etc.;} \\
\text{if} & n = 9, & O = 0 & A & \text{is undefined,} & B = -5A, & C = \frac{3\cdot 4}{1\cdot 2}A & \\
 & & & & & D = -\frac{1\cdot 2\cdot 3}{1\cdot 2\cdot 3}A, & E = 0 \quad \text{etc.;} &
\end{array}$$

if $n = 10$, $O = 0$ A is undefined, $B = -6A$, $C = \frac{4 \cdot 5}{1 \cdot 2} A$

$D = -\frac{2 \cdot 3 \cdot 4}{1 \cdot 2 \cdot 3} A$, $E = 0$ etc.;

if $n = 11$, $O = 0$ A is undefined, $B = -7A$, $C = \frac{5 \cdot 6}{1 \cdot 2} A$,

$D = -\frac{3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3} A$, $E = \frac{1 \cdot 2 \cdot 3 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4} A$, $F = 0$ etc.;

etc.

§28 So behold the particular integrals for all cases, in which n is a positive integer but not $n = 2$, whence the rational parts of the formulas found above for $\frac{dP}{dx}$ can be calculated:

$$\begin{array}{lll}
\text{if} & n = 1, & s = O; \\
\text{if} & n = 3, & s = Ax; \\
\text{if} & n = 4, & s = Ax; \\
\text{if} & n = 5, & s = A(x - xx); \\
\text{if} & n = 6, & s = A(x - 2xx); \\
\text{if} & n = 7, & s = A(x - 3xx + x^3); \\
\text{if} & n = 8, & s = A(x - 4xx + 3x^3); \\
\text{if} & n = 9, & s = A(x - 5xx + 6x^3 - x^4); \\
\text{if} & n = 10, & s = A(x - 6xx + 10x^3 - 4x^4); \\
\text{if} & n = 11, & s = A(x - 7xx + 15x^3 - 10x^4 + x^5); \\
\text{if} & n = 12, & s = A(x - 8xx + 21x^3 - 20x^4 + 5x^6).
\end{array}$$

§29 Hence for any number n this particular integral is

$$s = A \left\{ \begin{array}{c} x - \frac{n-4}{1} xx + \frac{(n-5)(n-6)}{1 \cdot 2} x^3 - \frac{(n-6)(n-7)(n-8)}{1 \cdot 2 \cdot 3} x^4 \\ + \frac{(n-7)(n-8)(n-9)(n-10)}{1 \cdot 2 \cdot 3 \cdot 4} x^5 - \text{etc.} \end{array} \right\},$$

which series, even if continued to infinity, solves the differential equation; nevertheless, although a certain term vanished, all following ones can be omitted, which taken alone would yield another particular integral. In addition, it is evident from this that any of these formulas is defined by the two preceding ones in such a way that, if for the cases $n = \nu$, $n = \nu + 1$, $n = \nu + 2$ the values of s are put s, s', s'', it will be

$$s'' = s' - sx,$$

if the constant A has the same value in all of them. And according to this rule one has to put $s = 0$ for the case $n = 2$. But, as I already mentioned, these particular integrals do not fulfill our conditions, but give irrational parts, as we will see soon.

§30 But to find the complete integrals, let us investigate other particular integrals, which yield irrational parts. For this purpose, let us put

$$s = \frac{t}{\sqrt{1-4x}};$$

it will be

$$ds = \frac{dt}{\sqrt{1-4x}} + \frac{2tdx}{(1-4x)^{\frac{3}{2}}}$$

and

$$dds = \frac{ddt}{\sqrt{1-4x}} + \frac{4dxdt}{(1-4x)^{\frac{3}{2}}} + \frac{12tdx^2}{(1-4x)^{\frac{5}{2}}};$$

having substituted these values our differential equation will be converted into this form

$$\begin{gathered} xx(1-4x)ddt - (n-1)xdtdx + (n-1)tdx^2 \\ +2(2n-3)xxdtdx - n(n-1)txdx^2 = 0. \end{gathered}$$

