A new way to express irrational quantities approximately*

Leonhard Euler

§1 It is known that every simple irrational quantity can be reduced to this form $(1+x)^n$, if the exponent n is assumed to denote an arbitrary fractional number, of course; for, whatever number N, which is to be raised to the fractional exponent $n=\frac{\mu}{\nu}$, is given, it is always possible to reduce it to this form

$$a^\nu + b,$$

whence the given formula becomes

$$(a^\nu + b)^{\frac{\mu}{\nu}} = a^\mu \left(1+\frac{b}{a^\nu}\right)^{\frac{\mu}{\nu}};$$

and so the irrationality is contained in the expression $\left(1+\frac{b}{a^\nu}\right)^{\frac{\mu}{\nu}}$, which is identical to the given form $(1+x)^n$, if one puts $\frac{b}{a^\nu}=x$ and $\frac{\mu}{\nu}=n$. And if we want to admit fractions for a and take b positively and negatively, the quantity $\frac{b}{a^\nu}$ can be made sufficiently small this way in each case, whence even the formula $(1+x)^n$ is converted into a rapidly converging series in usual manner.

*Original Title: "Nova ratio quantitates irrationales proxime exprimendi", first published in: *Novi Commentarii academiae scientiarum Petropolitanae, Volume 18* (1774, written 1772): pp. 136 – 170, reprint in: Opera Omnia: Series 1, Volume 7, pp. 316 – 349, Eneström Number E450, translated by: Alexander Aycock for the "Euler-Kreis Mainz".

§2 By the NEWTONIAN expansion of the binomial this formula $(1+x)^n$ is expanded into an infinite series in two ways, of course; first, directly

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1 \cdot 2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \text{etc.},$$

but then, since

$$(1+x)^n = \frac{1}{(1+x)^{-n}},$$

it will also be

$$(1+x)^n = \frac{1}{1 - \frac{n}{1}x + \frac{n(n+1)}{1 \cdot 2}x^2 - \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}x^3 + \text{etc.}}$$

Hence further multiplying these expressions and writing n for $2n$ a third expression converging a lot more rapidly will be derived

$$(1+x)^n = \frac{1 + \frac{n}{2}x + \frac{n(n-2)}{2 \cdot 4}x^2 + \frac{n(n-2)(n-4)}{2 \cdot 4 \cdot 6}x^3 + \text{etc.}}{1 - \frac{n}{2}x + \frac{n(n+2)}{2 \cdot 4}x^2 - \frac{n(n+2)(n+4)}{2 \cdot 4 \cdot 6}x^3 + \text{etc.}}$$

§3 To anyone paying attention it will quickly become clear that infinite many expressions similar to this last one can be exhibited, which are equal to the given formula $(1+x)^n$; for, if we put

$$(1+x)^n = \frac{1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + Fx^6 + \text{etc.}}{1 - \alpha x + \beta x^2 - \gamma x^3 + \delta x^4 - \varepsilon x^5 + \zeta x^6 - \text{etc.}},$$

the determination of the coefficients yields an undetermined problem and, if either the denominator or the numerator is assumed arbitrarily, the coefficients of the other are determined from this. Hence the question of greatest importance emerges, how so the numerator as the denominator must be determined such that both converge rapidly at the same time; and here it is certainly possible to attribute a finite number of terms to the denominator, where the question reduces to this, how the coefficients of the denominator have to be assumed such that a highly converging series results.

§4 But if in the denominator a given number of terms is constituted, the numerator will be a highly converging series, if one or more of each its second terms vanish completely; for, then the following terms will become so small, if it was $x < 1$, of course, such that they can be thrown out without any noticeable error. And here it is convenient to note that, if the binomial $1 - \alpha x$ is taken for the denominator, any arbitrary term of the numerator can be made zero; but if the denominator is set to be a trinomial, two successive terms of the numerator can be made equal to zero; but then three etc., if one assumes a quadrionomial or multinomial for the denominator. But then it is even clear that the approximation will be the closer the further the vanishing terms of the numerator are way from the initial terms; hence the following problems that have to be resolved arise.

PROBLEM 1

§5 *To transform the power of the binomial $(1+x)^n$ into a rapidly converging series of this kind*

$$(1+x)^n = \frac{1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + Fx^6 + \text{etc.}}{1 - \alpha x}$$

while the denominator is a binomial.

SOLUTION

If the power $(1+x)^n$ is expanded into a series and it is multiplied by the denominator $1 - \alpha x$, the following equations will arise that have to hold

$$\begin{array}{llllll} 0 = 1 + & \frac{n}{1}x + & \frac{n(n-1)}{1\cdot 2}x^2 + & \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + & \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}x^4 & + \text{etc.} \\ & -\ \alpha\ - & \frac{n}{1}\alpha\ - & \frac{n(n-1)}{1\cdot 2}\alpha\ - & \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\alpha & - \text{etc.} \\ -1 & -\ A\ - & B\ - & C\ - & D & - \text{etc.} \end{array}$$

Now depending on whether the second or third or fourth etc. term of the numerator has to vanish, one will the following determinations for the coefficients.

I. If $A = 0$, one immediately has $\alpha = \frac{n}{1}$ and the following terms of the numerator will be

$$B = -\frac{n(n+1)}{1\cdot 2}, \quad C = -\frac{2n(n-1)(n+1)}{1\cdot 2\cdot 3}, \quad D = -\frac{3n(n-1)(n-2)(n+1)}{1\cdot 2\cdot 3\cdot 4} \quad \text{etc.}$$

II. If $B = 0$, one immediately has $\alpha = \frac{n-1}{2}$ and for the numerator

$$A = \frac{n+1}{2\cdot 1}, \quad C = -\frac{1(n+1)n(n-1)}{2\cdot 1\cdot 2\cdot 3}, \quad D = -\frac{2(n+1)n(n-1)(n-2)}{2\cdot 1\cdot 2\cdot 3\cdot 4} \quad \text{etc.}$$

III. If $C = 0$, one has $\alpha = \frac{n-2}{3}$ and for the numerator

$$A = \frac{2(n+1)}{3\cdot 1}, \quad B = \frac{1(n+1)n}{3\cdot 1\cdot 2}, \quad D = -\frac{1(n+1)n(n-1)(n-2)}{3\cdot 1\cdot 2\cdot 3\cdot 4} \quad \text{etc.}$$

IV. If $D = 0$, one has $\alpha = \frac{n-3}{4}$ and for the numerator

$$A = \frac{3(n+1)}{4\cdot 1}, \quad B = \frac{2(n+1)n}{4\cdot 1\cdot 2}, \quad C = \frac{1(n+1)n(n-1)}{4\cdot 1\cdot 2\cdot 3} \quad \text{etc.}$$

Hence it is already clear in general, if any other of the following terms in the denominator has to vanish, that first one has

$$\alpha = \frac{n-\omega}{\omega+1}$$

and for the numerator

$$A = \frac{\omega}{\omega+1}\cdot\frac{n+1}{1}, \quad B = \frac{\omega-1}{\omega+1}\cdot\frac{(n+1)n}{1\cdot 2}, \quad C = \frac{\omega-2}{\omega+1}\cdot\frac{(n+1)n(n-1)}{1\cdot 2\cdot 3},$$

$$D = \frac{\omega-3}{\omega+1}\cdot\frac{(n+1)n(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 4}, \quad E = \frac{\omega-4}{\omega+1}\cdot\frac{(n+1)n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4\cdot 5},$$

the law of which progression is obvious.

COROLLARY 1

§6 If one omits the terms, which follow the vanishing one, in the numerator, one will have finite and rational expressions that exhibit the value $(1+x)^n$ continuously closer; therefore, if one puts $A=0$, one will have this approximation

$$(1+x)^n = \frac{1}{1-nx},$$

which, even if it hardly recedes from the truth, nevertheless deviates more than the following.

COROLLARY 2

§7 Let $B=0$ and the second case will yield this approximation

$$(1+x)^n = \frac{1+\frac{n+1}{2\cdot 1}x}{1-\frac{n-1}{2}x} = \frac{1+\frac{n+1}{2}x}{1-\frac{n-1}{2}x}.$$

Hence, if $n=\frac{\mu}{\nu}$, it will be

$$(1+x)^{\frac{\mu}{\nu}} = \frac{1+\frac{\mu+\nu}{2}x}{1-\frac{\mu-\nu}{2\nu}x}.$$

COROLLARY 3

§8 Let $C=0$ and the third case will give

$$(1+x)^n = \frac{1+\frac{2(n+1)}{3\cdot 1}x+\frac{1(n+1)n}{3\cdot 1\cdot 2}x^2}{1-\frac{n-2}{3}x},$$

whence, if it was $n=\frac{\mu}{\nu}$, it will be

$$(1+x)^{\frac{\mu}{\nu}} = \frac{1+\frac{2(\mu+\nu)}{3\cdot 1\nu}x+\frac{1(\mu+\nu)\mu}{3\cdot 1\cdot 2\nu^2}x^2}{1-\frac{\mu-2\nu}{3\nu}x}.$$

COROLLARY 4

§9 Let $D=0$ and the forth case gives

$$(1+x)^n = \frac{1+\frac{3(n+1)}{4\cdot 1}x+\frac{2(n+1)n}{4\cdot 1\cdot 2}x^2+\frac{1(n+1)n(n-1)}{4\cdot 1\cdot 2\cdot 3}x^3}{1-\frac{n-3}{4}x}$$

and hence, if $n=\frac{\mu}{\nu}$, it will be

$$(1+x)^{\frac{\mu}{\nu}} = \frac{1+\frac{3(\mu+\nu)}{4\cdot 1\nu}x+\frac{2(\mu+\nu)\mu}{4\cdot 1\cdot 2}x^2+\frac{1(\mu+\nu)\mu(\mu-\nu)}{4\cdot 1\cdot 2\cdot 3}x^3}{1-\frac{\mu-3\nu}{4\nu}x};$$

hence it is clear, how formulas of this kind have to be continued, for which reason I will not exhibit more of them here.

