Further Investigation on vibrating chords *

Leonhard Euler

Even though there seems to be hardly any room left for controversies in analytical questions of this kind, nevertheless, since the determination of all motions a chord can perform while vibrating requires a completely new branch of calculus Mathematicians are still hardly used to, it is not surprising that the complete solution, which derived from these principles I had once given, seemed to be rather suspect to most people. Therefore, I will make an effort here to explain all foundations and fundamental ideas this solution is based on very carefully and free them from all doubts and objections, and since these doubts mostly concern the method I used, it will suffice to have expanded the simplest cases, in which the chord has the same density throughout its whole length, with all eagerness. Furthermore, I will indeed consider all vibrations to be infinitely small, which hypothesis has been assumed by everyone treating this subject.

1. Therefore, first I will introduce the density and the mass of the chord, the motion of which I will investigate here, into the calculation in such a way that I set the mass or the weight of the portion of the chord, the length of which is $= k$, $= K$; for, it is not necessary to consider the material the chord was made out of here as long as the chord was perfectly flexible, since the whole motion depends only on its length and mass, but especially on the straining force, i.e. the tension; there is certainly no doubt about this.

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2. Therefore, let the chord be fixed at the points A and B (Fig. 1), the length of which chord we want to put $AB = a$, the mass or weight of which will therefore be $= \frac{Ka}{k}$; but then let it be strained by an arbitrary force, which we want to put equal to a weight $= \pi$; this can be mentally represented as follows, since the chord is pulled into both directions by equal forces $Aa = Bb = \pi$; but then

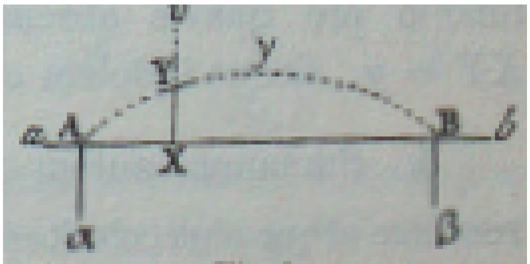

Fig. 1

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for it to remain motionless at the endpoints A and B while vibrating, it is necessary that at the points A and B the chord is additionally attacked by certain forces $A\alpha$ and $b\beta$ which are normal to the first and which are not given per se but at each moment must be of such a nature that both endpoints A and B of the chord remain at their places; but it is perspicuous, as long as the chord maintains its natural straight extended position AB, that these forces will be zero; but while, for the sake of an example, it is pulled upwards into the position AyB, it is evident that those forces $A\alpha$ and $B\beta$ must act downwards; but it is not necessary to know these forces, but they will later easily be determined by the solution for each state of the chord. Thus, we have a more clear idea about the forces by which the chord remains motionless at the points A and B.

3. Let us put that, after a certain time $= t$ has passed, which we assume to be given in seconds, the chord is bent into the shape AyB, for which we want to call the coordinates $AX = x$ and $XY = y$, so that $BX = a - x$; and since all vibrations are considered to be infinitely small, all ordinates $XY = y$ will be as small as possible; thus, we immediately obtain two extraordinary simplifications for the calculation: 1°) the portion AY of the chord can be considered to be equal to the abscissa $AX = x$, the weight of which will thus be $= \frac{Kx}{2}$, 2°) the point Y of the chord can only perform a motion into the direction of the ordinate XY during the vibration, while it goes back into its natural position AB; but if it recedes from it, the direction of the motion will be in the opposite direction, i.e. in the direction Yv. Having constituted this, it is evident that the angles AYX will be a right one nowhere, in other words, the tangent in the point Y will not be parallel to the axis AB. But also this hypothesis must deviate from the truth immediately when the vibrations are not infinitely small, nevertheless it is never doubted by any sceptic.

4. Therefore, since at each time t the shape AyB of the chord must be determined, it is evident that the ordinate y must be considered as a function of two variables, namely the time t and the abscissa x, from which the ordinate differentiable with respect to two variables, depending on whether only the time t or only the ordinate is considered as a variable. Of course, having assumed the abscissa x to be constant, that function will indicate how large the distance from the point Y to the axis AB will be at the time t, and after how much time this point Y will return to its initial position; thus, the durations of the vibrations can be calculated; but, having assumed the time t to be constant, the same function, for a certain abscissa x, will yield the quantity of the ordinate $XY = y$, and will thus reveal the nature of the curve AyB at a given time t.

5. But let us at the present time assume the point Y of the chord to recede from the axis and its velocity will be expressed by the formula $\left(\frac{dy}{dt}\right)$, the differential of which, divided by dt, will give the acceleration $= \left(\frac{ddy}{dt^2}\right)$; in order to be able to compare this to natural gravity expressed by 1, one has to divide by $2h$, while $2h$ denotes the altitude, from which heavy bodies fall down in one second such that this acceleration will be $= \frac{1}{2h}\left(\frac{ddy}{dt^2}\right)$; therefore, to this the force, which forces this element of the chord towards Yv, divided by the weight of this element must be equal; thus, since this weight is $\frac{Kdx}{k}$, the element $Yy = dx$ will be attacked from the direction Yv by a force $= \frac{Kdx}{2hk}\left(\frac{ddy}{dt^2}\right)$; of course, such a large force is necessary in each element that such a motion results which the analytical formulas describe. But we assume that the true motion is defined by our formulas, and so it is necessary that each element of the chord is attacked from the direction Yv.

6. But since such forces are actually not present, it is necessary that these fictional forces are equivalent forces to those forces actually acting on the chord, and so the question is reduced to this, of what nature those invented forces, which we will call elementary, since they are conceived to be applied in each element, must be that they are perfectly equivalent to the forces actually acting one the chord; or if in each element of the chord the same forces are imagined to be applied in the opposite direction, it is necessary that these forces are in an equilibrium with those actually acting on the chord: and so our investigation has been reduced to the investigation of the equilibrium state.