Hence put

$$t = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6 + \text{etc.}$$

here and after the substitution one gets to this equation

$$
\begin{array}{c}
0 = (n-1)A \\
-n(n-1)Ax - (n-3)Cx \qquad -2(n-4)Dx^3 \qquad -3(n-5)Ex^4 - \text{etc.}; \\
-(n-2)(n-3)B \;\; -(n-4)(n-5)C \;\; -(n-6)(n-7)D
\end{array}
$$

so if n is not $= 1$, A has to be $= 0$ and for the remaining

$$
\begin{aligned}
C &= -\frac{(n-2)(n-3)}{1(n-3)}B = -\frac{n-2}{1}B, \\
D &= -\frac{(n-4)(n-5)}{2(n-4)}C = -\frac{(n-2)(n-5)}{1\cdot 2}B, \\
E &= -\frac{(n-6)(n-7)}{3(n-5)}D = -\frac{(n-2)(n-6)(n-7)}{1\cdot 2\cdot 3}B.
\end{aligned}
$$

§31 Hence the finite values of t for the single integer numbers n will be:

$$
\begin{array}{lll}
\text{if} & n=1, & t = A; \\
\text{if} & n=2, & t = Bx; \\
\text{if} & n=3, & t = Bx; \\
\text{if} & n=4, & t = B(x-2xx); \\
\text{if} & n=5, & t = B(x-3xx); \\
\text{if} & n=6, & t = B(x-4xx+2x^3); \\
\text{if} & n=7, & t = B(x-5xx+5x^3); \\
\text{if} & n=8, & t = B(x-6xx+9x^3-2x^4); \\
\text{if} & n=9, & t = B(x-7xx+14x^3-7x^4); \\
\text{if} & n=10, & t = B(x-8xx+20x^3-16x^4+2x^5)
\end{array}
$$

and in general

$$
t = B\left\{
\begin{array}{c}
x - \dfrac{n-2}{1}xx + \dfrac{(n-2)(n-5)}{1\cdot 2}x^3 - \dfrac{(n-2)(n-6)(n-7)}{1\cdot 2\cdot 3}x^4 \\
+ \dfrac{(n-2)(n-7)(n-8)(n-9)}{1\cdot 2\cdot 3\cdot 4}x^5 - \text{etc.}
\end{array}
\right\},
$$

where, as above, again it is $t'' = t' - tx$.

§32 As we denoted the series P by $S(n)$ above, let us denote the series $s = \frac{dP}{dx}$ by $\Sigma(n)$; then it will be in general

$$\begin{aligned}
\Sigma(1) &= -A + \frac{B}{\sqrt{1-4x}},\\
\Sigma(2) &= 0 + \frac{2Bx}{\sqrt{1-4x}},\\
\Sigma(3) &= Ax + \frac{Bx}{\sqrt{1-4x}},\\
\Sigma(4) &= Ax + \frac{B(x-2xx)}{\sqrt{1-4x}},\\
\Sigma(5) &= A(x-xx) + \frac{B(x-3xx)}{\sqrt{1-4x}},\\
\Sigma(6) &= A(x-2xx) + \frac{B(x-4xx+2x^3)}{\sqrt{1-4x}},\\
\Sigma(7) &= A(x-3xx+x^3) + \frac{B(x-5xx+5x^3)}{\sqrt{1-4x}},\\
\Sigma(8) &= A(x-4xx+3x^3) + \frac{B(x-6xx+9x^3-2x^4)}{\sqrt{1-4x}},\\
\Sigma(9) &= A(x-5xx+6x^3-x^4) + \frac{B(x-7xx+14x^3-7x^4)}{\sqrt{1-4x}},\\
\Sigma(10) &= A(x-6xx+10x^3-4x^4) + \frac{B(x-8xx+20x^3-16x^4+2x^5)}{\sqrt{1-4x}}
\end{aligned}$$

etc.;

here it must be $A = -\frac{1}{2}$ and $B = \frac{1}{2}$ in order to accommodate these forms to the propounded series.

