On the formation of continued fractions [*]

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§1 The universal principle leading to continued fractions is found in an infinite series of quantities

$$A, \quad B, \quad C \quad \text{etc.},$$

three subsequent terms of which depend on each other according to a certain law, which is either constant or somehow variable, in such a way that

$$fA = gB + hC, \quad f'B = g'C + h'D, \quad f''C = g''D + h''E,$$
$$f'''D = g'''E + h'''F \quad \text{etc.}$$

For, hence the following equalities are deduced:

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$$\begin{aligned}
\frac{fA}{B} &= g &+ \frac{hC}{B} &= g &+ \frac{f'h}{f'B:C}, \\
\frac{f'B}{C} &= g' &+ \frac{h'D}{C} &= g' &+ \frac{f''h'}{f''C:D}, \\
\frac{f''C}{D} &= g'' &+ \frac{h''E}{D} &= g'' &+ \frac{f'''h''}{f'''D:E}, \\
\frac{f'''D}{E} &= g''' &+ \frac{h'''F}{E} &= g''' &+ \frac{f''''h'''}{f''''E:F},
\end{aligned}$$

etc.

If now the later values are continuously substituted in the earlier ones, immediately the following continued fraction will emerge

$$\frac{fA}{B} = g + \cfrac{f'h}{g' + \cfrac{f''h'}{g'' + \cfrac{f'''h''}{g''' + \cfrac{f''''h'''}{g'''' + \text{etc.}}}}}$$

whose value is therefore determined by the first two terms A and B of the series only.

§2 Therefore, as often as one has such a progression of quantities A, B, C, D, E etc., whose law is of such a nature that three subsequent terms of it depend on each other according to an arbitrary law of the above kind, hence a continued fraction is deduced whose value can be assigned. Therefore, if a formula was of such a nature that its expansion leads to a series of quantities A, B, C, D, E etc. of this kind, one can hence derive continued fractions; how this happens, it most conveniently shown in some examples.

I. Expansion of the Formula $s = x^n(\alpha - \beta x - \gamma xx)$

§3 In this formula the exponent n is considered as a variable successively obtaining all values

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6 \quad \text{etc.},$$

whence, as long as $n > 0$, this formula vanishes for $x = 0$, but then it also vanishes for

$$x = \frac{-\beta \pm \sqrt{\beta\beta + 4\alpha\gamma}}{2\gamma}.$$

Having noted these things, differentiate this formula that

$$ds = n\alpha x^{n-1}dx - (n+1)\beta x^n dx - (n+2)\gamma x^{n+1}dx,$$

whence by integrating term by term and just indicating the integration it will be

$$n\alpha \int x^{n-1}dx = (n+1)\beta \int x^n dx + (n+2)\gamma \int x^{n+1}dx + s.$$

Hence, if after each integration done in such a way that the integral vanishes for $x = 0$ one sets

$$x = \frac{-\beta \pm \sqrt{\beta\beta + 4\alpha\gamma}}{2\gamma},$$

in which case $s = 0$, of course, it will be

$$n\alpha \int x^{n-1}dx = (n+1)\beta \int x^n dx + (n+2)\gamma \int x^{n+1}dx,$$

which is a relation of such a kind among three subsequent integral formulas, as we desire it for the formation of a continued fraction; thus, these integral formulas, if one successively writes the numbers 1, 2, 3, 4, 5, 6 etc. instead of n, provide us with the quantities A, B, C, D etc.

§4 Therefore, let us write the natural numbers 1, 2, 3, 4 etc. in order instead of n that these relations result

$$\begin{aligned}
\alpha \int dx &= 2\beta \int xdx + 3\gamma \int xxdx, \\
2\alpha \int xdx &= 3\beta \int xxdx + 4\gamma \int x^3 dx, \\
3\alpha \int xxdx &= 4\beta \int x^3 dx + 5\gamma \int x^4 dx, \\
4\alpha \int x^3 dx &= 5\beta \int x^4 dx + 6\gamma \int x^5 dx
\end{aligned}$$

etc.,

Therefore, one will hence have

$$\begin{aligned}
A &= \int dx = \frac{-\beta \pm \sqrt{\beta\beta + 4\alpha\gamma}}{2\gamma}, \\
B &= \int xdx = \frac{1}{2}xx = \frac{1}{2}\left(\frac{-\beta \pm \sqrt{\beta\beta + 4\alpha}}{2\gamma}\right)^2, \\
C &= \int xxdx = \frac{1}{3}x^3, \\
D &= \int x^3 dx = \frac{1}{4}x^4
\end{aligned}$$

etc.

