Observations on harmonic Progressions *

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§1 Under the name of harmonic progressions all series of fractions are understood, whose numerators are equal to each other but whose denominators on the other hand constitute an arithmetic progression. Therefore, a general series of this kind is

$$\frac{c}{a}, \quad \frac{c}{a+b}, \quad \frac{c}{c+2b}, \quad \frac{c}{a+3b} \quad \text{etc.}$$

For, each three contiguous terms, as

$$\frac{c}{a+b}, \quad \frac{c}{a+2b}, \quad \frac{c}{a+3b},$$

have this property that the differences of the outer term from the middle term are proportional to the outer terms themselves. Of course, it is

$$\frac{c}{a+b}-\frac{c}{a+2b}:\frac{c}{a+2b}-\frac{c}{a+3b}=\frac{c}{a+c}:\frac{c}{a+3b}.$$

But because this is the property of the harmonic proportion, series of fractions of this kind were called *harmonic progressions*. They could also have been called reciprocals of first order, since in the general term $\frac{c}{a+(n-1)b}$ the index n has one, more precisely one negative, dimension.

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§2 Although in these series the terms continuously decrease, the sum of an infinite series of this kind is nevertheless always infinite. To demonstrate this no method to sum these series is necessary, but the truth of the statement will easily be seen using the following principle. An infinite series, which has a finite sum, even if it is extended twice as far, will not become larger, but everything, what is added after the infinitesimal term, will in reality be infinitely small. For, if this would not be the case, the sum of the infinite series would not be determined and therefore not finite. Hence it follows, if that, what results from the continuation beyond the infinitesimal term, is of finite magnitude, that the sum of the series must necessarily be infinite. Therefore, applying this principle we will be able to decide, whether the sum of a given series is infinite or finite.

§3 Therefore, let the series

$$\frac{c}{a}, \quad \frac{c}{a+b}, \quad \frac{c}{a+2b} \quad \text{etc.}$$

be continued to infinity and let the infinitesimal term be $\frac{c}{a+(i-1)b}$, while i is an infinite number, which is the index of this term. Now, continue this series further from the term $\frac{c}{a+ib}$ to the term $\frac{c}{a+(ni-1)b}$ corresponding to the index ni. Therefore, the number of additionally added terms is $(n-1)i$. But their sum will be smaller than

$$\frac{(n-1)ic}{a+ib},$$

but larger than

$$\frac{(n-1)ic}{a+(ni-1)b}.$$

But because i is infinitely large, a can be neglected in each of both denominators. Hence the sum will be greater than

$$\frac{(n-1)c}{nb},$$

but smaller than

$$\frac{(n-1)c}{b}.$$

Hence it is perspicuous that this sum is finite and as a logical consequence the sum of the propounded series $\frac{c}{a}$, $\frac{c}{a+b}$ etc., if it is continued to infinity, is infinitely large.

§4 But closer boundaries of this sum of the terms from i to ni are found using the following properties of the harmonic proportion. Of course, every harmonic proportion is of such nature that the middle term is smaller than the third part of the sum of all three. Therefore, the term in the middle of $\frac{c}{a+ib}$ and $\frac{c}{a+(ni-1)b}$, which is $\frac{c}{a+\frac{ni+i-1}{2}b}$, multiplied by the number of terms $(n-i)i$ or

$$\frac{(n-1)ic}{a+\frac{ni+i-1}{2}b}$$

will be smaller than the sum of the terms. Or the sum of the terms hence will be greater than

$$\frac{2(n-1)c}{(n+1)b}$$

because of the infinite i. Furthermore, the arithmetic mean of the most outer terms is greater than the third part of the sum of the terms. Hence it follows that also in the harmonic series the sum of terms will be smaller than $(n-1)i$ times the arithmetic mean of the most outer terms, which is

$$\frac{(2a+(ni+i-1)b)c}{2(a+ib)(a+(ni-1)b)} \quad \text{or} \quad \frac{(n+1)c}{2nib}.$$

Hence the sum will be smaller than

$$\frac{(n^2-1)c}{2nb},$$

so that these two boundaries are

$$\frac{2(n-1)c}{(n+1)b} \quad \text{and} \quad \frac{(n^2-1)c}{2nb}$$

and hence the sum approximately is

$$=\frac{(n-1)c}{b\sqrt{n}},$$

which is the geometric mean of the boundaries.