§33 Because these values constitute a recurring series, since any of them becomes equal the last of the preceding less the penultimate multiplied by x, one will be able to express the general term or the value $\Sigma(n)$ in finite terms; for, by the property of recurring series it will be

$$\Sigma(n) = M\left(\frac{1+\sqrt{1-4x}}{2}\right)^n + N\left(\frac{1-\sqrt{1-4x}}{2}\right)^n,$$

where the coefficients M and N are determined from the first two members of that series in such a way that

$$M = \frac{(A+B)(1-2x-\sqrt{1-4x})}{2x\sqrt{(1-4x}} = \frac{A+B}{x\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2}\right)^n,$$

$$N = \frac{(B-A)(1-2x+\sqrt{1-4x})}{2x\sqrt{(1-4x}} = \frac{B-A}{x\sqrt{1-4x}}\left(\frac{1+\sqrt{1-4x}}{2}\right)^n.$$

But because for our case it is $A = -\frac{1}{2}$ and $B = \frac{1}{2}$, it will be

$$\Sigma(n) = \frac{1}{x\sqrt{1-4x}}\left(\frac{1+\sqrt{1-4x}}{2}\right)^2\left(\frac{1+\sqrt{1-4x}}{2}\right)^n$$

or

$$\Sigma(n) = \frac{x}{\sqrt{1-4x}}\left(\frac{1+\sqrt{1-4x}}{2}\right)^{n-2} = \frac{dP}{dx}$$

while

$$P = \frac{1}{n}x^n + \frac{n}{n+1}x^{n+1} + \frac{n+1}{2}x^{n+2} + \frac{(n+2)(n+4)}{2\cdot 3}x^{n+3} + \frac{(n+3)(n+5)(n+6)}{2\cdot 3\cdot 4}x^{n+4} + \text{etc.}$$

§34 Hence the value of this series we put P, is

$$P = \int \frac{xdx}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2}\right)^{n-2};$$

to find this integral just put

$$\frac{1-\sqrt{1-4x}}{2} = y;$$

it will be

$$dy = \frac{dx}{\sqrt{1-4x}} \quad \text{and} \quad x = y - yy,$$

whence it is

$$P=\int dy(y-yy)y^{n-2}=\frac{y^n}{n}-\frac{y^{n+1}}{n+1}$$

and hence

$$P=\frac{n+1-ny}{n(n+1)}\cdot y^n$$

or

$$P=\frac{n+2+n\sqrt{1-4x}}{2n(n+1)}\left(\frac{1-\sqrt{1-4x}}{2}\right)^n=S(n)$$

for $x=\frac{1}{4}$, whence for the formulas given above (§ 14) one concludes

$$S(n)=\frac{n+2}{2^{n+1}n(n+1)}$$

and hence, as the formulas were there,

$$\frac{1}{(n+1)2^{n+1}}-S(n)=-\frac{1}{2^n(n+1)n},$$

which expression agrees completely with those we gave above (§ 19) based on induction alone; therefore, they can not further be in any doubt.

§35 Further, it is remarkable that the complete integral of this differential equation

$$\begin{gathered}xx(1-4x)dds-(n-1)xdxds+(n-1)sdx^2\\+2(2n-5)xxdxds-(n-1)(n-2)sxdx^2=0\end{gathered}$$

can be assigned and moreover algebraically; this integral using the preceding results is as follows:

$$s=\frac{Cx}{\sqrt{1-4x}}\left(\frac{1+\sqrt{1-4x}}{2}\right)^{n-2}+\frac{Dx}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2}\right)^{n-2};$$

how this can be found by integration using those results, is indeed not obvious at all. Hence it is nevertheless immediately understood that the substitution