7. Therefore, let us apply our elementary forces in opposite direction, so that now (Fig. 2) the element Y of the chord is acted upon in the direction YU by

$$\text{a force} = \frac{Kdx}{2hk}\left(\frac{ddy}{dt^2}\right),$$

and since these forces must be in equilibrium an with those actually acting on the chord and, as we saw, are first the straining forces $Aa = Bb = \pi$, second those unknown forces $A\alpha$ and $B\beta$,

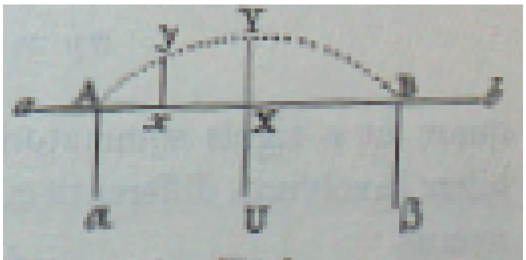

Fig. 2

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which will certainly be functions of the time t, as we will see in the following, it is especially necessary that the moments of all forces acting on the point Y cancel each other, since the chord is assumed to be perfectly flexible. Therefore, let us investigate the moments of all our forces, which from a preceding part or one to the left act on the point Y, that afterwards we can cancel them; then, from the other side to the right the moments will also immediately cancel, since all forces taken together at the same time are in an equilibrium.

8. Therefore, from the left first the straining force $Aa = \pi$ acts, the moment of which in direction XA on the point Y is πy, but the moment acting in the opposite direction of the other force $A\alpha = F$ will be Fx and all those elementary forces acting towards the axis will also act in that same direction. Therefore, to find the moment of these forces let us consider the point Y as fixed and consider the point y as variable to be moved successively from the limit A to Y, for which point the coordinates shall be $Ax = X$ and $xy = Y$ and the elementary force acting in the direction yx is $= \frac{KdX}{2hk} \cdot \frac{ddY}{dt^2}$, the moment of which on the point Y results by multiplying by the interval $xX = x - X$, so that this moment is $\frac{(x-X)KdX}{2hk} \cdot \frac{ddY}{dt^2}$, the integral of which, because of the constant x, will be

$$= \frac{Kx}{2hk} \int \frac{dXddY}{dt^2} - \frac{K}{2hk} \int \frac{XdXddY}{dt^2}$$

and express the moment of all elementary forces applied on the arc Ay, if these integrals are to be taken in such a way that they vanish for $X = 0$. Now, let us move the point y forward to Y and the moments of all elementary forces applied throughout the whole arc Ay will be

$$\frac{K}{2hk} \left(x \int \frac{dxddy}{dt^2} - \int \frac{xdxddy}{dt^2} \right),$$

which is reduced to this expression

$$\frac{K}{2hk} \int dx \int \frac{dxddy}{dt^2},$$

which double integration must be performed in such a way that for $x = 0$ the integral vanishes.

9. Therefore, having collected these moments, since their sum must be zero, we obtain the following equation

$$\pi y = Fx + \frac{K}{2hk} \int dx \int \frac{dxddy}{dt^2},$$

to free which from the integration signs involving the abscissa x as a variable, let us differentiate with respect to x and, diving by dx, we will obtain

$$\pi \left(\frac{dy}{dx} \right) = F + \frac{K}{2hK} \int \frac{dxddy}{dt^2},$$

which differentiated again in like manner yields this nice equation

$$\pi\left(\frac{ddy}{dx^2}\right)=\frac{K}{2hk}\left(\frac{ddy}{dt^2}\right).$$

Therefore, for the sake of brevity having put $\frac{2hk\pi}{K}=c^2$, we will eventually have this elegant equation

$$\left(\frac{ddy}{dt^2}\right)=c^2\left(\frac{ddy}{dx^2}\right),$$

which equation contains the whole motion the chord is capable of, so that the resolution of our question depends on the integration of this differential equation of second order and which differs from the usual equations of this order mainly in this regard that here a function of the two variables t and x is in question and for this reason this question is to be referred to that new branch of integral calculus, which is accommodated to functions of two or more variables.

10. But here, it immediately very conveniently happens that this equation allows a complete integration, while its complete integral is found to be

$$y=\varphi(ct+x)+\psi(ct-x),$$

the correctness of which will become clear quickly to anyone checking it; for, if we differentiate functions of this kind in usual manner, we will have

$$\left(\frac{dy}{dt}\right)=c\varphi'(ct+x)+c\psi'(ct-x)$$

and

$$\left(\frac{ddy}{dt^2}\right)=c^2\varphi''(ct+x)+c^2\psi''(ct-x)$$

and in like manner, taking only x as a variable, it will be

$$\left(\frac{dy}{dx}\right)=\varphi'(ct+x)-\psi'(ct-x)$$

and

$$\left(\frac{ddy}{dx^2}\right) = \varphi''(ct + x) + \psi''(ct - x),$$

whence it will manifestly be

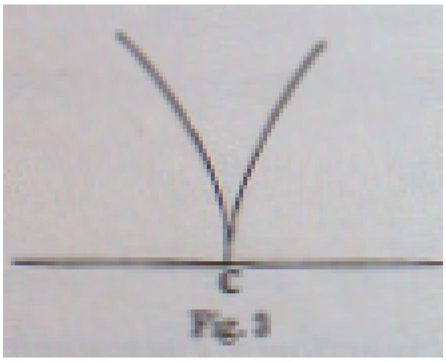
$$\left(\frac{ddy}{dt^2}\right) = c^2 \left(\frac{ddy}{dx^2}\right).$$