COROLLARY 5

§10 But in general one will have this form

$$(1+x)^n = \frac{1+\frac{(\omega-1)(n+1)}{\omega\cdot 1}x+\frac{(\omega-2)(n+1)n}{\omega\cdot 1\cdot 2}x^2+\frac{(\omega-3)(n+1)n(n-1)}{\omega\cdot 1\cdot 2\cdot 3}x^3+\text{etc.}}{1-\frac{n-\omega+1}{\omega}x},$$

where one can take any number for ω; and if this expression is continued to infinity, it not only comes closer to the truth, but will also exhibit the true value of the formula $(1+x)^n$.

COROLLARY 6

§11 If one takes $\omega=n+1$, the denominator will go over into one and the known NEWTONIAN series

$$(1+x)^n = 1+\frac{n}{1}x+\frac{n(n-1)}{1\cdot 2}x^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3+\text{etc.}$$

arise. But if one takes an infinite number for ω, it will be

$$(1+x)^n = \frac{1+\frac{n+1}{1}x+\frac{(n+1)n}{1\cdot 2}x^2+\frac{(n+1)n(n-1)}{1\cdot 2\cdot 3}x^3+\text{etc.}}{1+x},$$

the reason for which is also obvious from the NEWTONIAN binomial.

COROLLARY 7

§12 If one puts $\omega = n$, one will have

$$(1+x)^n$$
$$= \frac{1 + \frac{(n+1)(n-1)}{1\cdot n}x + \frac{(n+1)(n-2)}{1\cdot 2}x^2 + \frac{(n+1)(n-1)(n-3)}{1\cdot 2\cdot 3}x^3 + \frac{(n+1)(n-1)(n-2)(n-4)}{1\cdot 2\cdot 3\cdot 4}x^4 + \text{etc.}}{1 - \frac{1}{n}x}$$

or, multiplying the numerator and denominator by n,

$$(1+x)^n = \frac{n + \frac{(n+1)(n-1)}{1}x + \frac{(n+1)n(n-2)}{1\cdot 2}x^2 + \frac{(n+1)n(n-1)(n-3)}{1\cdot 2\cdot 3}x^3 + \text{etc.}}{n - x}.$$

COROLLARY 8

§13 If one puts $\omega = x$, the denominator will become $= x - n$ and one will obtain

$$(1+x)^n$$
$$= \frac{1 + \frac{n+1}{1}(x-1) + \frac{(n+1)n}{1\cdot 2}x(x-2) + \frac{(n+1)n(n-1)}{1\cdot 2\cdot 3}x^2(x-3) + \frac{(n+1)n(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 4}x^3(x-4) + \text{etc.}}{x - n}$$

and in like manner innumerable series can be deduced from this expression, the reason for which cannot be seen so easily from other sources; hence this investigation seems to amplify the doctrine of series quite substantially.

PROBLEM 2

§14 *To transform the power of the binomial $(1+x)^n$ into a rapidly converging series of this kind*

$$(1+x)^n = \frac{1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + Fx^6 + \text{etc.}}{1 - \alpha x + \beta x^2},$$

while the denominator is a trinomial.

SOLUTION

Having resolved the power $(1+x)^n$ into a series in usual manner, one will have to construct the following equation

$$\begin{array}{rcccccccl}
0 = 1 + & \frac{n}{1}x & + & \frac{n(n-1)}{1\cdot 2}x^2 & + & \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 & + & \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}x^4 & + \text{etc.} \\
-\ \alpha & - & & \frac{n}{1}\alpha & - & \frac{n(n-1)}{1\cdot 2}\alpha & - & \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\alpha & - \text{etc.} \\
 & + & & \beta & + & \frac{n}{1}\beta & + & \frac{n(n-1)}{1\cdot 2}\beta & + \text{etc.} \\
-1 - A & - & & B & - & C & - & D & - \text{etc.}
\end{array}$$

and here it is possible to define the denominator $1-\alpha x+\beta x^2$ in such a way that in the numerator two terms vanish successively, whence it will be rendered more convergent.

I. Let $A=0$ and $B=0$; it will be $\alpha=\frac{n}{1}$ and $\beta=\frac{n(n+1)}{1\cdot 2}$, whence one has

$$\begin{aligned}
C &= \frac{n}{1}\left(\frac{(n-1)(n-2)}{2\cdot 3}-\frac{(n-1)n}{2\cdot 1}+\frac{(n+1)n}{1\cdot 2}\right)=\frac{1(n+2)(n+1)n}{3\cdot 1\cdot 2\cdot 1},\\
D &= \frac{n(n-1)}{1\cdot 2}\left(\frac{(n-2)(n-3)}{3\cdot 4}-\frac{(n-2)n}{3\cdot 1}+\frac{(n+1)n}{1\cdot 2}\right)=\frac{2(n+2)(n+1)n(n-1)}{4\cdot 1\cdot 2\cdot 1\cdot 2},\\
E &= \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\left(\frac{(n-3)(n-4)}{4\cdot 5}-\frac{(n-3)n}{4\cdot 1}+\frac{(n+1)n}{1\cdot 2}\right)=\frac{3(n+2)(n+1)n(n-1)(n-2)}{5\cdot 1\cdot 2\cdot 1\cdot 2\cdot 3}
\end{aligned}$$

and in general it will be

$$N=\cdots\left(\frac{(n-\nu)(n-\nu-1)}{(\nu+1)(\nu+2)}-\frac{(n-\nu)n}{(\nu+1)1}+\frac{(n+1)n}{1\cdot 2}\right)=\cdots\frac{\nu(n+2)(n+1)}{(\nu+2)\cdot 1\cdot 2},$$

from which general value those special cases are easily derived.

II. Let $B=0$ and $C=0$; for α and β it will be

$$\beta - \frac{n}{1}\alpha + \frac{n(n-1)}{1\cdot 2} = 0,$$

$$\beta - \frac{n-1}{2}\alpha + \frac{(n-1)(n-2)}{2\cdot 3} = 0;$$

hence

$$\alpha = \frac{2(n-1)}{3}, \quad \beta = \frac{n(n-1)}{2\cdot 3}.$$

But for the numerator one will have

$$A = \frac{n}{1} - \frac{2(n-1)}{3} = \frac{n+2}{3},$$

$$D = \frac{n(n-1)}{1\cdot 2}\left(\frac{(n-2)(n-3)}{3\cdot 4} - \frac{2(n-2)(n-1)}{3\cdot 3} + \frac{n(n-1)}{2\cdot 3}\right)$$

$$= \frac{1\cdot 2(n+2)(n+1)n(n-1)}{3\cdot 4\cdot 2\cdot 3\cdot 1\cdot 2},$$

$$E = \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\left(\frac{(n-3)(n-4)}{4\cdot 5} - \frac{2(n-3)(n-1)}{4\cdot 3} + \frac{n(n-1)}{2\cdot 3}\right)$$

$$= \frac{2\cdot 3(n+2)(n+1)n(n-1)(n-2)}{4\cdot 5\cdot 2\cdot 3\cdot 1\cdot 2\cdot 3},$$

$$F = \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}\left(\frac{(n-4)(n-5)}{5\cdot 6} - \frac{2(n-4)(n-1)}{5\cdot 3} + \frac{n(n-1)}{2\cdot 3}\right)$$

$$= \frac{3\cdot 4(n+2)(n+1)n(n-1)(n-2)(n-3)}{5\cdot 6\cdot 2\cdot 3\cdot 1\cdot 2\cdot 3\cdot 4}$$

and in general

$$N = \cdots\left(\frac{(n-\nu)(n-\nu-1)}{(\nu+1)(\nu+2)} - \frac{2(n-\nu)(n-1)}{(\nu+1)\cdot 3} + \frac{n(n-1)}{2\cdot 3}\right) = \cdots\frac{\nu(\nu-1)(n+2)(n+1)}{(\nu+2)(\nu+1)\cdot 2\cdot 3}.$$

III. Let $C = 0$ and $D = 0$ and for the denominator it will be

$$\beta - \frac{n-1}{2}\alpha + \frac{(n-1)(n-2)}{2\cdot 3} = 0,$$

$$\beta - \frac{n-2}{3}\alpha + \frac{(n-2)(n-3)}{3\cdot 4} = 0;$$

hence

$$\alpha = \frac{n-2}{2}, \quad \beta = \frac{(n-1)(n-2)}{3\cdot 4}$$

and hence for the numerator

$$A = \frac{n}{1} - \frac{n-2}{2} = \frac{n+2}{2},$$

$$B = \frac{n(n-1)}{1\cdot 2} - \frac{n(n-2)}{1\cdot 2} + \frac{(n-1)(n-2)}{3\cdot 4} = \frac{(n+2)(n+1)}{3\cdot 4},$$

$$E = \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\left(\frac{(n-3)(n-4)}{4\cdot 5} - \frac{(n-3)(n-2)}{4\cdot 2} + \frac{(n-1)(n-2)}{3\cdot 4}\right)$$
$$= \frac{(n+2)(n+1)n(n-1)(n-2)}{2\cdot 3\cdot 4\cdot 5\cdot 1\cdot 2\cdot 3}.$$

But since it suffices to know the terms, which precede the vanishing ones, I do not determine the following ones, since there law will become evident in the following.