$$s=\frac{xu}{\sqrt{1-4x}}$$

will be very helpful; for, then having set $t = ux$ in § 30 this equation results

$$xx(1-4x)ddu - (n-3)xdxdu - (n-2)(n-3)xudx^2$$
$$+2(2n-7)xxdxdu = 0$$

or

$$x(1-4x)ddu - (n-3)dxdu - (n-2)(n-3)udx^2$$
$$+2(2n-7)xdxdu = 0,$$

whose integral therefore is

$$u = C\left(\frac{1+\sqrt{1-4x}}{2}\right)^{n-2} + D\left(\frac{1-\sqrt{1-4x}}{2}\right)^{n-2}.$$

§36 If in this equation one puts

$$\sqrt{1-4x} = y$$

and the element dy is considered as a constant, this simpler equation will result

$$(1-yy)ddu + 2(n-3)ydydu - (n-2)(n-3)udy^2 = 0,$$

whose integral is already known to be

$$u = C\left(\frac{1+y}{2}\right)^{n-2} + C\left(\frac{1-y}{2}\right)^{n-2}.$$

In order to show this clearly, how this can be found from there, let us put $n = m+2$ that we have

$$(1-yy)ddu + 2(m-1)ydydu - m(m-1)udy^2 = 0,$$

where it is plain that this substitution will turn out to be very convenient

$$u = (\alpha + \beta y)^m,$$

whence it becomes

$$du = m\beta dy(\alpha+\beta y)^{m-1} \quad \text{and} \quad ddu = m(m-1)\beta\beta dy^2(\alpha+\beta y)^{m-2};$$

then it will be

$$m(m-1)(\alpha+\beta y)^{m-2}(\beta\beta(1-yy)+2\beta(\alpha y+\beta yy)-\alpha\alpha-2\alpha\beta y-\beta\beta yy)=0$$

and hence $\beta\beta=\alpha\alpha$, therefore

$$u=C(1\pm y)^m.$$

And because of the ambiguous sign one will obtain the complete integral

$$u=C(1+y)^m+D(1-y)^m.$$

§37 Moreover, it will be helpful to notice that this last equation

$$(1-yy)ddu+2(m-1)ydydu-m(m-1)udy^2=0$$

becomes integrable, if it is divided by $(1\pm y)^m$. The first equation on the other hand

$$x(1-4x)ddu-(n-3)dxdu-(n-2)(n-3)udx^2+2(2n-7)xdxdu=0$$

will become integrable, if it is multiplied by

$$x^{-n+3}du-\frac{n-2}{2}x^{-n+2}udx.$$

But in general having propounded this equation

$$\begin{aligned}xx(A+Bx)ddu&+\frac{1}{2}(2\alpha+\lambda)Axdxdu+\frac{1}{2}\alpha(\lambda-2)Audx^2\\&+\frac{1}{2}(2\alpha+\lambda+1)Bxxdxdu+\frac{1}{2}\alpha(\lambda-1)Bxudx^2=0,\end{aligned}$$

if it is multiplied by

$$x^{\lambda-2}du+\alpha x^{\lambda-3}udx,$$

it will become integrable and the integral will be

$$\frac{1}{2}x^{\lambda}(A+Bx)du^2+\alpha x^{\lambda-1}(A+Bx)ududx+\frac{1}{2}\alpha\alpha x^{\lambda-2}(A+Bx)u^2dx^2=\frac{1}{2}Cdx^2$$

or

$$x^\lambda du^2 + 2\alpha x^{\lambda-1} ududx + \alpha\alpha x^{\lambda-2} u^2 dx^2 = \frac{Cdx^2}{A+Bx},$$

therefore

$$x^{\frac{1}{2}\lambda} du + \alpha x^{\frac{1}{2}\lambda-1} udx = \frac{dx\sqrt{C}}{\sqrt{A+Bx}}$$

and hence

$$u = x^{-\alpha} \int \frac{x^{\alpha-\frac{1}{2}\lambda} dx\sqrt{C}}{\sqrt{A+Bx}}.$$