Here it is to be carefully noted that the characters φ and ψ denote any regular or somehow irregular functions; this kind of analysis differs mainly from the ordinary kind in that regard, since here even arbitrarily irregular functions and such not restricted to the law of continuity enter; this does not happen in usual analysis; to make this more clear, since Geometers usually represent functions by curved lines, let the formula $\varphi(ct + x)$ denote the ordinate, corresponding to the abscissa $ct + x$, of a certain curve; and in like manner, the formula $\psi(ct - x)$ will denote the ordinate, corresponding to the abscissa $ct - x$, of another curve and it is not necessary at all that these two curved lines can be expressed by a certain analytical equation, but even curves somehow conflated from different portions of different curves and even curves drawn by hand arbitrarily come into consideration here, as long as all parts are connected and are never abruptly disconnected. Therefore, there is no obstruction that these curves are composed of several straight lines or even circular arcs or mixed arcs of other curves.

11. But for this reason my solution seems to be suspect to D'Lambert and other Geometers, who do not want to admit other curved lines than those, which are expressed by certain analytic expressions, and which are contained in the law of continuity in the strict sense. But hence especially portions of different curves, which have cusps, are excluded, while cuspidal angle of this kind seem to contradict the nature of the differential equation of second order completely; I often responded that here such angles, which he fears, can not occur, since here only infinitely small vibrations are considered, where, as we already noted, all tangents must not be parallel to the axis, but even this seemed not to be satisfactory for him and he even shuns infinitely small inclinations. I certainly hope to end this dispute completely, when I will have shown that very sharp bends in those curves representing the functions φ and ψ do not cause any trouble in the task, to have demonstrated what in this one example will suffice. Of course, let the function φ be represented by a regular curve of such a kind, to the abscissa u of which the following ordinate corresponds

$$\varphi u = \sqrt[3]{(a-u)^2 a},$$

the shape of which curve will be of the following nature;



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it is of course the very well known Neilian cubic parabola, which in the point C (Fig. 3) has an infinitely sharp cusp; nevertheless, this cusp is not by any means a reason that this function does not satisfy our equation; for, having taken the abscissa $u = ct + x$, that we at least for the first function have:

$$y = \sqrt[3]{a(a-ct-x)^2}, \quad \left(\frac{dy}{dt}\right) = -\frac{2c\sqrt[3]{a}}{3\sqrt[3]{a-ct-x}} \quad \text{and} \quad \left(\frac{ddy}{dt^2}\right) = -\frac{2c^2\sqrt[3]{a}}{9\sqrt[3]{(a-ct-x)^4}},$$

and in like manner

$$\left(\frac{dy}{dx}\right) = -\frac{2\sqrt[3]{a}}{3\sqrt[3]{a-ct-x}} \quad \text{and} \quad \left(\frac{ddy}{dx^2}\right) = -\frac{2\sqrt[3]{a}}{9\sqrt[3]{(a-ct-x)^4}},$$

which formula perfectly satisfies the equation

$$\left(\frac{ddy}{dt^2}\right) = c^2\left(\frac{ddy}{dx^2}\right);$$

not even the case $a = ct + x$, where the cusp occurs, is excluded, from which one can without a doubt correctly conclude, if cusps do not cause any difficulty in the task, that sharp bends, moreover infinitely small ones, will to be worried about even less.

12. Having mentioned these things in advance, let us accommodate the complete integral of our equation to the propounded case of the vibrating chord, where two conditions are to be satisfied. Of course, first that in A, where $x = 0$, the ordinate y always vanishes for every time t; further, that the same happens at the other endpoint B, where $x = a$. But the first condition, having put $x = 0$, yields

$$y = \varphi ct + \psi ct,$$

since which value must be $= 0$, it is necessary that

$$\psi ct = -\varphi ct,$$

i.e., the curve to be represented by the function ψ must be of such a nature that the ordinate corresponding to a certain abscissa is the negative of that one, which corresponds to the same abscissa in the curve φ, whence it follows that it will in general be

$$\psi(ct - x) = -\varphi(ct - x);$$

hence that condition gives us this equation

$$y = \varphi(ct + x) - \varphi(ct - x),$$

for, so, having put $x = 0$, manifestly $y = 0$ results. But for the other condition let us now set $x = a$ and it must again be

$$\varphi(ct + a) - \varphi(ct - a) = 0$$

or

$$\varphi(ct + a) = \varphi(ct - a),$$

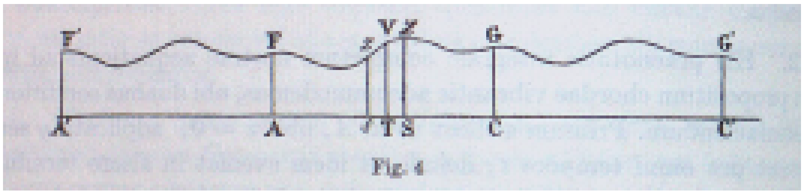
whence, if we set $ct - a = u$, it will be $ct + a = u + 2a$, so that it must in general be

$$\varphi(u + 2a) = \varphi u.$$

Of course, the curve to be represented by this function φ must be of such a nature, that, whatever ordinate corresponds to the abscissa u, the same also corresponds to all abscissas $u + 2a$, $u + 4a$, $u + 6a$ etc. and the same, going backwards, to these abscissas

$$u - 2a, \quad u - 4a, \quad u - 6a \quad \text{etc.},$$

from which it is seen, how this curve must be continued to infinity. Therefore, such a curve is conveniently constructed as follows (Fig. 4); over the the portion $AC = 2a$ of the axis arbitrarily construct a curved line FG, in such a way nevertheless, that the ordinate CG becomes equal to the ordinate AF, but then continue this same curve



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FG beyond C to the right and in like manner beyond F to the left, that over each portion of the axis of length $= 2a$ similar and equal portions of the curve FG correspond; and besides this prescription the description of the curve is completely arbitrary and can be composed either form several straight lines or arcs of arbitrary curved or even curves drawn by hand, as long as it is just continued to both sides as we described it.