IV. Let $D = 0$ and $E = 0$; for the denominator it will be

$$\beta - \frac{n-2}{3}\alpha + \frac{(n-2)(n-3)}{3\cdot 4} = 0,$$

$$\beta - \frac{n-3}{4}\alpha + \frac{(n-3)(n-4)}{4\cdot 5} = 0;$$

hence

$$\alpha = \frac{2(n-3)}{5}, \quad \beta = \frac{(n-2)(n-3)}{4\cdot 5};$$

but for the numerator one will find

$$A = \frac{n}{1} - \frac{2(n-3)}{5} = \frac{3(n+2)}{5},$$

$$B = \frac{n(n-1)}{1\cdot 2} - \frac{2n(n-3)}{1\cdot 5} + \frac{(n-2)(n-3)}{4\cdot 5} = \frac{3(n+2)(n+1)}{5\cdot 4},$$

$$C = \frac{n}{1}\left(\frac{(n-1)(n-2)}{2\cdot 3} - \frac{2(n-1)(n-3)}{2\cdot 5} + \frac{(n-2)(n-3)}{4\cdot 5}\right) = \frac{(n+2)(n+1)n}{5\cdot 4\cdot 3}.$$

V. Let $E = 0$ and $F = 0$ and from the things mentioned before we easily conclude that first it will be

$$\alpha = \frac{2(n-4)}{6}, \quad \beta = \frac{(n-3)(n-4)}{5 \cdot 6},$$

but then

$$A = \frac{4(n+2)}{6}, \quad B = \frac{6(n+2)(n+1)}{6 \cdot 5}, \quad C = \frac{4(n+2)(n+1)n}{6 \cdot 5 \cdot 4}$$

and

$$D = \frac{1(n+2)(n+1)n(n-1)}{6 \cdot 5 \cdot 4 \cdot 3}.$$

Therefore, finally we will in general find these determinations

$$\alpha = \frac{2(n-\omega)}{\omega+2}, \quad \beta = \frac{(n-\omega)(n-\omega+1)}{(\omega+2)(\omega+1)};$$

$$A = \frac{\omega}{1} \cdot \frac{n+2}{\omega+2},$$

$$B = \frac{\omega(\omega-1)}{1 \cdot 2} \cdot \frac{(n+2)(n+1)}{(\omega+2)(\omega+1)},$$

$$C = \frac{\omega(\omega-1)(\omega-2)}{1 \cdot 2 \cdot 3} \cdot \frac{(n+2)(n+1)n}{(\omega+2)(\omega+1)\omega},$$

$$D = \frac{\omega(\omega-1)(\omega-2)(\omega-3)}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{(n+2)(n+1)n(n-1)}{(\omega+2)(\omega+1)\omega(\omega-1)},$$

$$E = \frac{\omega(\omega-1)(\omega-2)(\omega-3)(\omega-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot \frac{(n+2)(n+1)n(n-1)(n-2)}{(\omega+2)(\omega+1)\omega(\omega-1)(\omega-2)}$$

etc.

Hence even the coefficients of the terms following after the vanishing ones are easily formed.

Corollary 1

§15 Whenever for the denominator it is in general

$$\alpha = \frac{2(n-\omega)}{\omega+2} \quad \text{and} \quad \beta = \frac{(n-\omega)(n-\omega+1)}{(\omega+2)(\omega+1)},$$

for the numerator we will have

$$\begin{aligned}
A &= \frac{\omega}{\omega+2} \cdot \frac{n+2}{1}, \\
B &= \frac{\omega(\omega-1)}{(\omega+2)(\omega+1)} \cdot \frac{(n+2)(n+1)}{1 \cdot 2}, \\
C &= \frac{(\omega-1)(\omega-2)}{(\omega+2)(\omega+1)} \cdot \frac{(n+2)(n+1)n}{1 \cdot 2 \cdot 3}, \\
D &= \frac{(\omega-2)(\omega-3)}{(\omega+2)(\omega+1)} \cdot \frac{(n+2)(n+1)n(n-1)}{1 \cdot 2 \cdot 3 \cdot 4}, \\
E &= \frac{(\omega-3)(\omega-4)}{(\omega+2)(\omega+1)} \cdot \frac{(n+2)(n+1)n(n-1)(n-2)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \\
&\text{etc.}
\end{aligned}$$

The analogies of these values to those that were found in the first problem, already indicates the structure of the following, where the denominator consists of more terms, quite clearly.

COROLLARY 2

§16 Having neglected the terms following after the vanishing ones in the numerator, we will have the following approximations: If $\omega = 0$, it will be

$$(1+x)^n = \frac{1}{1 - nx + \frac{n(n+1)}{1 \cdot 2}xx},$$

which certainly differs a lot from the truth in this class.

COROLLARY 3

§17 Let us put $\omega = 1$ and it will approximately be

$$(1+x)^n = \frac{1+\frac{n+2}{3}x}{1-\frac{2(n-1)}{3}x+\frac{n(n-1)}{2\cdot 3}xx};$$

but if $\omega = 2$, it will be even closer

$$(1+x)^n = \frac{1+\frac{2(n+2)}{4}x+\frac{(n+2)(n+1)}{4\cdot 3}xx}{1-\frac{2(n-2)}{4}x+\frac{(n-2)(n-1)}{4\cdot 3}xx};$$

and if $\omega = 3$, it will be

$$(1+x)^n = \frac{1+\frac{3(n+2)}{5}x+\frac{3(n+2)(n+1)}{5\cdot 4}xx+\frac{(n+2)(n+1)n}{5\cdot 4\cdot 3}x^3}{1-\frac{2(n-3)}{5}x+\frac{(n-3)(n-2)}{5\cdot 4}xx};$$

if $\omega = 4$, it will be

$$(1+x)^n = \frac{1+\frac{4(n+2)}{6}x+\frac{6(n+2)(n+1)}{6\cdot 5}xx+\frac{4(n+2)(n+1)n}{6\cdot 5\cdot 4}x^3+\frac{(n+2)(n+1)n(n-1)}{6\cdot 5\cdot 4\cdot 3}x^4}{1-\frac{2(n-4)}{6}x+\frac{(n-4)(n-3)}{6\cdot 5}xx};$$

if $\omega = 5$, it will be

$$(1+x)^n = \frac{\left\{\begin{aligned}&1+\frac{5(n+2)}{7}x+\frac{10(n+2)(n+1)}{7\cdot 6}x^2+\frac{10(n+2)(n+1)n}{7\cdot 6\cdot 5}x^3+\frac{5(n+2)(n+1)n(n-1)}{7\cdot 6\cdot 5\cdot 4}x^4\\ &\qquad+\frac{(n+2)(n+1)n(n-1)(n-2)}{7\cdot 6\cdot 5\cdot 4\cdot 3}x^5\end{aligned}\right\}}{1-\frac{2(n-5)}{7}x+\frac{(n-5)(n-4)}{7\cdot 6}xx}$$

These expressions are easily continued even further from the coefficients of the powers of the binomial. The further they are continued, the less they deviate from the truth.

COROLLARY 4

§18 But it is not possible to exhibit this transformation of the formula $(1+x)^n$ in a more convenient way in general than that we say that

$$(1+x)^n = \frac{1+Ax+Bx^2+Cx^3+Dx^4+Ex^5+\text{etc.}}{1-\alpha x+\beta xx},$$

while the values of the coefficients are

$$\alpha = \frac{2(n-\omega)}{\omega+2}, \quad \beta = \frac{(n-\omega)(n-\omega+1)}{(\omega+2)(\omega+1)};$$

$$A = \frac{(\omega+1)\omega}{(\omega+2)(\omega+1)} \cdot \frac{n+2}{1},$$

$$B = \frac{\omega(\omega-1)}{(\omega+2)(\omega+1)} \cdot \frac{(n+2)(n+1)}{1\cdot 2},$$

$$C = \frac{(\omega-1)(\omega-2)}{(\omega+2)(\omega+1)} \cdot \frac{(n+2)(n+1)n}{1\cdot 2\cdot 3},$$

$$D = \frac{(\omega-2)(\omega-3)}{(\omega+2)(\omega+1)} \cdot \frac{(n+2)(n+1)n(n-1)}{1\cdot 2\cdot 3\cdot 4},$$

$$E = \frac{(\omega-3)(\omega-4)}{(\omega+2)(\omega+1)} \cdot \frac{(n+2)(n+1)n(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 4}$$

etc.

COROLLARY 5

§19 Here it is clear again, since the quantity ω can be chosen arbitrarily, if one takes $\omega = n$, that $\alpha = 0$, $\beta = 0$ and

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \text{etc.}$$

result. But if $\omega = \infty$, it will be $\alpha = -2$ and $\beta = 1$; hence

$$(1+x)^n = \frac{1 + \frac{n+2}{1}x + \frac{(n+2)(n+1)}{1\cdot 2}x^2 + \frac{(n+2)(n+1)n}{1\cdot 2\cdot 3}x^3 + \text{etc.}}{1+2x+xx}$$

or

$$(1+x)^n = \frac{(1+x)^{n+2}}{(1+x)^2},$$

the reason for which is manifest.

Problem 3

§20 *To transform the power of the binomial* $(1+x)^n$ *into a highly convergent series of this kind*

$$(1+x)^n = \frac{1+Ax+Bx^2+Cx^3+Dx^4+Ex^5+Fx^6+\text{etc.}}{1-\alpha x+\beta x^2-\gamma x^3},$$

while the denominator is a quadrominal.