§38 But our equation considered above is not contained in this general equation; hence let us investigate conditions of this equation

$$xx(A+Bx)ddu + x(C+Dx)dudx + (E+Fx)udx^2 = 0$$

more accurately, so that it multiplied by

$$x^{\lambda-2} du + \alpha x^{\lambda-3} udx$$

becomes integrable. And at first the integral is indeed

$$\frac{1}{2}x^\lambda (A+Bx)du^2 + \alpha x^{\lambda-1}(A+Bx)ududx + \frac{\alpha E}{\lambda-2}x^{\lambda-2}uudx^2$$
$$+\frac{\alpha F}{\lambda-1}x^{\lambda-1}u^2dx^2 = Gdx^2;$$

but it is required at first that it is

$$C = \left(\alpha + \frac{1}{2}\lambda\right)A, \quad D = \left(\alpha + \frac{1}{2}\lambda + \frac{1}{2}\right)B,$$

but then indeed in three ways

I. either $E = \frac{1}{2}\alpha(\lambda-2)A$ and $F = \frac{1}{2}\alpha(\lambda-1)B$, which is the upper case,

II. or $\lambda = 2\alpha+2$, $F = \frac{1}{2}\alpha(2\alpha+1)B$ while E remains undefined;

III. or $\lambda = 2\alpha+1$, $E = \frac{1}{2}\alpha(2\alpha-1)B$ while F remains undefined.

§39 So behold these two far-reaching differential equations, which can be integrated by this method:

$$\text{I.}\quad xx(A+Bx)ddu+(2\alpha+1)Axdxdu+Eudx^2$$
$$+\left(2\alpha+\frac{3}{2}\right)Bxxdxdu+\frac{1}{2}(2\alpha+1)Bxudx^2=0,$$

which multiplied by

$$x^{2\alpha}du+\alpha x^{2\alpha-1}udx$$

give the integral

$$\frac{1}{2}x^{2\alpha+2}(A+Bx)du^2+\alpha x^{2\alpha+1}(A+Bx)ududx+\frac{1}{2}Ex^{2\alpha}uudx^2$$
$$+\frac{1}{2}\alpha\alpha Bx^{2\alpha+1}uudx^2=Gdx^2.$$

The other form on the other hand is

$$\text{II.}\quad xx(A+Bx)ddu+\left(2\alpha+\frac{1}{2}\right)Axdxdu+\frac{1}{2}\alpha(2\alpha-1)Audx^2$$
$$+(2\alpha+1)Bxxdxdu+Fxudx^2=0,$$

which multiplied by

$$x^{2\alpha-1}du+\alpha x^{2\alpha-2}udx$$

yields this integral

$$\frac{1}{2}x^{2\alpha+1}(A+Bx)du^2+\alpha x^{2\alpha}(A+Bx)ududx+\frac{1}{2}\alpha\alpha Ax^{2\alpha-1}u^2dx^2$$
$$+\frac{1}{2}Fx^{2\alpha}u^2dx^2=Gdx^2.$$

If in the first one puts

$$A=1,\quad B=-4\quad\text{und}\quad 2\alpha+1=-n+3\quad\text{and}\quad E=0,$$

the equation propounded in § 35 emerges.

§40 But there is another way to investigate the sum of the progression in § 23

$$\frac{dP}{dx} = x^{n-1} + \frac{n}{1}x^n + \frac{(n+1)(n+2)}{1\cdot 2}x^{n+1} + \frac{(n+2)(n+3)(n+4)}{1\cdot 2\cdot 3}x^{n+2} + \text{etc.};$$

because x is considered as a constant, let us consider this series

$$s = 1 + \frac{n}{2}a^2 + \frac{(n+1)(n+2)}{2\cdot 4}a^4 + \frac{(n+2)(n+3)(n+4)}{2\cdot 4\cdot 6}a^6 + \text{etc.},$$