13. But having constructed such a curved line, which can be called the scale of construction, a vibrational motion will always be very easily defined in the following way, while the shape of the chord, which it has at a certain time t, is assigned, of course. For, since c denotes a certain straight line, t on the other hand an absolute number, from the point A of the axis take the section $AT = ct$ and from this point T to both sides separate the sections $TS = Ts = x$ and the ordinates SZ and sz; having done this, for the shape of our chord to the abscissa $AX = x$ the ordinate $XY = SZ - sz$ will correspond, since in our scale $AS = ct + x$ and $As = ct - x$; but here it is per se clear that for the endpoint A of the chord, where $x = 0$ and thus $TS = 0$, $Ts = 0$, the ordinate will be $= TV - TV = 0$. But for the other endpoint B, where $x = a$, and hence one has to take

$$TS = Ts = a,$$

the ordinates SZ and sz will be equal, since then the distance of the intervals AS and As is $2a$, as a logical consequence their difference will be $= 0$, yielding the the ordinate in the endpoint B for the chord; since hence this way for each time the shape of the chord is most easily assigned, the complete motion of the chord becomes perfectly known.

14. Since on the scale to the abscissas continuously growing by the interval $= 2a$ equal ordinates correspond, it is manifest that for another time t', if it was $ct' = ct + 2a$, the chord will have the same shape again it had at the time t; on the other hand, the chord is considered to have performed two vibrations in such a way that $t' - t$ exhibits the duration of two vibrations; but since $t' - t = \frac{2a}{c}$, the duration of one single vibration will be $= \frac{a}{c}$ and this already expressed in seconds, and here it most conveniently happens that the duration of each vibrations does not depend on the nature of the shapes, which the chord will have had while vibrating, but it always expressed by this very simple formula $\frac{a}{c}$. But having put $c^2 = \frac{2hk\pi}{K}$, it is now clear that the duration of one single vibration will be $\frac{a\sqrt{K}}{\sqrt{2hk\pi}}$ and so for each chord of the same density, for which $\frac{K}{k}$ obviously has the same value, the duration of the vibrations will be as $\frac{a}{\sqrt{\pi}}$; this means, they have the ratio composited from the simple length to the square root reciprocal of the straining forces, which ratio is certainly known for a long time and confirmed by experiments. Furthermore, since the sounds created by the chord are usually estimated from the number of vibrations of a given time, it will be helpful to have noted here that the number of vibrations created by our chord in each second will be $= \frac{c}{a} = \frac{\sqrt{2hk\pi}}{a\sqrt{K}}$ and so the sounds themselves have a ratio composited from the square root of the tensions π and the simple reciprocal of the length a.

15. If in the above formulas we put the time $t = 0$, we will find the shape the chord had initially, for which thus to the abscissa x the following ordinate corresponds:

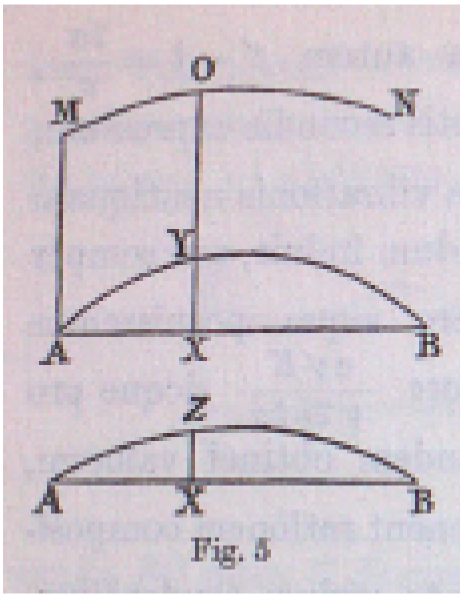
$$y = \varphi x - \varphi. - x,$$

whence vice versa from the initial shape of the chord the nature of the scale of construction can be concluded, but it is nevertheless not completely determined, the reason for which is per se obvious, since in the initial state, aside from the shape, also the motion, the chord could have had, must be considered, so that the initial state is contained in two conditions. Therefore, to take this circumstance into account, let us, starting from the general formulas, derive the velocity of the point Y, which will be

$$\left(\frac{dy}{dt}\right) = c\varphi'(ct+x) - c\varphi'(ct-x),$$

which exhibits the velocity of the point y moving away from the axis and the indicates the velocity in terms of spaces to be passed in one minute.

16. Therefore, to accommodate our whole investigation to the initial and known state of the chord, let (Fig. 5) AYB represent the



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initial shape of the chord and let us put the ordinate XY corresponding to the abscissa $x = \Gamma x$; further, for the initial motion over the same axis $AB = a$ construct the scale of the velocities AZB, each ordinate XZ of which shall exhibit the velocity, by which a point of the chord moves away from the axis, since which is a certain function of the abscissa $AX = x$, represent it by the function $\Delta' x$; thus, now it is necessary that for our general formulas

$$\varphi x - \varphi. - x = \Gamma x$$

and in like manner for the velocity, having put $t = 0$ there,

$$c\varphi' x - c\varphi'. - x = \Delta' x,$$

multiplying which last equation by dx and integrating we reduce it to this form

$$c\varphi' x - c\varphi'. - x = \Delta x + cf,$$

where Δx expresses the area of the curve AXZ, and so it will be

$$\varphi x + \varphi. - x = \frac{\Delta x}{c} + f = \text{area}\frac{AXZ}{c} + f.$$