Solution

Therefore, the following equation has to be constructed

$$\begin{array}{rcccccccl}
0 = 1 + & \frac{n}{1}x & + & \frac{n(n-1)}{1\cdot 2}x^2 & + & \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 & + & \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}x^4 & + \text{ etc.} \\
 - \ \alpha & & - & \frac{n}{1}\alpha & - & \frac{n(n-1)}{1\cdot 2}\alpha & - & \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\alpha & - \text{ etc.} \\
 & & + & \beta & + & \frac{n}{1}\beta & + & \frac{n(n-1)}{1\cdot 2}\beta & - \text{ etc.} \\
 & & & & - & \gamma & - & \frac{n}{1}\gamma & - \text{ etc.} \\
 -1 - A & & - & B & - & C & - & D & - \text{ etc.}
\end{array}$$

Here one can even achieve that in the series of coefficients A, B, C, D etc. three successive terms vanish. Therefore, take any three successively vanishing terms and one will obtain three equations of this kind

$$\gamma - \frac{n-\omega+2}{\omega-1}\beta + \frac{(n-\omega+2)(n-\omega+1)}{(\omega-1)\omega}\alpha - \frac{(n-\omega+2)(n-\omega+1)(n-\omega)}{(\omega-1)\omega(\omega+1)} = 0,$$

$$\gamma - \frac{n-\omega+1}{\omega-1}\beta + \frac{(n-\omega+1)(n-\omega)}{\omega(\omega+1)}\alpha - \frac{(n-\omega+1)(n-\omega)(n-\omega-1)}{\omega(\omega+1)(\omega+2)} = 0,$$

$$\gamma - \frac{n-\omega}{\omega+1}\beta + \frac{(n-\omega)(n-\omega-1)}{(\omega+1)(\omega+2)}\alpha - \frac{(n-\omega)(n-\omega-1)(n-\omega-2)}{(\omega+1)(\omega+2)(\omega+3)} = 0.$$

Hence, taking differentials, one will have

$$\frac{n+1}{(\omega-1)\omega}\beta - \frac{2(n+1)(n-\omega+1)}{(\omega-1)\omega(\omega+1)}\alpha + \frac{3(n+1)(n-\omega+1)(n-\omega)}{(\omega-1)\omega(\omega+1)(\omega+2)} = 0,$$
$$\frac{n+1}{\omega(\omega+1)}\beta - \frac{2(n+1)(n-\omega)}{\omega(\omega+1)(\omega+2)}\alpha + \frac{3(n+1)(n-\omega)(n-\omega-1)}{\omega(\omega+1)(\omega+2)(\omega+3)} = 0$$

or

$$\beta - \frac{2(n-\omega+1)}{\omega+1}\alpha + \frac{3(n-\omega+1)(n-\omega)}{(\omega+1)(\omega+2)} = 0,$$
$$\beta - \frac{2(n-\omega)}{\omega+2}\alpha + \frac{3(n-\omega)(n-\omega-1)}{(\omega+2)(\omega+3)} = 0,$$

the difference of which equations gives

$$\frac{2(n+2)}{(\omega+1)(\omega+2)}\alpha - \frac{2\cdot 3(n+2)(n-\omega)}{(\omega+1)(\omega+2)(\omega+3)} = 0;$$

and hence

$$\alpha = \frac{3(n-\omega)}{\omega+3}, \quad \beta = \frac{3(n-\omega)(n-\omega+1)}{(\omega+3)(\omega+2)} \quad \text{and} \quad \gamma = \frac{(n-\omega)(n-\omega+1)(n-\omega+2)}{(\omega+3)(\omega+2)(\omega+1)}.$$

But having found these values for the denominator, for the numerator one will get

$$A = \frac{\omega}{\omega+3} \cdot \frac{n+3}{1},$$

$$B = \frac{\omega(\omega-1)}{(\omega+3)(\omega+2)} \cdot \frac{(n+3)(n+2)}{1 \cdot 2},$$

$$C = \frac{\omega(\omega-1)(\omega-2)}{(\omega+3)(\omega+2)(\omega+1)} \cdot \frac{(n+3)(n+2)(n+1)}{1 \cdot 2 \cdot 3},$$

$$D = \frac{(\omega-1)(\omega-2)(\omega-3)}{(\omega+3)(\omega+2)(\omega+1)} \cdot \frac{(n+3)(n+2)(n+1)n}{1 \cdot 2 \cdot 3 \cdot 4},$$

$$E = \frac{(\omega-2)(\omega-3)(\omega-4)}{(\omega+3)(\omega+2)(\omega+1)} \cdot \frac{(n+3)(n+2)(n+1)n(n-1)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5},$$

$$F = \frac{(\omega-3)(\omega-4)(\omega-5)}{(\omega+3)(\omega+2)(\omega+1)} \cdot \frac{(n+3)(n+2)(n+1)n(n-1)(n-2)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$$

etc.

and the denominator will be formed from these values

$$\alpha = \frac{3(n-\omega)}{\omega+3},$$

$$\beta = \frac{3(n-\omega)(n-\omega+1)}{(\omega+3)(\omega+2)},$$

$$\gamma = \frac{(n-\omega)(n-\omega+1)(n-\omega+2)}{(\omega+3)(\omega+2)(\omega+1)}.$$

Having subsituted these, it will be

$$(1+x)^n = \frac{1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + \text{etc.}}{1 - \alpha x + \beta x^2 - \gamma x^3}.$$

Corollary 1

§21 Here it is manifest, whatever positive integer number is assumed for ω, that in the numerator always three consecutive terms vanish. Therefore, if $\omega = 0$, it will be

$$(1+x)^n = \frac{1+\star+\star+\star-\frac{(n+3)(n+2)(n+1)n}{1\cdot2\cdot3\cdot4}x^4-\frac{4(n+3)(n+2)(n+1)n(n-1)}{1\cdot2\cdot3\cdot4\cdot5}x^5-\text{etc.}}{1-\frac{n}{1}x+\frac{n(n+1)}{1\cdot2}x^2-\frac{n(n+1)(n+2)}{1\cdot2\cdot3}x^3},$$

whence, after rejecting the terms that follow after the vanishing ones in the numerator, it will approximately be

$$(1+x)^n = \frac{1}{1-\frac{n}{1}x+\frac{n(n+1)}{1\cdot2}x^2-\frac{n(n+1)(n+2)}{1\cdot2\cdot3}x^3}.$$

COROLLARY 2

§22 In like manner, putting $\omega = 1$, it will approximately be

$$(1+x)^n = \frac{1+\frac{n+3}{4}x}{1-\frac{3(n-1)}{4}x+\frac{3(n-1)n}{4\cdot3}x^2-\frac{(n-1)n(n+1)}{4\cdot3\cdot2}x^3};$$

but if one takes $\omega = 2$, it will be

$$(1+x)^n = \frac{1+\frac{2(n+3)}{5}x+\frac{(n+3)(n+2)}{5\cdot4}x^2}{1-\frac{3(n-2)}{5}x+\frac{3(n-2)(n-1)}{5\cdot4}x^2-\frac{(n-2)(n-1)n}{5\cdot4\cdot3}x^3};$$

but having put $\omega = 3$, it will be

$$(1+x)^n = \frac{1+\frac{3(n+3)}{6}x+\frac{3(n+3)(n+2)}{6\cdot5}x^2+\frac{(n+3)(n+2)(n+1)}{6\cdot5\cdot4}x^3}{1-\frac{3(n-3)}{6}x+\frac{3(n-3)(n-2)}{6\cdot5}x^2-\frac{(n-3)(n-2)(n-1)}{6\cdot5\cdot4}x^3}.$$

COROLLARY 3

§23 This last formula is remarkable for this reason, since the numerator and the denominator consist of the same number of terms and since the one goes over into the other, if the exponent n is taken negatively. Therefore, this expression is to be brought into agreement with the the same ones arising in the above problems

$$(1+x)^n = \frac{1+\frac{(n+1)x}{2}}{1-\frac{(n-1)}{2}x} \quad (\S\ 7),$$

$$(1+x)^n = \frac{1+\frac{2(n+2)}{4}x+\frac{(n+2)(n+1)}{4\cdot 3}x^2}{1-\frac{2(n-2)}{4}x+\frac{(n-2)(n-1)}{4\cdot 3}x^2} \quad (\S\ 17),$$

whence at the same time the structure of formulas of this kind is easily concluded.

PROBLEM 4

§24 *To transform the power of the binomial $(1+x)^n$ into a rapidly converging progression of this kind*

$$(1+x)^n = \frac{1+Ax+Bx^2+Cx^3+Dx^4+Ex^5+Fx^6+\text{etc.}}{1-\alpha x+\beta x^2-\gamma x^3+\delta x^4-\varepsilon x^5+\zeta x^6-\text{etc.}},$$

while the denominator is an arbitrary multinomial.

SOLUTION

If we console the solutions of the preceding problems, paying a little attention we calculate the following general solution:

$$A = \frac{\omega}{\omega+\varphi}\cdot\frac{n+\varphi}{1},$$

$$B = \frac{\omega(\omega-1)}{(\omega+\varphi)(\omega+\varphi-1)}\cdot\frac{(n+\varphi)(n+\varphi-1)}{1\cdot 2},$$

$$C = \frac{\omega(\omega-1)(\omega-2)}{(\omega+\varphi)(\omega+\varphi-1)(\omega+\varphi-2)}\cdot\frac{(n+\varphi)(n+\varphi-1)(n+\varphi-2)}{1\cdot 2\cdot 3},$$

$$D = \frac{\omega(\omega-1)(\omega-2)(\omega-3)}{(\omega+\varphi)(\omega+\varphi-1)(\omega+\varphi-2)(\omega+\varepsilon-3)}\cdot\frac{(n+\varphi)(n+\varphi-1)(n+\varphi-2)(n+\varphi-3)}{1\cdot 2\cdot 3\cdot 4}$$

etc.