§5 From these things is it possible to conclude, in which cases this more general series

$$\frac{c}{a}, \quad \frac{c}{a+b}, \quad \frac{c}{a+2^\alpha b} \quad \text{etc. to infinity to} \quad \frac{c}{a+i^\alpha b}$$

has a finite or infinite sum. For, let $(n-1)i$ terms follow after the last term and the sum of these will be smaller than

$$\frac{(n-1)c}{i^{\alpha-1}b},$$

but greater than

$$\frac{(n-1)c}{n^\alpha i^{\alpha-1}b}.$$

Hence, if α was a number greater than 1, the sum of these following terms will be $=0$ and therefore the sum of the progression will be finite. But if it is $\alpha<1$, the sum of the following terms will be infinite, which is why the sum of the progression itself will be infinite of an infinitely larger degree. Therefore, among these progressions only the harmonic progression, in which it is $\alpha=1$, has this property that its sum, if continued to infinity, is infinitely large but the sum of the terms following after the infinitesimal term on the other hand is finite.

§6 But I investigate in the following way, how large the sum of terms from the term of the index i to the term of the index ni is. Put the sum of the series

$$\frac{c}{a}, \quad \frac{c}{a+b}, \quad \cdots \quad , \frac{c}{a+(i-1)b}$$

up to the term of the index $i=s$, which is a quantity to be determined from a, b, c and i. Let i grow by 1 and s will be augmented by the following term $\frac{c}{a+ib}$. Hence it will be

$$di:ds = 1:\frac{c}{a+ib} \quad \text{or} \quad ds=\frac{cdi}{a+ib}.$$

Hence one finds

$$s = C+\frac{c}{b}\log(a+ib),$$

while C denotes a certain constant quantity. But it is also clear from this form that the sum of the same series continued from the beginning to the term of the index ni will be

$$= C + \frac{c}{b}\log(a + nib).$$

Therefore, the difference of these sums

$$\frac{c}{b}\log\frac{a+nib}{a+ib} = \frac{c}{b}\log n \quad (\text{while } a \text{ vanishes})$$

will give the sum of the terms from $\frac{c}{a+ib}$ to $\frac{c}{a+nib}$. But because we assigned the boundaries of this sum above, $\frac{c}{b}\log n$ will be greater than $\frac{2(n-1)c}{(n+1)b}$ and smaller than $\frac{(n^2-1)c}{2nb}$, or

$$\log > \frac{2(n-1)}{n+1} \quad \text{and} \quad \log n < \frac{n^2-1}{2n}.$$

§7 Below we will show that that quantity C is finite and we will try to determine it. Therefore, C can be neglected in the sum and the sum of the progression

$$\frac{c}{a}, \quad \frac{c}{a+b}, \quad \cdots \quad \frac{c}{a+(i-1)b},$$

while the number of terms is infinite $= i$, will become

$$= \frac{c}{b}\log(a+ib) = \frac{c}{b}\log i.$$

Therefore, the sum will approximately be the logarithm of the number of terms and hence infinitely smaller than the root of arbitrary large power of the number of terms; nevertheless the sum is infinitely large.

§8 From this consideration innumerable series result which can be used to express the logarithms of certain numbers. At first, let us take this harmonic progression

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{etc.},$$

for which it is $a = 1$, $b = 1$, $c = 1$. Therefore, the difference of this series

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{i}$$
continued to the term of the index i to the same
$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{ni}$$
continued to the term of the index ni will be $=\log n$. Hence that series subtracted from this one gives $\log n$. But since the number of terms of the second series is n times greater than the number of the first, from n terms of the series
$$1+\frac{1}{2}+\cdots+\frac{1}{ni}$$
one has to subtract one of the other series
$$1+\frac{1}{2}+\cdots+\frac{1}{i},$$
that the subtraction to infinity can be done uniformly. Hence it will be
$$\begin{array}{l}\log n = 1+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{2n}+\frac{1}{2n+1}+\cdots+\frac{1}{3n}+\text{etc.}\\ \qquad\qquad\qquad\qquad -1 \qquad\qquad\qquad\qquad -\frac{1}{2} \qquad\qquad\qquad\qquad -\frac{1}{3}\end{array}$$
Therefore, if the single terms of the lower series are actually subtracted from the terms written over them of the upper series and the integer numbers 2, 3, 4 $\cdots$ etc. are substituted for n, we will successively obtain the following series of logarithms:
$$\begin{aligned}
\log 2 &= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\text{etc.},\\
\log 3 &= 1+\frac{1}{2}-\frac{2}{3}+\frac{1}{4}+\frac{1}{5}-\frac{2}{6}+\frac{1}{7}+\frac{1}{8}-\frac{2}{9}-\frac{1}{10}+\frac{1}{11}-\frac{2}{12}+\text{etc.},\\
\log 4 &= 1+\frac{1}{2}+\frac{1}{3}-\frac{3}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}-\frac{3}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}-\frac{3}{12}+\text{etc.},\\
\log 5 &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{4}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}-\frac{4}{10}+\frac{1}{11}+\frac{2}{12}+\text{etc.},\\
\log 6 &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{5}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}-\frac{5}{12}+\text{etc.}
\end{aligned}$$