Therefore, now from these two equations first we find

$$2\varphi x = \Gamma x + \frac{\Delta x}{c} + f$$

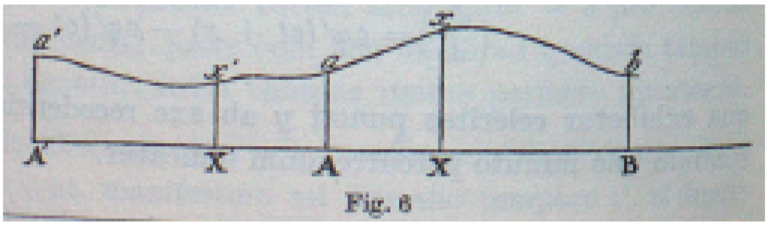
and

$$2\varphi. - x = \frac{\Delta x}{c} - \Gamma x + f.$$

To this end, in figure 5 over the axis AB describe the curve MON, by taking the first ordinate AM of arbitrary length $= f$, and for the abscissa x let

$$XO = f + \text{area}\frac{AXZ}{c}.$$

17. Now from figure 5 the scale of construction will easily be defined as follows:



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For, since $\varphi x = \frac{1}{2}XO + \frac{1}{2}XY$ and $\varphi. - x = \frac{1}{2}XO - \frac{1}{2}XY$, first, having taken $x = 0$ (Fig. 6), in A the ordinate will be $Aa = \frac{1}{2}AM$; further, having taken $AX = AX' = x$ to both sides, so that actually $AX' = -x$, for x the ordinate will be $Xx = \frac{1}{2}XO + \frac{1}{2}XY$ and for the other $X'x' = \frac{1}{2}XO - \frac{1}{2}XY$ and finally, having taken the abscissas $AB = a = AA'$ to both sides, the ordinate in the point B will be $= \frac{1}{2}BN$ and $A'a' = \frac{1}{2}BN$ and so the most outer ordinates Bb and $A'a'$ become equal and so the whole curve constructed over the base $A'B = 2a$ is completely constructed from the initial state of the chord; and nothing else remains except to replicate this curve $a'b$ so to the right as to the left arbitrarily often; and this way the whole scale of construction will be defined, from which thereafter the shape of the chord at each time will become known.

18. Therefore, after we had determined the motion of the chord, now we will be able to determine those forces $A\alpha$ and $B\beta$ mentioned initially, which are required to keep the endpoints A and B of the chord fixed and which were unknown. To this end, let us consider all forces acting on each point X of the chord separately. But above in paragraph 8 we saw that from the left the following forces act on the point Y: 1. the straining force $Aa = \pi$, 2. that unknown force $A\alpha = F$ and 3. all elementary forces applied to the portion Ay, since any of this is $= \frac{Kdx}{2hk} \cdot \frac{ddy}{dt^2}$, the sum of all of them will be

$$\frac{K}{2hk}\int \frac{dxddy}{dt^2} = \pi \int dx \frac{ddy}{dx^2},$$

the integral of which is

$$\pi \frac{dy}{dx} + \text{Const.},$$

which must be of such a nature that the integral vanishes for $x = 0$. Further, we even contemplated the moments of all these forces for the point Y; and the moment of the first force $Aa = \pi$ was πy, but the moment of the force F acting in the opposite direction was Fx, to which one must add the moment resulting from all elementary forces

$$= \frac{K}{2hk}\int dx \int \frac{dxddy}{dt^2} = \pi \int dx \left(\frac{dy}{dx} + \text{Const.}\right) = \pi(y + Cx).$$

Therefore, since we already found,

$$y = \varphi(ct + x) - \varphi(ct - x),$$

it will be

$$\frac{dy}{dx} = \varphi'(ct+x) + \varphi'(ct-x),$$

whence that sum of the elementary forces, because of

$$\frac{dy}{dx} + C = \varphi'(ct+x) + \varphi'(ct-x) - 2\varphi' ct,$$

will be

$$= \pi(\varphi'(ct+x) + \varphi'(ct-x) - 2\varphi' ct).$$

But hence the the moments will already be generated, the moment of the straining force will be

$$= \pi(\varphi(ct+x) - \varphi(ct-x)),$$

but the moment of the other force F will be $= Fx$; but the moment of all elementary forces

$$\pi(\varphi(ct+x) - \varphi(ct-x) - 2x\varphi' ct),$$

which last two act in the opposite direction, and since they have to cancel each other, this equation results

$$\begin{aligned} &\pi(\varphi(ct+x) - \varphi(ct-x)) \\ &= Fx + \pi(\varphi(ct+x) - \varphi(ct-x) - 2x\varphi' ct), \end{aligned}$$

whence manifestly that unknown force F is found to be $= 2\pi\varphi' ct$, to which nevertheless the force at the other endpoint $B\beta = G$ is not equal, since the ordinate y for the abscissa $Bx = a - x$ is expressed by another function. But we will be able to calculate this force G in like manner by the following reasoning, since the sum of all elementary forces applied throughout the whole chord Ay will be

$$\pi(\varphi'(ct+a) + \varphi'(ct-a) - 2\varphi' ct) - G,$$

since which must be $= 0$, we conclude that

$$G = \pi(\varphi'(ct+a) + \varphi'(ct-a) - 2\varphi' ct).$$

19. Up to this point, we have given the solution for the following general problem.

General Problem

Given the initial state of a chord, i.e. so the shape as the motion forced upon it, of uniform density, to define the shape the chord will have at a certain time, if its vibrations were infinitely small.