Further, for the denominator:

$$\alpha = \frac{\varphi}{1} \cdot \frac{n-\omega}{\omega+\varphi},$$

$$\beta = \frac{\varphi(\varphi-1)}{1\cdot 2} \cdot \frac{(n-\omega)(n-\omega+1)}{(\omega+\varphi)(\omega+\varphi-1)},$$

$$\gamma = \frac{\varphi(\varphi-1)(\varphi-2)}{1\cdot 2\cdot 3} \cdot \frac{(n-\omega)(n-\omega+1)(n-\omega+2)}{(\omega+\varphi)(\omega+\varphi-1)(\omega+\varphi-2)},$$

$$\delta = \frac{\varphi(\varphi-1)(\varphi-2)(\varphi-3)}{1\cdot 2\cdot 3\cdot 4} \cdot \frac{(n-\omega)(n-\omega+1)(n-\omega+2)(n-\omega+3)}{(\omega+\varphi)(\omega+\varphi-1)(\omega+\varphi-2)(\omega+\varphi-3)}$$

etc.

These values are reduced to a form closer to the preceding one such that

$$\alpha = \frac{\varphi}{\varphi+\omega} \cdot \frac{n-\omega}{1},$$

$$\beta = \frac{\varphi(\varphi-1)}{(\varphi+\omega)(\varphi+\omega+1)} \cdot \frac{(n-\omega)(n-\omega+1)}{1\cdot 2},$$

$$\gamma = \frac{\varphi(\varphi-1)(\varphi-2)}{(\varphi+\omega)(\varphi+\omega-1)(\varphi+\omega-2)} \cdot \frac{(n-\omega)(n-\omega+1)(n-\omega+2)}{1\cdot 2\cdot 3},$$

$$\delta = \frac{\varphi(\varphi-1)(\varphi-2)(\varphi-3)}{(\varphi+\omega)(\varphi+\omega-1)(\varphi+\omega-2)(\varphi+\omega-3)} \cdot \frac{(n-\omega)(n-\omega+1)(n-\omega+2)(n-\omega+3)}{1\cdot 2\cdot 3\cdot 4}$$

etc.

But even if from this law the denominator can also be continued to infinity, nevertheless from the principle, from which we deduced it, it is clear that it does not have be produced beyond the vanishing terms, if one takes an integer positive number for φ, of course.

COROLLARY 1

§25 Therefore, the denominator can be formed from the numerator, if the numbers φ and ω are exchanged and one writes $-n$ instead of n. But having put $-n$ for n, the formula $(1+x)^n$ goes over into $(1+x)^{-n}$; hence, if it was $(1+x)^n = \frac{P}{Q}$, it will be $(1+x)^{-n} = \frac{Q}{P}$, from which the reason of this conversion is understood more clearly.

COROLLARY 2

§26 Therefore, since the numerator and the denominator can be exchanged, even the numerator can be terminated at the vanishing terms; but then the denominator has to be continued to infinity such that a fraction equal to the power $(1+x)^n$ is obtained.

COROLLARY 3

§27 If one takes $\varphi = \omega$, the numerator and the denominator become a lot more similar to each other and will only differ in regard of the sign of the exponent $-n$. But then it will be

$$A = \frac{\omega}{2\omega} \cdot \frac{n+\omega}{1},$$

$$B = \frac{\omega(\omega-1)}{2\omega(2\omega-1)} \cdot \frac{(n+\omega)(n+\omega-1)}{1\cdot 2},$$

$$C = \frac{\omega(\omega-1)(\omega-2)}{2\omega(2\omega-1)(2\omega-2)} \cdot \frac{(n+\omega)(n+\omega-1)(n+\omega-2)}{1\cdot 2\cdot 3},$$

$$D = \frac{\omega(\omega-1)(\omega-2)(\omega-3)}{2\omega(2\omega-1)(2\omega-2)(2\omega-3)} \cdot \frac{(n+\omega)(n+\omega-1)(n+\omega-2)(n+\omega-3)}{1\cdot 2\cdot 3\cdot 4}$$

etc.

$$\alpha = \frac{\omega}{2\omega} \cdot \frac{n-\omega}{1},$$

$$\beta = \frac{\omega(\omega-1)}{2\omega(2\omega-1)} \cdot \frac{(n-\omega)(n-\omega+1)}{1\cdot 2},$$

$$\gamma = \frac{\omega(\omega-1)(\omega-2)}{2\omega(2\omega-1)(2\omega-2)} \cdot \frac{(n-\omega)(n-\omega+1)(n-\omega+2)}{1\cdot 2\cdot 3},$$

$$\delta = \frac{\omega(\omega-1)(\omega-2)(\omega-3)}{2\omega(2\omega-1)(2\omega-2)(2\omega-3)} \cdot \frac{(n-\omega)(n-\omega+1)(n-\omega+2)(n-\omega+3)}{1\cdot 2\cdot 3\cdot 4}$$

etc.

COROLLARY 4

§28 Hence the above (§ 23) formulas that are more than suitable for approximation are derived:

$$(1+x)^n = \frac{1+\frac{1}{2}\cdot\frac{n+1}{1}x}{1-\frac{1}{2}\cdot\frac{n-1}{1}x},$$

$$(1+x)^n = \frac{1+\frac{2}{4}\cdot\frac{n+2}{1}x+\frac{2\cdot 1}{4\cdot 3}\cdot\frac{(n+2)(n+1)}{1\cdot 2}x^2}{1-\frac{2}{4}\cdot\frac{n-2}{1}x+\frac{2\cdot 1}{4\cdot 3}\cdot\frac{(n-2)(n-1)}{1\cdot 2}x^2},$$

$$(1+x)^n = \frac{1+\frac{3}{6}\cdot\frac{n+3}{1}x+\frac{3\cdot 2}{6\cdot 5}\cdot\frac{(n+3)(n+2)}{1\cdot 2}x^2+\frac{3\cdot 2\cdot 1}{6\cdot 5\cdot 4}\cdot\frac{(n+3)(n+2)(n+1)}{1\cdot 2\cdot 3}x^3}{1-\frac{3}{6}\cdot\frac{n-3}{1}x+\frac{3\cdot 2}{6\cdot 5}\cdot\frac{(n-3)(n-2)}{1\cdot 2}x^2+\frac{3\cdot 2\cdot 1}{6\cdot 5\cdot 4}\cdot\frac{(n-3)(n-2)(n-1)}{1\cdot 2\cdot 3}x^3};$$

how these have to be continued, is clear without further explanation.

SCHOLION 1

§29 These formulas are the more remarkable the less obvious their origin is; for, even though so in the numerator as the denominator the law of progression is clear, according to which each of both is continued to infinity, we nevertheless already noted that just the one of them must be produced to infinity, while the other consists of a finite number of terms; more precisely, the one has to terminate, where some terms start to vanish, even if later again terms of finite magnitude occur. But these remarks are to be interpreted in such a way, if in the values of the letters A, B, C etc., α, β, γ etc. a vanishing factor of the numerator is considered to be thrown out by a vanishing factor of the denominator, that the fraction $\frac{\omega-m}{2\omega-2m}$ is put equal to 1 in the case $\omega = m$. But if, as the reasoning in the calculation requires, this fraction is only taken equal the half of unity, then the law of continuity is not violated anymore; and if, having retained this law, so the numerator as the denominator are continued even beyond the vanishing terms to infinity, the resulting fraction will be perfectly equal to the formula $(1+x)^n$. Exactly the same is to be said in general, as long as a certain relation between the numbers φ and ω is assumed such that, if $\varphi = \lambda\omega$, the value of the fraction $\frac{\omega-m}{(\lambda+1)\omega-(\lambda+1)m}$ is also taken to be $= \frac{1}{\lambda+1}$ in the case $\omega = m$, such that this caveat is not to be considered to be adverse to the principle of continuity.