where

$$aa = 2x \quad \text{and} \quad \frac{dP}{dx} = x^{n-1}s.$$

Now recall this series

$$\frac{(1+ay)^{-n+1} + (1-ay)^{-n+1}}{2} = 1 + \frac{(n-1)n}{1\cdot 2}aay^2 + \frac{(n-1)n(n+1)(n+2)}{1\cdot 2\cdot 3\cdot 4}a^4y^4 + \text{etc.},$$

for which, for the sake of brevity, we want to write

$$1 + Aa^2y^2 + Ba^4y^4 + Ca^6y^6 + \text{etc.},$$

and it will be

$$s = 1 + \frac{1}{n-1}Aa^2 + \frac{1\cdot 3}{(n-1)n}Ba^4 + \frac{1\cdot 3\cdot 5}{(n-1)(n+1)}Ca^6 + \text{etc.}$$

Now put

$$s = \frac{1}{z}\int dz(1 + Aa^2y^2 + Ba^4y^4 + Ca^6y^6 + \text{etc.})$$

and it has to be

$$\begin{aligned}
\int yydz &= \frac{1}{n-1}\int dz, \\
\int y^4dz &= \frac{3}{n}\int yydz, \\
\int y^6dz &= \frac{5}{n+1}\int y^4dz
\end{aligned}$$

and hence in general

$$\int y^{2\alpha} dz = \frac{2\lambda - 1}{n + \lambda - 2} \int y^{2\lambda - 2} dz,$$

if after the integration a certain value is attributed to y.

§41 Hence let us put that it is in general

$$\int y^{2\lambda} dz = \frac{2\lambda - 1}{n + \lambda - 2} \int y^{2\lambda - 2} dz + \frac{Qy^{2\lambda - 1}}{n + \lambda - 2}$$

and by differentiating and dividing by $y^{2\lambda-2}$ one calculates

$$(n - \lambda - 2) yydz = (2\lambda - 1) dz + ydQ + (2\lambda - 1) Qdy,$$

which equation has to hold for all numbers λ, whence it will be so

$$yydz = 2dz + 2Qdy$$

as

$$(n - 2) yydz = -dz + ydQ - Qdy,$$

therefore

$$dz = \frac{2Qdy}{yy - 2} = \frac{ydQ - Qdy}{(n - 2) yy + 1},$$

whence it becomes

$$\frac{dQ}{Q} = -\frac{(2n - 3) ydy}{2 - yy} \quad \text{and} \quad Q = (2 - yy)^{n - \frac{3}{2}}$$

and hence

$$dz = -2dy (2 - yy)^{n - \frac{1}{2}}.$$

Therefore, having put $y = \sqrt{2}$ after the integration it is

$$\int y^{2\lambda} dz = \frac{2\lambda - 1}{n + \lambda - 2} \int y^{2\lambda - 2} dz$$

and one finds

$$s = \frac{\int dy(2-yy)^{n-\frac{1}{2}}((1+ay)^{-n+1}+(1-ay)^{-n+1}}{2\int dx(2-yy)^{n-\frac{1}{2}}},$$

if one puts $y=\sqrt{2}$ after the integration.

§42 Even though this method immediately exhibits the integral for the desired sum, it nevertheless does not show the true value in an algebraic expression. But above we saw that

$$\frac{dP}{dx} = \frac{x}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2}\right)^{n-2},$$

whence we conclude, that here it will be

$$s = \frac{x^{2-n}}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2}\right)^{n-2} = \frac{1}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2x}\right)^{n-2}.$$

Hence if we put $2x = aa$, also the value of the upper integral formula in the case $y=\sqrt{2}$ will be algebraic

$$s = \frac{1}{\sqrt{1-2aa}}\left(\frac{1-\sqrt{1-2aa}}{aa}\right)^{n-2},$$

which circumstance seems to be of great importance, because it is maybe possible to derive many other beautiful results in like manner from it.