Indeed, it is perspicuous that the solution we gave is not only quite easy and nice, but also conform to the nature of the question, since it is accommodated to all initial states, while other solutions, which often appear, are restricted to certain species of curves, which the chord can describe during the vibration. But nobody can deny, that the initial state of the chord is completely arbitrary, and nobody has ever tried to show that a chord must always change into other species of curves, after initially other shapes were imprinted on it; even since D'Alembert did not perform this task, but rather said, that in those cases, in which initial shapes differing from those curves were forced upon the chord, the following motion can simply not be assigned at all by Analysis; I certainly immediately concede this concerning ordinary Analysis; and in this regard, I believe to have achieved a lot, since I have successfully applied this new branch of Analysis, which concerns functions of two variables, to the motion of chords. But to remove every doubt, I will expand several simple cases, which seem to contradict Analysis, and will further show that my solution not only agrees with experience but also is perfect agreement with all principles of motion.

First Case

If initially from the natural state AB the chord was pulled in such a way that it constitutes the isosceles triangle ADB (Fig. 7), the altitude CD of which is infinitely small, and is suddenly released from this position, that it begins the motion from the state of rest, to find its vibrational motion.

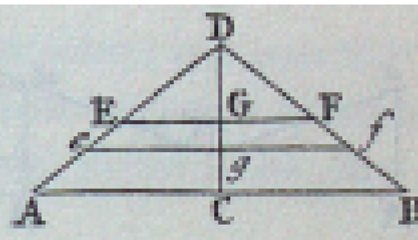


Fig. 7

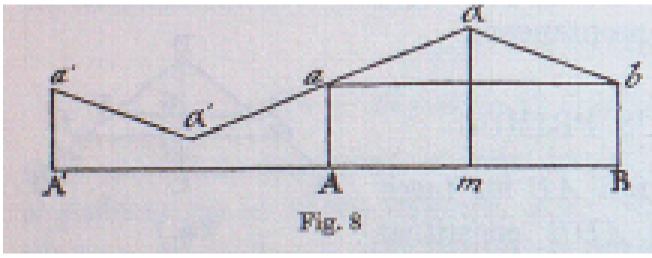
Scan taken from the Opera Omnia Version

EXPANSION

20. Therefore, since initially all elements of the chord have been at rest and each point at both sides AD and BD is not excited by any forces, since the tensions of the chords are the same everywhere and act in opposite directions, it is evident that at the first instant all these points do not have any motion at all, but will remain at rest; only the element at the vertex D, because of the tension of the chord, will be excited obliquely into both directions DA and DB; thus, hence a force acting in the direction DC will result, from which the point D will start to move in direction DC, while all remaining points of the chord are still at rest. But as soon as this point D started to move and it almost instantaneously got to G, now the points E and F will start to move, since the tensions to both sides around these points are not in an equilibrium anymore, while the remaining points from E to A and from F to B are even still at rest. But the points in the segment EGF, since they are not excited any longer, will remain in the acquired motion towards the axis AB, and so again, after an infinitely small amount of time, the shape $AefB$ will be forced upon the chord, and this way it will eventually get to the natural state AB; hence in like manner it will go over into the opposite position.

21. Anyone who wants to study this more carefully, will, without any doubt, be forced to concede that the chord will perform a motion of such a kind we just described. Therefore, let us see, a motion of what kind our general solution will yield; for, if it leads to the identical motion, there will certainly be no doubt that it is true. But if we find another and different motion from this, then it can justly be considered to be suspect.

22. Since all initial velocities vanish and so the scale of the velocities coincides with the axis AB, also those areas AXZ (Fig. 5) vanish and hence that line FVG will be a line parallel to the axis AB



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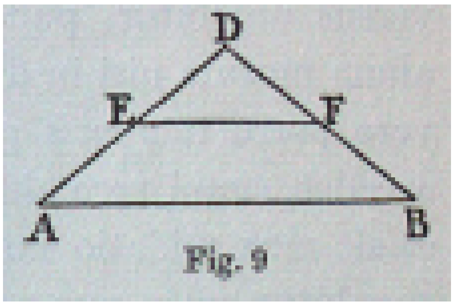
and removed from it by the arbitrary distance $AF = CG$; from this the scale of construction will be of the following nature. Over the principal segment AB (Fig. 8) there is an isosceles triangle $A\alpha b$, the altitude of which is just the half of the initial shape CD. But over the segment AA' to the left there is a similar isosceles triangle $a'\alpha' a$ in inverse position and this way on the right one has to construct triangles alternately pointing downwards and upwards over equally long segments and from this scale for $t = 0$ the initial shape results.

23. Since here one vibration has the duration $= \frac{a}{c}$, having taken the interval $AT = ct = a$, the point T will fall onto the point B, whence the shape of the chord will result similar to the initial one but just pointing in the opposite direction, as the nature of the vibrations demands it; but after the half of that the time has passed, i.e. for $t = \frac{a}{2c}$, that point T will be under the vertex in this scale of construction and here it is manifest that for the shape of the chord all ordinates must become zero or that in this moment the chord will pass through its natural state, which is not in any doubt.