SCHOLION 2

§30 To see all this more clearly, let us consider the case $\omega = 0$ and because of $\frac{\omega}{2\omega} = \frac{1}{2}$ the numerator of our fraction will be

$$1 + \frac{1}{2} \cdot \frac{n}{1}x + \frac{1}{2} \cdot \frac{n(n-1)}{1 \cdot 2}x^2 + \frac{1}{2} \cdot \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \frac{1}{2} \cdot \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}x^4 + \text{etc.}$$

and the denominator

$$1 - \frac{1}{2} \cdot \frac{n}{1}x + \frac{1}{2} \cdot \frac{n(n+1)}{1 \cdot 2}x^2 - \frac{1}{2} \cdot \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}x^3 + \frac{1}{2} \cdot \frac{n(n+1)(n+2)(n+3)}{1 \cdot 2 \cdot 3 \cdot 4}x^4 - \text{etc.};$$

but it is manifest that the value of the numerator is $= \frac{1}{2} + \frac{1}{2}(1+x)^n$, but of the denominator $= \frac{1}{2} + \frac{1}{2}(1+x)^{-n}$ and the first divided by the latter yields $(1+x)^n$. If in like manner one sets $\omega = 1$, it will be

for the numerator	for the denominator
$A = \frac{1}{2} \cdot \frac{n+1}{1}$,	$\alpha = \frac{1}{2} \cdot \frac{n-1}{1}$,
$B = 0$	$\beta = 0$,
$C = -\frac{1}{4} \cdot \frac{(n+1)n(n-1)}{1 \cdot 2 \cdot 3}$	$\gamma = -\frac{1}{4} \cdot \frac{(n-1)n(n+1)}{1 \cdot 2 \cdot 3}$,
$D = -\frac{2}{4} \cdot \frac{(n+1)n(n-1)(n-2)}{1 \cdot 2 \cdot 3 \cdot 4}$	$\gamma = -\frac{2}{4} \cdot \frac{(n-1)n(n+1)(n+2)}{1 \cdot 2 \cdot 3 \cdot 4}$,
$E = -\frac{3}{4} \cdot \frac{(n+1)n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$	$\gamma = -\frac{3}{4} \cdot \frac{(n-1)n(n+1)(n+2)(n+3)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$
etc.	etc.

and hence one calculates that

$$\text{the numerator} \quad = \frac{1}{2} + \frac{1}{4}(n+1)x + \left(\frac{1}{2} - \frac{1}{4}(n-1)x\right)(1+x)^n,$$

$$\text{the denominator} \quad = \frac{1}{2} - \frac{1}{4}(n-1)x + \left(\frac{1}{2} + \frac{1}{4}(n-1)x\right)(1+x)^{-n};$$

the first of them divided by the second obviously yields the propounded formula $(1+x)^n$. But if in the denominator the terms affected with the letters γ, δ, ε etc. would be omitted, then in the numerator one would have to write the unity instead of the fraction $\frac{\omega-1}{2\omega-2} = \frac{0}{0}$ because of the law established above, whence the values C, D, E etc. will result twice as large; and the value of the numerator would be $= \left(1 - \frac{1}{2}(n-1)x\right)(1+x)^n$, the denominator on the other hand $= 1 - \frac{1}{2}(n-1)x$, which fraction again leads to the truth. Therefore, let us see, how by means of formulas of this kind so radical as exponential as logarithmic can be exhibited approximately in a convenient way, since it is known that so logarithmic as exponential quantities can be reduced to the form $(1+x)^n$.

PROBLEM 5

§31 *To assign the square root of an arbitrary given non-square number approximately by means of the exhibited formulas.*

SOLUTION

Let the given non-square number be $= aa + b$ and set $\frac{b}{aa} = x$; it will be

$$aa + b = aa(1+x)$$

and hence

$$\sqrt{aa+b} = a(1+x)^{\frac{1}{2}}.$$

Therefore, we will have $n = \frac{1}{2}$ and from the preceding problem the formulas coming continuously closer to $\sqrt{aa+b}$ will be

$$\sqrt{aa+b} = \frac{1+\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{b}{a^2}}{1+\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{b}{a^2}}a,$$

$$\sqrt{aa+b} = \frac{1+\frac{2}{4}\cdot\frac{5}{2}\cdot\frac{b}{a^2}+\frac{2\cdot1}{4\cdot3}\cdot\frac{5\cdot3}{2\cdot4}\cdot\frac{b^2}{a^4}}{1+\frac{2}{4}\cdot\frac{3}{2}\cdot\frac{b}{a^2}+\frac{2\cdot1}{4\cdot3}\cdot\frac{3\cdot1}{2\cdot4}\cdot\frac{b^2}{a^4}}a,$$

$$\sqrt{aa+b} = \frac{1+\frac{3}{6}\cdot\frac{7}{2}\cdot\frac{b}{a^2}+\frac{3\cdot2}{6\cdot5}\cdot\frac{7\cdot5}{2\cdot4}\cdot\frac{b^2}{a^4}+\frac{3\cdot2\cdot1}{6\cdot5\cdot4}\cdot\frac{7\cdot5\cdot3}{2\cdot4\cdot6}\cdot\frac{b^3}{a^6}}{1+\frac{3}{6}\cdot\frac{5}{2}\cdot\frac{b}{a^2}+\frac{3\cdot2}{6\cdot5}\cdot\frac{5\cdot3}{2\cdot4}\cdot\frac{b^2}{a^4}+\frac{3\cdot2\cdot1}{6\cdot5\cdot4}\cdot\frac{5\cdot3\cdot1}{2\cdot4\cdot6}\cdot\frac{b^3}{a^6}}a$$

etc.

But having expanded these factors and, for the sake of brevity, having put $\frac{bb}{a} = x$, we will obtain the following forms:

$$\sqrt{aa+b} = \frac{1+\frac{3}{4}x}{1+\frac{1}{4}x}a,$$

$$\sqrt{aa+b} = \frac{1+\frac{5}{4}x+\frac{5}{16}xx}{1+\frac{3}{4}x+\frac{1}{16}xx}a,$$

$$\sqrt{aa+b} = \frac{1+\frac{7}{4}x+\frac{7}{8}xx+\frac{7}{64}x^3}{1+\frac{5}{4}x+\frac{3}{8}xx+\frac{1}{64}x^3}a,$$

$$\sqrt{aa+b} = \frac{1+\frac{9}{4}x+\frac{27}{16}xx+\frac{15}{32}x^3+\frac{9}{256}x^4}{1+\frac{7}{4}x+\frac{15}{16}xx+\frac{5}{32}x^3+\frac{1}{256}x^4}a,$$

$$\sqrt{aa+b} = \frac{1+\frac{11}{4}x+\frac{11}{4}xx+\frac{77}{64}x^3+\frac{55}{256}x^4+\frac{11}{1024}x^5}{1+\frac{9}{4}x+\frac{7}{4}xx+\frac{35}{64}x^3+\frac{15}{256}x^4+\frac{1}{1024}x^5}a$$

etc.

But if we put $\frac{x}{4} = y$ or $y = \frac{b}{4aa}$, it will be

$$\sqrt{aa+b} = \frac{1+3y}{1+y}a,$$

$$\sqrt{aa+b} = \frac{1+5y+5yy}{1+3y+yy}a,$$

$$\sqrt{aa+b} = \frac{1+7y+14yy+7y^3}{1+5y+6yy+y^3}a,$$

$$\sqrt{aa+b} = \frac{1+9y+27yy+30y^3+9y^4}{1+7y+15yy+10y^3+y^4}a,$$

$$\sqrt{aa+b} = \frac{1+11y+44yy+77y^3+55y^4+11y^5}{1+9y+28yy+35y^3+15y^4+y^5}a$$

etc.

COROLLARY 1

§32 If we consider the numerators and denominators of these formulas more carefully, we will note without any difficulty that each of them constitute a recurring progression of second order and each term depends on the preceding two in such a way that, if three in order are P, Q, R, always

$$R = (1+2y)Q - yyP$$

or one has the relation scale $1+2y$, $-yy$.

COROLLARY 2

§33 If we set the value for $\frac{b}{4aa}$ for y and free the numerator and the denominator from fractions, we will have the following formulas

$$\sqrt{aa+b} = \frac{4a^2+3b}{4a^2+b}a,$$

$$\sqrt{aa+b} = \frac{16a^4+20a^2b+5bb}{16a^4+12a^3b+bb}a,$$

$$\sqrt{aa+b} = \frac{64a^6+112a^4b+56a^2b^2+7b^3}{64a^6+80a^4b+24a^2b^2+b^3}a,$$

$$\sqrt{aa+b} = \frac{256a^8+576a^6b+432a^4b^2+120a^2b^3+9b^4}{256a^8+448a^6b+240a^4b^2+40a^2b^3+b^4}$$

etc.

COROLLARY 3

§34 In these formulas again so the numerators as the denominators constitute a recurring series, whose relation scale is $2(2aa+b)$, $-bb$, such that, if P, Q, R denote three consecutive terms, it will be

$$R = 2(2aa+b)Q - bbP.$$

But the two initial terms of the numerators are 1 and $4aa+3bb$, of the denominators on the other hand 1 and $4aa+b$, whence the remaining ones are easily found.

COROLLARY 4

§35 If the fraction $\frac{b}{4aa}$ can be reduced to smaller terms, it will be more convenient to use them instead of b and $4aa$. Therefore, in smallest terms set

$$\frac{b}{4aa} = \frac{y}{z}$$

and we will have

$$\sqrt{aa+b} = \frac{z+3y}{z+y}a,$$

$$\sqrt{aa+b} = \frac{zz+5yz+5yy}{zz+3yz+yy}a,$$

$$\sqrt{aa+b} = \frac{z^3+7yz^2+14y^2z+7y^3}{z^3+5yz^2+6y^2z+y^3}$$

and hence it will be

$$R = (z+2y)Q - yyP.$$

COROLLARY 5

§36 These fractions can be expressed even more conveniently this way

$$\sqrt{aa+b} = \frac{z+2y+y\cdot 1}{z+2y-y\cdot 1}a,$$

$$\sqrt{aa+b} = \frac{z^2+4yz+3yy+y(z+2y)}{z^2+4yz+3yy-y(z+2y)}a,$$

$$\sqrt{aa+b} = \frac{z^3+6yz^2+10y^2z+4y^3+y(z^2+4yz+3y^2)}{z^3+6yz^2+10y^2z+4y^3-y(z^2+4yz+3y^2)}a,$$

$$\sqrt{aa+b} = \frac{z^4+8yz^3+21y^2z^2+20y^3z+5y^4+y(z^3+6yz^2+10y^2z+4y^3)}{z^4+8yz^3+21y^2z^2+20y^3z+5y^4-y(z^3+6yz^2+10y^2+4y^3)}a$$

etc.