But then one will have these values for the letters f, g, h etc.:

$$\begin{array}{lllll}
f = \alpha, & f' = 2\alpha, & f'' = 3\alpha, & f''' = 4\alpha & \text{etc.;} \\
g = 2\beta, & g' = 3\beta, & g'' = 4\beta, & g''' = 5\beta & \text{etc.;} \\
h = 3\gamma, & h' = 4\gamma, & h'' = 5\gamma, & h''' = 6\gamma & \text{etc.;}
\end{array}$$

from these values the following continued fraction results

$$\frac{\alpha A}{B} = 2\beta + \cfrac{6\alpha\gamma}{3\beta + \cfrac{12\alpha\gamma}{4\beta + \cfrac{20\alpha\gamma}{5\beta + \cfrac{30\alpha\gamma}{6\beta + \text{etc.}}}}}$$

whose value therefore is

$$\frac{4\alpha\gamma}{-\beta + \sqrt{\beta\beta + 4\alpha\gamma}} = \beta + \sqrt{\beta\beta + 4\alpha\gamma}.$$

§5 To simplify this continued fraction, let us write $\frac{1}{2}\delta$ instead of $\alpha\gamma$ and it results

$$\beta + \sqrt{\beta\beta + 2\delta} = 2\beta + \cfrac{3\delta}{3\beta + \cfrac{6\delta}{4\beta + \cfrac{10\delta}{5\beta + \cfrac{15\delta}{6\beta + \text{etc.}}}}}$$

But since this expression seems to be truncated by its head, having added this head, let us set

$$s = \beta + \cfrac{\delta}{2\beta + \cfrac{3\delta}{3\beta + \cfrac{6\delta}{4\beta + \cfrac{10\delta}{5\beta + \text{etc.}}}}}$$

and it will be

$$s = \beta + \frac{\delta}{\beta + \sqrt{\beta\beta + 2\delta}},$$

which expression is reduced to this one

$$s = \frac{1}{2}\beta + \frac{1}{2}\sqrt{\beta\beta + 2\delta}.$$

§6 But this continued fraction can be simplified even further, if we write 2ε instead of δ, such that

$$\frac{1}{2}\beta + \frac{1}{2}\sqrt{\beta\beta + 4\varepsilon} = \beta + \cfrac{2\varepsilon}{2\beta + \cfrac{6\varepsilon}{3\beta + \cfrac{12\varepsilon}{4\beta + \cfrac{20\varepsilon}{5\beta + \text{etc.}}}}}$$

If now the denominator and numerator of the first fraction are divided by 2, of the second by 3, of the third by 4, of the fourth by 5, the following form will result

$$\frac{1}{2}\beta + \frac{1}{2}\sqrt{\beta\beta + 4\varepsilon} = \beta + \cfrac{\varepsilon}{\beta + \cfrac{\varepsilon}{\beta + \cfrac{\varepsilon}{\beta + \cfrac{\varepsilon}{\beta + \text{etc.}}}}}$$

which is very simple; if its sum is considered to be unknown and is called $= z$, it will obviously be $z = \beta + \frac{\varepsilon}{z}$ and hence $zz = \beta z + \varepsilon$, whence

$$z = \frac{\beta + \sqrt{\beta\beta + 4\varepsilon}}{2},$$

which is the same.

§7 But that very simple sum can be deduced immediately from the form assumed initially, i.e.

$$s = x^n(\alpha - \beta x - \gamma xx),$$

since we put which equal to zero, it will obviously be

$$\alpha = \beta x + \gamma xx$$

and in like manner

$$\alpha x = \beta xx + \gamma x^3, \quad \alpha xx = \beta x^3 + \gamma x^4 \quad \text{etc.},$$

such that for the series A, B, C, D etc. we have this simple series of powers

$$1, \quad x, \quad x^2, \quad x^3, \quad x^4 \quad \text{etc.};$$

but then the letters f, g, h etc. become α, β, γ etc., whence this continued fractions originates

$$\frac{\alpha}{x} = \beta + \cfrac{\alpha\gamma}{\beta + \cfrac{\alpha\gamma}{\beta + \cfrac{\alpha\gamma}{\beta + \text{etc.}}}}$$

where

$$\frac{1}{x} = \frac{\beta + \sqrt{\beta\beta + 4\alpha\gamma}}{2\alpha}.$$

Therefore, the value of this fraction is

$$\frac{1}{2}\beta + \frac{1}{2}\sqrt{\beta\beta + 4\alpha\gamma}$$

as before, since $\alpha\gamma = \varepsilon$.