24. But let us find the shape of the chord for the time $t = \frac{a}{4c}$ so that

$$AT = ct = \frac{a}{4} = \frac{1}{4}AB,$$

and as long as the abscissas x are smaller than $ATx = \frac{1}{4}a$, the ordinates y must continuously grow and indeed just as much as the grew in the initial figure from the point A; but as soon as the abscissa x becomes $= \frac{1}{4}a$, the ordinate y will become equal to the half of the altitude of the initial shape. Therefore, now take the abscissa x greater than $\frac{1}{4}a$, and it will easily become clear that the ordinates y are equal to each other and $= m\alpha - Aa$, which is the case up to $x = \frac{3}{4}a$, and so, throughout this segment



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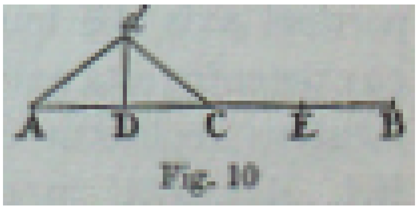
the portion of the chord will be parallel to the axis AB; but if $x > \frac{3}{4}a$, the ordinates will decrease uniformly again until the vanish at B. Thus, after the time $t = \frac{a}{4c}$, the chord will have the shape $AEFB$ (Fig. 9), as if the initial shape were ADB, so that the legs AD and BD are bisected in the points E and F and the portion EF is parallel to the axis; therefore, it is clear, as we remarked initially, that only the portion EF was moved up to this point, but the most outer portions AE and BF still remain at rest; since this motion was now clarified, the solidity of our general solution is strongly confirmed, so that now there seems to remain no doubt.

25. It is certainly immediately objected, since initially the chord was bent into the shape ADB, that the angle in the following vibrations becomes continuously wider; and this might even be motivated by experiments, which I will not deny by any means; but here one has to keep in mind that in our case the chord was assumed to be perfectly flexible so that it does not resist the bending into the triangle. But since there is no chord formed from any material with those properties, it is to attributed to this, if those angles become wider during the vibrations and this, even though in agreement with experiment, can not falsify our method such that it could be considered to be suspect.

Second Case

If initially not the whole chord AB but only the half of it AC (Fig. 10)



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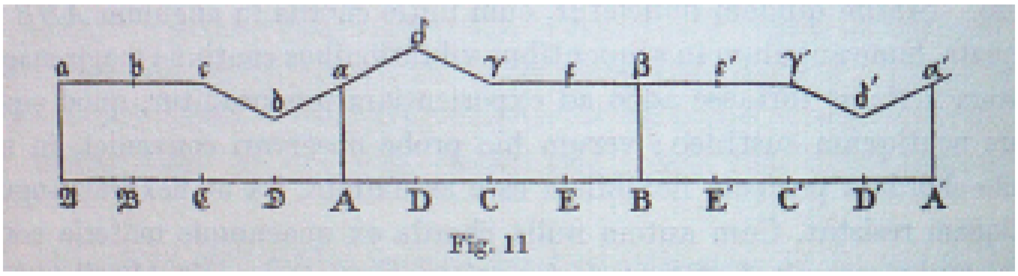
is bent into the isosceles triangle ACd, while the other part of the chord remains at rest, but then the chord is suddenly released from this state, to investigate the following vibrational motion.

Expansion

26. This case is even more remarkable, since not only the illustrious D'Alembert, but even others treating this subject, have not dared to tackle this case and believed it to be unsolveable by analysis. But before we take on its expansion, using heuristic reasoning, let us see, a motion of which kind must follow, and at first certainly, if the chord was initially at rest, it is evident that only the two points d and C are excited to move, since in all remaining points of the chord the tensions to both sides are in an equilibrium; thus, the point d will be forced towards the axis AB, but the point C upwards away from the axis, since it can only perform a motion normal to the axis. But in the following the continuously larger portion of the triangle AdC will move towards the axis, but at the same time the portion Cd of the other part will be elevate over the axis, and so soon the wave AdC moving forwards will be carried towards the other point B; since we can not deny this, let us see, what kind of motion our solution must produce.

27. Since at the first instant the chord starts its motion from the state of rest, the scale of construction will be of such a nature as seen in figure 11; over the segment AB, to which the chord is referred, there is the scale $\alpha\beta\gamma\delta\varepsilon$, completely similar to the initial shape,



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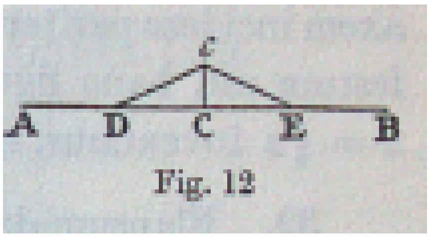
consisting of the isosceles triangle $\alpha\beta\gamma$ and the remaining straight portion $\gamma\beta$, just with the difference that here the triangle, concerning the altitude, is twice as small as in the initial figure; but to the left the same figure is to be drawn in inverse position, so that the triangle $\alpha\mathfrak{b}\mathfrak{C}$, pointing downwards, is over the portion $A\mathfrak{C}$, while the remaining straight part $\mathfrak{ca}$ is parallel to the axis; having done this, repeat this figure $\mathfrak{acb}\cdots\beta$ over the axis $\mathfrak{A}B = 2a$ to the right arbitrarily far; further, split each of the segments $\mathfrak{A}A$, AB, BA' into four parts, so that we can investigate the following states of the chord more easily.

28. Since the duration of one vibration was found above to be $= \frac{a}{c}$, consider the length $AB = a$ of the chord to exhibit the duration of one single vibration, and hence those four parts expressed in the figure will also exhibit one fourth of the whole duration of one vibration and hence let us us explore the shape of the chord after the time $\frac{1}{4}a$, $\frac{1}{2}a$, $\frac{3}{4}a$ and a has passed.