COROLLARY 6

§37 For these fractions it suffices to have constituted this one series

$$1, \quad z+2y, \quad z^2+4yz+3y^2, \quad z^3+6yz^2+10y^2z+4y^3, \cdots P, Q, R, \cdots$$

which likewise is recurring with the law

$$R = (z + 2y)Q - yyP.$$

But having formed this series, it will approximately be

$$\sqrt{aa + b} = \frac{Q + yP}{Q - yP}a,$$

which fraction is very easily formed from two consecutive terms of that series, of course.

Example 1

§38 *To exhibit the square root of* 2 *approximately.*
Since $aa + b = 2$, it will be $a = 1$ and $b = 1$, whence $\frac{b}{4aa} = \frac{1}{4} = \frac{y}{z}$; therefore, $y = 1$ and $z = 4$ and $z + 2y = 6$. Hence from the relation scale

$$R = 6Q - P$$

form this recurring series

$$1, \quad 6, \quad 35, \quad 204, \quad 1189, \quad 6930, \quad 40391, \cdots P, Q, R, \cdots$$

and the fractions $\frac{Q+P}{Q-P}$ coming continuously closer to $\sqrt{2}$ are

$$\sqrt{2} = \frac{7}{5}, \quad \frac{41}{29}, \quad \frac{239}{169}, \quad \frac{1393}{985}, \quad \frac{8119}{5741}, \quad \frac{47321}{33461} \quad \text{etc.}$$

Example 2

§39 *To exhibit the square root of* 3 *approximately.*
Since $aa + b = 3$, set $a = 1$; it will be $b = 2$ and $\frac{b}{4aa} = \frac{2}{4} = \frac{1}{2}$, whence $y = 1$ and $z = 2$, therefore, $z + 2y = 4$. Hence from the relation scale

$$R = 4Q - P$$

form this recurring series

$$1, \quad 4, \quad 15, \quad 56, \quad 209, \quad 780, \quad 2911, \quad 10864, \cdots P, Q, R, \cdots$$

and it will approximately be $\sqrt{3} = \frac{Q+P}{Q-P}$ or

$$\sqrt{3} = \frac{5}{3}, \quad \frac{19}{11}, \quad \frac{71}{41}, \quad \frac{265}{153}, \quad \frac{989}{571}, \quad \frac{3691}{2131}, \quad \frac{13775}{7953} \quad \text{etc.}$$

Alternatively. Or let us set $a = 2$ such that $b = -1$; it will be $\frac{b}{4aa} = -\frac{1}{16} = \frac{y}{z}$, whence $y = -1$, $z = 16$ and $z + 2y = 14$. Hence from the relation scale

$$R = 14Q - P$$

form the recurring series

$$1, \quad 14, \quad 195, \quad 2716, \quad 37829, \quad 526890, \cdots P, Q, R, \cdots$$

it will approximately be $\sqrt{3} = \frac{Q-P}{Q+P} \cdot 2$ or

$$\sqrt{3} = \frac{13}{15} \cdot 2, \quad \frac{181}{209} \cdot 2, \quad \frac{2521}{2911} \cdot 2, \quad \frac{35113}{40545} \cdot 2 \quad \text{etc.}$$

or

$$\sqrt{3} = \frac{26}{15}, \quad \frac{362}{209}, \quad \frac{5042}{2911}, \quad \frac{70226}{40545} \quad \text{etc.}$$

PROBLEM 6

§40 *To assign the cube root of a given non-cubic number approximately by means of the formulas exhibited just before.*

SOLUTION

Let the given non-cubic number be $= a^3 + b$ and put $\frac{b}{a^3} = x$; it will be

$$a^3 + b = a^3(1 + x)$$

and hence

$$\sqrt[3]{a^3 + b} = a(1 + x)^{\frac{1}{3}}.$$

Therefore, we have $n = \frac{1}{3}$, whence from § 28 we will have these approximations

$$\sqrt[3]{a^3+b} = a \cdot \frac{1+\frac{1}{2}\cdot\frac{4}{3}x}{1+\frac{1}{2}\cdot\frac{2}{3}x},$$

$$\sqrt[3]{a^3+b} = a \cdot \frac{1+\frac{2}{4}\cdot\frac{7}{3}x+\frac{2\cdot 1}{4\cdot 3}\cdot\frac{7\cdot 4}{3\cdot 6}x^2}{1+\frac{2}{4}\cdot\frac{5}{3}x+\frac{2\cdot 1}{4\cdot 3}\cdot\frac{5\cdot 2}{3\cdot 6}x^2},$$

$$\sqrt[3]{a^3+b} = a \cdot \frac{1+\frac{3}{6}\cdot\frac{10}{3}x+\frac{3\cdot 2}{6\cdot 5}\cdot\frac{10\cdot 7}{3\cdot 6}xx+\frac{3\cdot 2\cdot 1}{6\cdot 5\cdot 4}\cdot\frac{10\cdot 7\cdot 4}{3\cdot 6\cdot 9}x^3}{1+\frac{3}{6}\cdot\frac{8}{3}x+\frac{3\cdot 2}{6\cdot 5}\cdot\frac{8\cdot 5}{3\cdot 6}xx+\frac{3\cdot 2\cdot 1}{6\cdot 5\cdot 4}\cdot\frac{8\cdot 5\cdot 2}{3\cdot 6\cdot 9}x^3}$$

etc.

But having expanded these coefficients, we will have

$$\sqrt[3]{a^3+b} = a \cdot \frac{1+\frac{2}{3}x}{1+\frac{1}{3}x},$$

$$\sqrt[3]{a^3+b} = a \cdot \frac{1+\frac{7}{6}x+\frac{7}{27}x^2}{1+\frac{5}{6}x+\frac{5}{54}x^2},$$

$$\sqrt[3]{a^3+b} = a \cdot \frac{1+\frac{5}{3}x+\frac{7}{9}xx+\frac{7}{81}x^3}{1+\frac{4}{3}x+\frac{4}{9}xx+\frac{2}{81}x^3}$$

etc.

COROLLARY 1

§41 If we substitute the value $\frac{b}{a^3}$ instead of x and get rid of the fractions, we will obtain the following formulas

$$\sqrt[3]{a^3+b} = \frac{3a^3+2b}{3a^3+b}a,$$

$$\sqrt[3]{a^3+b} = \frac{54a^6+63a^3b+14bb}{54a^6+45a^3b+5bb}a,$$

$$\sqrt[3]{a^3+b} = \frac{81a^9+135a^6b+63a^3b^2+7b^3}{81a^9+108a^6b+36a^3b^2+2b^3}$$

etc.

But here it is not possible to find a convenient law of progression.

COROLLARY 2

§42 But it suffices to use the first form; for, hence a cube coming closer to the given number is calculated, whose root for given a will give a new value for b and x. So if the cube root of 2 is in question, it will immediately be $a = 1$ and approximately $\sqrt[3]{2} = \frac{5}{4}$. Now let $a = \frac{5}{4}$ and it is $b = 2 - a^3 = \frac{3}{64}$ and $x = \frac{3}{125}$, whence again by means of the first series it will be

$$\sqrt[3]{2} = \frac{125+2}{125+1} \cdot \frac{5}{4} = \frac{127}{126} \cdot \frac{5}{4} = \frac{635}{504};$$

the cube of this fraction is approximately $2 - \frac{1}{2 \cdot 63^3}$, which therefore differs from the truth just by the part $\frac{1}{500094}$.

COROLLARY 3

§43 In like manner formulas for the extraction of roots of higher powers can be constructed. So if $\sqrt[m]{a^m+b}$ is in question, put $x = \frac{b}{a^m}$ and $n = \frac{1}{m}$ and hence one will have

$$\sqrt[m]{a^m+b} = \frac{2ma^m+(m+1)b}{2ma^m+(m-1)b}a,$$

which can even suffice to define roots to an arbitrary level of precision.

PROBLEM 7

§44 *By means of the formulas found above, to express the logarithm of any given number approximately.*

SOLUTION

Let $1 + x$ be the given number and its known that its hyperbolic logarithm is

$$\log(1+x) = \frac{(1+x)^n - 1}{n},$$

while $n = 0$. If in the formulas found above we consider n as an infinitely small number, we will have

$$(1+x)^n = \frac{1+\frac{1}{2}(1+n)x}{1+\frac{1}{2}(1-n)x},$$

$$(1+x)^n = \frac{1+\frac{2}{4}(2+n)x+\frac{2\cdot1}{4\cdot3}\left(1+\frac{3}{2}n\right)x^2}{1+\frac{2}{4}(2-n)x+\frac{2\cdot1}{4\cdot3}\left(1-\frac{3}{2}n\right)x^2},$$

$$(1+x)^n = \frac{1+\frac{3}{6}(3+n)x+\frac{3\cdot2}{6\cdot5}\left(3+\frac{5}{2}n\right)x^2+\frac{3\cdot2\cdot1}{6\cdot5\cdot4}\left(1+\frac{11}{6}n\right)x^3}{1+\frac{3}{6}(3-n)x+\frac{3\cdot2}{6\cdot5}\left(3-\frac{5}{2}n\right)x^2+\frac{3\cdot2\cdot1}{6\cdot5\cdot4}\left(1-\frac{11}{6}n\right)x^3},$$

$$(1+x)^n = \frac{1+\frac{4}{8}(4+n)x+\frac{4\cdot3}{8\cdot7}\left(6+\frac{7}{2}n\right)x^2+\frac{4\cdot3\cdot2}{8\cdot7\cdot6}\left(4+\frac{13}{3}n\right)x^3+\frac{4\cdot3\cdot2\cdot1}{8\cdot7\cdot6\cdot5}\left(1+\frac{25}{12}n\right)x^4}{1+\frac{4}{8}(4-n)x+\frac{4\cdot3}{8\cdot7}\left(6-\frac{7}{2}n\right)x^2+\frac{4\cdot3\cdot2}{8\cdot7\cdot6}\left(4-\frac{13}{3}n\right)x^3+\frac{4\cdot3\cdot2\cdot1}{8\cdot7\cdot6\cdot5}\left(1-\frac{25}{12}n\right)x^4}$$

etc.