etc.,

Hence a convergent series is easily found for the logarithm of each number.

§9 From these series others of the same form, which have a rational sum, can be derived. For, since the double of the series $= \log 2$ is $\log 4$, if the series

$$1 + \frac{1}{2} + \frac{1}{3} - \frac{3}{4} + \text{etc.}$$

is subtracted from this one

$$2 - \frac{2}{2} + \frac{2}{3} - \frac{2}{4} + \text{etc.},$$

the remainder, namely this series

$$1 - \frac{3}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{3}{6} + \text{etc.},$$

will be $= 0$, or

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{3}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{3}{10} + \text{etc.}$$

Similarly, if the series exhibiting $\log 6$ is subtracted from the sum of the series exhibiting $\log 2$ and $\log 3$, the residue, namely

$$1 - \frac{1}{2} - \frac{2}{3} - \frac{1}{4} + \frac{1}{5} + \frac{2}{6} + \frac{1}{7} - \frac{1}{8} - \frac{2}{9} - \frac{1}{10} + \text{etc.},$$

will be $= 0$, or

$$1 = \frac{1}{2} + \frac{2}{3} + \frac{1}{4} - \frac{1}{5} - \frac{2}{6} - \frac{1}{7} + \frac{1}{8} + \frac{2}{9} + \frac{1}{10} - \text{etc.}$$

In like manner, one will be able to find innumerable other series of this kind.

§10 Those series expressing the logarithms certainly converge, but very slowly; therefore, in order to find the logarithms conveniently using those series, a certain artifice is required. To find this artifice it must be noted that these series do not proceed uniformly, but have certain revolutions, which are absolved in so many terms as n has units; therefore, I will call that many terms taken simultaneously one member of the series. So in the series for $\log 2$ two terms will constitute one member, in the series for $\log 3$ three, in the series for $\log 4$ and so forth. Therefore, the members will constitute an equal series and to find logarithms it is necessary to add several members. For, let us put that m members have already been added to find the logarithm of two and instead of all the following ones one will be able to add $\frac{1}{4m}$, which will come the closer to the truth, the greater the number m was. To find $\log 3$ add $\frac{1}{9m}$ to m already added members instead of all the following ones. In like manner for $\log 4$ one must add $\frac{1}{16m}$ and so forth. These remarks follow from the summation method applied in § 6; since in this method m must be a very large quantity, I neglected the numbers added to m in the differential so that the integration does not depend on logarithms.

§11 But to determine the sum, even though it is infinite, of the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{i}$ accurately, I express the single terms in the following way. It is

$$1 = \log 2 + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \text{etc.}$$

and

$$\begin{aligned}
\frac{1}{2} &= \log\frac{3}{2} + \frac{1}{2\cdot 4} - \frac{1}{3\cdot 8} + \frac{1}{4\cdot 16} - \frac{1}{5\cdot 32} + \text{etc.},\\
\frac{1}{3} &= \log\frac{4}{3} + \frac{1}{2\cdot 9} - \frac{1}{3\cdot 27} + \frac{1}{4\cdot 81} - \frac{1}{5\cdot 243} + \text{etc.},\\
\frac{1}{4} &= \log\frac{5}{4} + \frac{1}{2\cdot 16} - \frac{1}{3\cdot 64} + \frac{1}{4\cdot 256} - \frac{1}{5\cdot 1024} + \text{etc.}\\
&\vdots\\
\frac{1}{i} &= \log\frac{i+1}{i} + \frac{1}{i^2} - \frac{1}{3\cdot i^3} + \frac{1}{4\cdot i^4} - \frac{1}{5\cdot i^5} + \text{etc.}
\end{aligned}$$

Having added these series it will result

$$\begin{aligned}
1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{i} = \log(i+1) &+ \frac{1}{2}\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\text{etc.}\right)\\
&- \frac{1}{3}\left(1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\text{etc.}\right)\\
&+ \frac{1}{4}\left(1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\text{etc.}\right)\text{etc.}
\end{aligned}$$

Since these series are convergent, if they are summed approximately, it will result

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{i}=\log(i+1)+0.577218.$$

If the sum is called s, it will be, as we did it above,

$$ds=\frac{di}{i+1} \quad \text{and hence} \quad s=\log(i+1)+C.$$

Therefore, we detected the value of this constant C, which is $C=0.577218$.