29. Therefore, for the time $= \frac{1}{4}a$ the ordinate TV (Fig. 4) is to be taken in the point D, and since for the abscissa $x = 0$ always the ordinate in the chord also is $= 0$, for the abscissa $x = \frac{1}{4}a = AD$ the ordinate will be $C\gamma - A\alpha = 0$, whence it is plain that the chord from A and D must fall onto the axis (Fig. 12). But then take $x = \frac{1}{2}a = AC$ and in the relation scale the ordinates, equally far away from TV, will be $E\varepsilon$ and $\mathfrak{D}\mathfrak{d}$; hence for the shape of the chord



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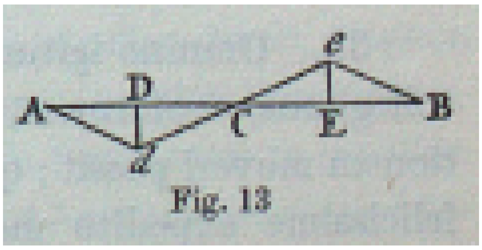
the ordinate will be $E\varepsilon = \mathfrak{D}\mathfrak{d} = Cc$, so that now Dc is a portion of the chord; further, let us take $x = \frac{3}{4}a = AE$ and in the scale of relation from the point D to both sides take as long segments $DB = D\mathfrak{C} = \frac{3}{4}a$ and now our ordinate will be $B\beta - \mathfrak{C}\mathfrak{c} = 0$ so that $x = a$ in the scale of relation from the point D to both sides the segments $DE' = D\mathfrak{B} = a$ are separated and now the difference of the ordinates in these points is $E'\varepsilon' - \mathfrak{B}\mathfrak{b} = 0$; therefore, we learn that after the time $\frac{1}{4}a$ the shape of the chord will be of such a nature that to both sides throughout the segments AD and BE the chord lies on the axis; but throughout the segment DE it forms the isosceles triangle DcE pointing up, the altitude Cc of which will be twice as small as in the initial state, so that the undulation has been moved forward to the segment DE.

30. Let the time $\frac{1}{2}a$ have passed and now the ordinate TV will correspond to the point C, from which point successively to both sides segments of the length $\frac{1}{4}a$, $\frac{1}{2}a$, $\frac{3}{4}a$ and a are to be taken and for the first $x = \frac{1}{4}a$ in the relation scale we will have the ordinates $E\varepsilon$ and $D\delta$, the difference $E\varepsilon - D\delta$ of which



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becomes negative or for the abscissa $x = \frac{1}{4}a$ it shows an ordinate pointing downwards (Fig. 13), which again will be twice as small as in the initial figure. Now take $x = AC = \frac{1}{2}a$ and in the scale of construction one will have the two ordinates $B\beta$ and $A\alpha$, the difference of which vanishes, so that in the point C of the chord the ordinate vanishes again, so that here the isosceles triangle below the axis points downwards. Further, having taken $x = \frac{3}{4}a$, in the scale of construction one will have the ordinate $E'\varphi'$ and $\mathfrak{D}\mathfrak{d}$, the difference of which will be positive and, concerning the altitude, will be twice as small as in the initial triangle. Therefore, after the time $= \frac{1}{2}a$ the chord will be bent into two isosceles triangles, the first pointing down, the second pointing up.

31. Now let the time $\frac{3}{4}a$ have passed and now the ordinate TV is to be placed in E;

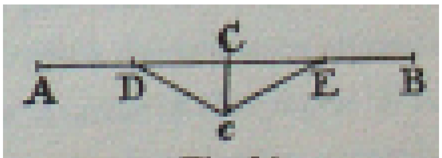

Fig. 14

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hence for the shape of the chord first take $x = \frac{1}{4}a$ and in the scale of construction we will have the two ordinates $B\beta$ and $C\gamma$, the difference $= 0$ of which shows that the segment AD of the chord will lie on the axis (Fig. 14); further take $x = \frac{1}{2}a = AC$, and on the scale of construction one will have the two ordinates $E'\varepsilon$ and $D\delta$, the negative difference of which will become equal to the half of the altitude of the triangle in the initial figure. But having taken $x = \frac{3}{4}a$ the two ordinates on the scale will be $C'\gamma'$ and $A\alpha$, the difference of which $= 0$ shows that the chord lies on the axis in the point E as the the whole segment EB and so it is manifest that this shape is completely equal to that which found for the passed time $x = \frac{1}{4}$, except that this has the inverse position.

32. Finally, let a time $= a$ have passed from the beginning such that here the shape of the chord around the end of the first vibration is to be defined and now that ordinate TV will fall on B, from which we want to separate the segments $\frac{1}{4}a$, $\frac{1}{2}a$ and $\frac{3}{4}a$ in both directions. Therefore, now for $x = \frac{1}{4}a$ on the scale we will have the two ordinates $E'\varepsilon'$ and $E\varepsilon$,

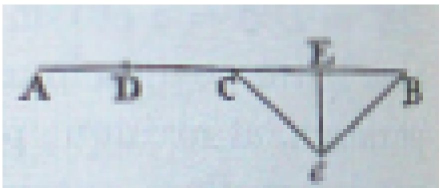

Fig. 15

Scan taken from the Opera Omnia Version

the difference of which $= 0$ yields $y = 0$ for the chord (Fig. 15). Now let $x = \frac{1}{2}a$ and the ordinates on the scale will be $C'\gamma'$ and $C\gamma$, the difference of which $= 0$ again yields $y = 0$ such that now the whole first half of the chord falls onto the axis. But having taken $x = \frac{3}{4}a$ the two ordinates of the scale will be $D'\delta'$ and $D\delta$, the difference of which is negative and equal to the altitude of the triangle in the initial state; from this it is plain that this shape of the chord is completely similar and equal o the initial figure, except that its position is inverse; and since now the first vibration has been defined, also the following motion will be understood per se such that it would be superfluous to prosecute these determinations any further.

33. Therefore, in total our expansion shows a motion in agreement to that one which we conjectured such that hence there can not be raised any more doubt against this solution; hence since that case seems to be most counterintuitive, having expedited it so luckily I think to have put my theory about the vibrating chords beyond any doubts abundantly; and hence I even hope that in the future all objections will be answered sufficiently such that it would be superfluous to expand still more cases in similar manner.