If one puts $n = 0$ here now, we will have the following approximations for $\log(1+x)$

$$\log(1+x) = \frac{x}{1+\frac{1}{2}x},$$

$$\log(1+x) = \frac{x+\frac{1}{2}x^2}{1+x+\frac{1}{6}x^2},$$

$$\log(1+x) = \frac{x+x^2+\frac{11}{60}x^3}{1+\frac{3}{2}x+\frac{3}{5}x^2+\frac{1}{20}x^3},$$

$$\log(1+x) = \frac{x+\frac{3}{2}x^2+\frac{13}{21}x^3+\frac{5}{84}x^4}{1+2x+\frac{9}{7}x^2+\frac{2}{7}x^3+\frac{1}{70}x^4}$$

etc.

Or if one puts $x = \frac{m}{n}$, since it is convenient to explore mainly the logarithms of fractions, and one gets rid of the partial fractions, it will be

$$\log\left(1+\frac{m}{n}\right) = \frac{2m}{2n+m},$$

$$\log\left(1+\frac{m}{n}\right) = \frac{6mn+3mm}{6nn+6mn+mm},$$

$$\log\left(1+\frac{m}{n}\right) = \frac{60mn^2+60m^2n+11m^3}{60n^3+90mn^2+36m^2n+3m^3},$$

$$\log\left(1+\frac{m}{n}\right) = \frac{420mn^3+630m^2n^2+260m^3n+25m^4}{420n^4+840mn^3+540m^2n^2+120m^3n+6m^4}$$

etc.

and these fractions come so close to the true value $\log\left(1+\frac{m}{n}\right)$ such that of usual series

$$\log\left(1+\frac{m}{n}\right) = \frac{m}{n} - \frac{m^2}{2n^2} + \frac{m^3}{3n^3} - \frac{m^4}{4n^4} + \text{etc.}$$

one would have to take a huge number of terms to obtain the same approximation.

COROLLARY 1

§45 Therefore, if we desire the hyperbolic logarithm of two, because of $m = 1$ and $n = 1$ the following approximations will result

$$\log 2 = \frac{2}{3}, \quad \frac{9}{13}, \quad \frac{131}{189}, \quad \frac{1335}{1926} = \frac{445}{642};$$

having converted these fractions into decimals, since

$$\log 2 = 0,6931471805599453,$$

it will approximately be

$$\log 2 = 0,666666,$$

$$\log 2 = 0,692308,$$

$$\log 2 = 0,693122,$$

$$\log 2 = 0,69314642,$$

but in reality

$$\log 2 = 0,69314718;$$

and so fourth fraction deviates form the truth just by the part $\frac{76}{100000000}$.

COROLLARY 2

§46 But the logarithms of numbers smaller than two will be found a lot more exactly. For, since $\log \frac{3}{2} = 0,405465108108164$, let us put $m = 1$ and $n = 2$ and our formulas will approximately give

$$\log \frac{3}{2} = \frac{2}{5} = 0,40000000,$$

$$\log \frac{3}{2} = \frac{15}{37} = 0,405405405,$$

$$\log \frac{3}{2} = \frac{371}{915} = 0,405464481,$$

$$\log \frac{3}{2} = \frac{6425}{15846} = 0,4054651016;$$

the error of the last fraction is $\frac{65}{10000000000}$, of course, and hence more than a hundred times smaller than in the preceding case.

COROLLARY 3

§47 Therefore, whenever the fraction $\frac{m}{n}$ is smaller than $\frac{1}{2}$, then it will be

$$\log\left(1+\frac{m}{n}\right) = \frac{420mn^3 + 630m^2n^2 + 260m^3n + 25m^4}{420n^4 + 840mn^3 + 540m^2n^2 + 120m^3n + 6m^4}$$

so exactly such that in the decimal fraction the error is seen just after the tenth digit. But using other methods it is certainly hardly possible to get to the truth so easily.

COROLLARY 4

§48 If the fraction $\frac{m}{n}$ was very small, then it will suffice to use the first or second formula; so, if $\frac{m}{n} = \frac{1}{8}$, the first formula gives

$$\log\frac{9}{8} = \frac{2}{17} = 0,11765$$

and the second

$$\log\frac{9}{8} = \frac{51}{433} = 0,11778291;$$

but actually

$$\log\frac{9}{8} = 0,11778303,$$

whence the second formula differs from the truth by about $\frac{1}{10000000}$.

PROBLEM 8

§49 *To express the exponential quantity e^x approximately by means of the formulas found before, while e is the number, whose hyperbolic logarithm becomes equal to* 1.

Solution

It is known that

$$e^x = \left(1 + \frac{x}{n}\right)^n,$$

if one takes an infinite number for n. Therefore, in the formulas of § 28 let us write x instead of $\frac{x}{n}$ and put $n = \infty$ at the same time, and we will obtain the following approximation

$$e^x = \frac{1 + \frac{1}{2}x}{1 - \frac{1}{2}x},$$

$$e^x = \frac{1 + \frac{2}{4}x + \frac{1}{4\cdot 3}x^2}{1 - \frac{2}{4}x + \frac{1}{4\cdot 3}x^2},$$

$$e^x = \frac{1 + \frac{3}{6}x + \frac{3}{6\cdot 5}x^2 + \frac{1}{6\cdot 5\cdot 4}x^3}{1 - \frac{3}{6}x + \frac{3}{6\cdot 5}x^2 - \frac{1}{6\cdot 5\cdot 4}x^3},$$

$$e^x = \frac{1 + \frac{4}{8}x + \frac{6}{8\cdot 7}x^2 + \frac{4}{8\cdot 7\cdot 6}x^3 + \frac{1}{8\cdot 7\cdot 6\cdot 5}x^4}{1 - \frac{4}{8}x + \frac{6}{8\cdot 7}x^2 - \frac{4}{8\cdot 7\cdot 6}x^3 + \frac{1}{8\cdot 7\cdot 6\cdot 5}x^4}$$

etc.

Hence the law, according to which the following formulas of this kind have to be constructed, is obvious. If we want to throw out of the partial fractions, we will have

$$e^x = \frac{2+x}{2-x},$$

$$e^x = \frac{12+6x+x^2}{12-6x+x^2},$$

$$e^x = \frac{120+60x+12x^2+x^3}{120-60x+12x^2-x^3},$$

$$e^x = \frac{1680+840x+180x^2+20x^3+x^4}{1680-840x+180x^2-20x^3+x^4}$$

etc.

COROLLARY 1

§50 Therefore, hence the number e itself will be in approximating fractions

$$e = \frac{3}{1}, \quad \frac{19}{7}, \quad \frac{193}{71}, \quad \frac{2721}{1001} \quad \text{etc.},$$

the law of which fractions should be noted that, if one puts

$$e = \frac{A}{\mathfrak{A}}, \quad \frac{B}{\mathfrak{B}}, \quad \frac{C}{\mathfrak{C}}, \quad \frac{D}{\mathfrak{D}}, \quad \frac{E}{\mathfrak{E}} \quad \text{etc.},$$

we have

$$A = 3, \quad B = 6A+1, \quad C = 10B + A, \quad D = 14C + B, \quad E = 18D + C \quad \text{etc.},$$

$$\mathfrak{A} = 1, \quad \mathfrak{B} = 6\mathfrak{A}+1, \quad \mathfrak{C} = 10\mathfrak{B} + \mathfrak{A}, \quad \mathfrak{D} = 14\mathfrak{C} + \mathfrak{B}, \quad \mathfrak{E} = 18\mathfrak{D} + \mathfrak{C} \quad \text{etc.},$$

where the multipliers 6, 10, 14, 18 etc. are even numbers of the form $4n+2$.

COROLLARY 2

§51 Therefore, since

$$e = 2,71828182845904523526,$$

let us see how close the found fractions come to the truth

$$e = \frac{3}{1} = 3,0000,$$

$$e = \frac{19}{7} = 2,714285714,$$

$$e = \frac{193}{71} = 2,718309859,$$

$$e = \frac{2721}{1001} = 2,718281718$$

etc.,

where the first differs in the tenth, the second in the thousands, the third in the hundred thousands and the fourth in the hundred times hundred thousands.

COROLLARY 3

§52 Such a law of progression is also detected in the general formulas for e^x. For, if in our formulas we put

$$e = \frac{A}{\mathfrak{A}}, \quad \frac{B}{\mathfrak{B}}, \quad \frac{C}{\mathfrak{C}}, \quad \frac{D}{\mathfrak{D}}, \quad \frac{E}{\mathfrak{E}} \quad \text{etc.,}$$

having taken $A = 1$ and $\mathfrak{A} = 1$, it will be

$$B = 2 + x, \quad C = 6B + Axx, \quad D = 10C + Bxx, \quad E = 14D + Cxx \quad \text{etc.,}$$

$$\mathfrak{B} = 2 - x, \quad \mathfrak{C} = 6\mathfrak{B} + \mathfrak{A}xx, \quad \mathfrak{D} = 10\mathfrak{C} + \mathfrak{B}xx, \quad \mathfrak{E} = 14\mathfrak{D} + \mathfrak{C}xx \quad \text{etc.}$$

Hence the series so of the numerators as the denominators are easily continued.