§12 If the series $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{i}$ is continued further to infinity and is subdivided into members, of which each as the series itself contains i terms, the member contained within $\frac{1}{i}$ and $\frac{1}{2i}$ will be $=\log 2$, the following $=\log\frac{3}{2}$, the third $=\log\frac{4}{3}$ etc. And because the series of the sum itself is the logarithm of infinity, one can analogously put $\log\frac{1}{0}$. And this way we will obtain the following rather curious scheme:

$$\begin{array}{l|c|c|c|c|c}
\text{Series} & 1+\frac{1}{2}+\cdots+\frac{1}{i} & +\cdots+\frac{1}{2i} & +\cdots+\frac{1}{3i} & +\cdots+\frac{1}{4i} & +\cdots+\frac{1}{5i} \\
\text{Sums} & \log\frac{1}{0} & \log\frac{2}{1} & \log\frac{3}{2} & \log\frac{4}{3} & \log\frac{5}{4}
\end{array} \quad \text{etc.}$$

§13 It might certainly seem to be difficult to find these same properties of harmonic and logarithmic expressions analytically and in the same way I did elsewhere to sum series[1]. But everyone considering the subject with more attention will easily see that this is not only possible but even possible in much more generality. For, I consider not the simple harmonic progression but the one connected to a geometric progression, namely this one

$$\frac{cx}{a}+\frac{cx^2}{a+b}+\frac{cx^3}{a+2b}+\frac{cx^4}{a+3b}+\text{etc.}$$

I put its sum s and having multiplied both expressions by $bx^{\frac{a-b}{b}}$ it will be

$$bx^{\frac{a-b}{b}}s=\frac{bcx^{\frac{a-b}{b}}}{a}+\frac{bcx^{\frac{a+b}{b}}}{a+b}+\frac{bcx^{\frac{a+2b}{b}}}{a+2b}+\text{etc.}$$

[1] Euler refers to E20 and E25 again.

And having taken the differentials one will have

$$bD.x^{\frac{a-b}{b}}s = dx\left(cx^{\frac{a-b}{b}} + cx^{\frac{a}{b}} + cx^{\frac{a+b}{b}} + \text{etc.}\right) = \frac{cx^{\frac{a-b}{b}}dx}{1-x}.$$

Having taken integrals again it will be

$$bx^{\frac{a-b}{b}}s = c\int \frac{x^{\frac{a-b}{b}}dx}{1-x}$$

and

$$s = \frac{c}{bx^{\frac{a-b}{b}}}\int \frac{x^{\frac{a-b}{b}}dx}{1-x}.$$

From this series I now subtract this one

$$\frac{fx^m}{g} + \frac{fx^{2m}}{g+h} + \frac{fx^{3m}}{g+2h} + \text{etc.},$$

whose sum we want to call t. Multiply it by

$$\frac{h}{m}x^{\frac{m(g-h)}{h}};$$

it will be

$$\frac{h}{m}x^{\frac{m(g-h)}{h}}t = \frac{fhx^{\frac{mg}{h}}}{mg} + \frac{fhx^{\frac{m(g+h)}{h}}}{m(g+h)} + \frac{fhx^{\frac{m(g+2h)}{h}}}{m(g+2h)} + \text{etc.}$$

And having taken differentials it will be

$$\frac{h}{m}D.x^{\frac{m(g-h)}{h}}t = dx\left(fx^{\frac{mg-h}{h}} + fx^{\frac{m(g+h)-h}{h}} + fx^{\frac{m(g+2h)-h}{h}} + \text{etc.}\right) = \frac{fx^{\frac{mg-h}{h}}dx}{1-x^m}.$$

Hence one will have

$$t = \frac{fm}{hx^{\frac{m(g-h)}{h}}}\int \frac{x^{\frac{mg-h}{h}}dx}{1-x^m}.$$

And hence

$$s - t = \frac{c}{bx^{\frac{a-b}{b}}}\int \frac{x^{\frac{a-b}{a}}dx}{1-x} - \frac{fm}{hx^{\frac{m(g-h)}{h}}}\int \frac{x^{\frac{mg-h}{h}}dx}{1-x^m}.$$

But this subtraction has to be done in such a way that from the term of the index m of the series s the first term of the series t is subtracted and from the term of the index $2m$ of that series the second of this series and so forth.

§14 To find our logarithmic series, let it be $a = b$ and $g = h$. Having done this it will be

$$s = \frac{c}{b}\int\frac{dx}{1-x} = \frac{c}{b}\log\frac{1}{1-x}$$

and

$$t = \frac{f}{h}\int\frac{mx^{m-1}dx}{1-x^m} = \frac{f}{h}\log\frac{1}{1-x^m}.$$

Therefore

$$s - t = \log\frac{(1-x^m)^{\frac{f}{h}}}{(1-x)^{\frac{c}{b}}}.$$

But in order for this expression to become finite for $x = 1$, it must be $\frac{f}{h} = \frac{c}{b}$; therefore, let all these letters become $= 1$ and it will be

$$s - t = \log\frac{1-x^m}{1-x} = \log(1 + x + x^2 + \cdots + x^{m-1}).$$

This expression gives the difference of these series

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \text{etc.} \quad \text{and} \quad \frac{x^m}{1} + \frac{x^{2m}}{2} + \frac{x^{3m}}{3} + \text{etc.}$$

Hence, if it is $m = 2$, it will be

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{etc.};$$

if it is $m = 3$, it will be

$$\log(1+x+x^2) = x + \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} - \frac{2x^6}{6} + \text{etc.}$$

and in like manner

$$\log(1+x+x^2+x^3) = x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{3x^4}{4} + \text{etc.}$$

If in these expressions it is $x = 1$, the same series for the logarithms of natural numbers as those we gave before [§ 8] will result.

§15 If it is $h = 2g$, it will be

$$t = \frac{fx^{\frac{m}{2}}}{h} \int \frac{mx^{\frac{m-2}{2}}dx}{1 - x^m}.$$

Put $x^m = y$; it will be

$$t = \frac{f\sqrt{y}}{h} \int \frac{dy}{(1-y)\sqrt{y}} = \frac{f\sqrt{y}}{h} \log \frac{1+\sqrt{y}}{1-\sqrt{y}} = \frac{fx^{\frac{m}{2}}}{h} \log \frac{1+x^{\frac{m}{2}}}{1-x^{\frac{m}{2}}}.$$

Furthermore, if it is $a = b$, it will be

$$s = \frac{c}{b} \log \frac{1}{1-x}.$$

But s is the sum of this series

$$\frac{cx}{a} + \frac{cx^2}{2a} + \frac{cx^3}{3c} + \text{etc.}$$

and

$$tx^{\frac{-m}{2}} = \frac{f}{h} \log \frac{1+x^{\frac{m}{2}}}{1-x^{\frac{m}{2}}}$$

gives this series

$$\frac{fx^{\frac{m}{2}}}{g} + \frac{fx^{\frac{3m}{2}}}{3g} + \frac{fx^{\frac{5m}{2}}}{5g} + \text{etc.}$$

Let $a = 1$ and $g = 1$; it will be

$$s - tx^{\frac{-m}{2}} = c \log \frac{1}{1-x} - \frac{f}{2} \log \frac{1+x^{\frac{m}{2}}}{1-x^{\frac{m}{2}}} = \log \frac{\left(1 - x^{\frac{m}{2}}\right)^{\frac{f}{2}}}{(1-x)^c \left(1 + x^{\frac{m}{2}}\right)^{\frac{f}{2}}}.$$

In order for this expression to become finite, if it is $x = 1$, it is necessary that it is $\frac{f}{2} = c$ or $f = 2c$. Therefore, let $c = 1$ and $m = 2n$; the difference of the series

$$x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\text{etc.}$$

and

$$\frac{2x^n}{1}+\frac{2x^{3n}}{3}+\frac{2x^{5n}}{5}+\text{etc.}$$

will be

$$=\log\frac{1-x^n}{(1-x)(1+x^n)}.$$

Put $n=2$; the difference will be $=\log\frac{1+x}{1+x^2}$ and for $x=1$ it will be $=0$, whence this series

$$1-\frac{3}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{3}{6}+\frac{1}{7}+\text{etc.}$$

will be $=0$, as we already found above [§ 9].

§16 One can now find infinitely many other series of this kind having a rational sum from this form $\log\frac{1+x}{1+xx}$ by assuming other similar forms, which vanish for $x=1$. For, from this form $\log\frac{1+x}{1+x^2}$, if it is expressed by means of a series, the found series itself immediately results. For, it is

$$\log(1+x)=\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\text{etc.}$$

and

$$\log(1+x^2)=\frac{x^2}{1}-\frac{x^4}{2}+\frac{x^6}{3}-\frac{x^8}{4}+\frac{x^{10}}{5}-\text{etc.}$$

Therefore, this series subtracted from the upper gives this series

$$\frac{x}{1}-\frac{3x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}-\frac{3x^6}{6}+\text{etc.},$$

whose sum will be $\log\frac{1+x}{1+x^2}$. In like manner, $\log\frac{1+x}{1+x^3}$ will give this series

$$\frac{x}{1}-\frac{x^2}{2}-\frac{2x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}+\frac{2x^6}{6}+\frac{x^7}{7}-\frac{x^8}{8}-\frac{2x^9}{9}-\text{etc.}$$

Therefore, having put $x=1$ it will be

$$0 = 1 - \frac{1}{2} - \frac{2}{3} - \frac{1}{4} + \frac{1}{5} + \frac{2}{6} + \frac{1}{7} - \frac{1}{8} - \frac{2}{9} - \text{etc.},$$

which same series we found already in § 9.

§17 In this way one will be able to find the sums of all irregular series of this kind which nevertheless proceed regularly in members; for, they are always to be considered as the difference of two series. Let, e.g., this series be propounded

$$1 - \frac{2}{2} + \frac{1}{3} + \frac{1}{4} - \frac{2}{5} + \frac{1}{6} + \frac{1}{7} - \frac{2}{8} + \text{etc.}$$

This is the difference of the series

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \text{etc.}$$

and

$$\frac{3x^2}{2} + \frac{3x^5}{5} + \frac{3x^8}{8} + \text{etc.}$$

for $x = 1$. But the sum of that series is $\log \frac{1}{1-x}$, the sum of the first on the other hand is $\int \frac{3xdx}{1-x^3}$ or

$$\log \frac{1}{1-x} + \frac{1}{2}\log(x^2+x+1) + \frac{\sqrt{-3}}{2}\log\frac{2x+1-\sqrt{-3}}{2x+1+\sqrt{-3}} - \frac{\sqrt{-3}}{2}\log\frac{1-\sqrt{-3}}{1+\sqrt{-3}}.$$

Therefore, having subtracted this one from that one and having put $x = 1$

$$-\frac{1}{2}\log 3 + \frac{\sqrt{-3}}{2}\log\frac{3+\sqrt{-3}}{3-\sqrt{-3}} - \frac{\sqrt{-3}}{2}\log\frac{1+\sqrt{-3}}{1-\sqrt{-3}}$$

will result for the sum of the propounded progression. But $\frac{\sqrt{-3}}{2}\log\frac{3+\sqrt{-3}}{3-\sqrt{-3}}$ is indeed the circumference of the circle divided by $\sqrt{3}$ having put the diameter $= 1$ and $\frac{\sqrt{-3}}{2}\log\frac{1-\sqrt{-3}}{1+\sqrt{-3}}$ is its half.

§18 But even if the members themselves do not enter the series uniformly, the sum is assigned without any difficulty. Let us take this series

$$1-\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{2}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}-\frac{3}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}-\frac{1}{14}+\text{etc.}$$

This is the difference of these series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{i\cdot\frac{i+3}{2}}$$

and

$$\frac{2}{2}+\frac{3}{5}+\frac{4}{9}+\frac{5}{14}+\cdots+\frac{i+1}{i\cdot\frac{i+3}{2}}$$

continued to infinity in such a way that the most outer terms have the same denominator $i\cdot\frac{i+3}{2}$. But the sum of the first of these series is $C+\log i+\log(i+3)-\log 2$, where C denotes the constant found in § 11, namely 0.577218. The other series which is to be subtracted is resolved into these two

$$\frac{2}{3}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{i}\right)$$

and

$$\frac{4}{3}\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots+\frac{1}{i+3}\right)$$

The sum of the first series is $\frac{2}{3}C+\frac{2}{3}\log i$, the sum of the second series on the other hand is $\frac{4}{3}C-\frac{22}{9}+\frac{4}{3}\log(i+3)$. These two, subtracted from that sum $C+\log i+\log(i+3)-\log 2$, give $-C+\frac{22}{9}-\log 2$ or approximately 1.174078 for the sum of the propounded series.