II. Expansion of the Formula $s = x^n(a-x)$

§8 Therefore, this formula vanishes putting $x = a$; but hence

$$ds = nax^{n-1}dx - (n+1)x^n dx,$$

which expression, since it consists only of two terms, must be reduced to a fraction, whose denominator is $\alpha + \beta x$, such that

$$ds = \frac{na\alpha x^{n-1}dx + (\beta na - \alpha(n+1))x^n dx - \beta(n+1)x^{n+1}dx}{\alpha + \beta x}.$$

Therefore, having integrated each term separately, it will be

$$s = na\alpha \int \frac{x^{n-1}dx}{\alpha + \beta x} + (n\beta a - (n+1)\alpha) \int \frac{x^n dx}{\alpha + \beta x} - \beta(n+1) \int \frac{x^{n+1}dx}{\alpha + \beta x};$$

hence, if we set $x = a$ after each integration, that $s = 0$, we will have this reduction

$$na\alpha\int\frac{x^{n-1}dx}{\alpha+\beta x}=((n+1)\alpha-n\beta a)\int\frac{x^{n}dx}{\alpha+\beta x}+(n+1)\beta\int\frac{x^{n+1}dx}{\alpha+\beta x}.$$

§9 Let us now successively substitute the numbers 1, 2, 3, 4 etc. for n and having made the comparison to the general formulas we will have

$$A=\int\frac{dx}{\alpha+\beta x},\quad B=\int\frac{xdx}{\alpha+\beta x},\quad C=\int\frac{xxdx}{\alpha+\beta x}\quad\text{etc.},$$

where after the integration it must be $x=a$, of course. Furthermore, we will have

$$\begin{aligned}&f=a\alpha,\quad f'=2a\alpha,\quad f''=3a\alpha,\quad f'''=4a\alpha\quad\text{etc.};\\&g=2\alpha-\beta a,\quad g'=3\alpha-2\beta a,\quad g''=4\alpha-3\beta a\quad\text{etc.};\\&h=2\beta,\quad h'=3\beta,\quad h''=4\beta,\quad h'''=5\beta\quad\text{etc.};\end{aligned}$$

and from these the following continued fraction results

$$\frac{\alpha aA}{B}=(2\alpha-\beta a)+\cfrac{4a\alpha\beta}{(3\alpha-2\beta a)+\cfrac{9a\alpha\beta}{(4\alpha-3\beta a)+\cfrac{16a\alpha\beta}{(5\alpha-4\beta a)+\text{etc.}}}}$$

§10 But having done the integration

$$\int\frac{dx}{\alpha+\beta x}=\frac{1}{\beta}\log\frac{\alpha+\beta x}{\alpha},$$

since the integrals must vanish for $x=0$. Therefore, now let

$$x=a$$

and it will be

$$A=\frac{1}{\beta}\log\frac{\alpha+\beta x}{\alpha}.$$

Further,

$$\int \frac{xdx}{\alpha+\beta x} = \frac{1}{\beta}\left(x - \frac{\alpha}{\beta}\log\frac{\alpha+\beta x}{\alpha}\right)$$

and for $x = a$ it will be

$$B = \frac{a}{\beta} - \frac{\alpha}{\beta\beta}\log\frac{\alpha+\beta a}{\alpha},$$

for which reason the value of our continued fraction will be

$$\frac{\alpha a\beta \log\frac{\alpha+\beta a}{\alpha}}{a\beta - \alpha\log\frac{\alpha+\beta a}{\alpha}};$$

but it is evident that without loss of generality one can take $a = 1$; for, then it will be

$$\frac{\alpha\beta\log\frac{\alpha+\beta}{\alpha}}{\beta - \alpha\log\frac{\alpha+\beta}{\alpha}} = (2\alpha-\beta) + \cfrac{4\alpha\beta}{(3\alpha-2\beta) + \cfrac{9\alpha\beta}{(4\alpha-3\beta)} + \text{etc.}}$$

§11 But this whole expression obviously just depends on the ratio of the numbers α and β; hence let us take $\alpha = 1$ and $\beta = n$ and this continued fraction will result

$$\frac{n\log(1+n)}{n-\log(1+n)} = (2-n) + \cfrac{4n}{(3-2n) + \cfrac{9n}{(4-3n) + \cfrac{16n}{(5-4n) + \text{etc.}}}}$$

if, according to the structure, we write $1+n$ in front of which and set the sum $= s$, that

$$s = 1 + \cfrac{n}{(2-n) + \cfrac{4n}{(3-2n) + \cfrac{9n}{(4-3n) + \cfrac{16n}{(5-4n) + \text{etc.}}}}}$$

it will be

$$s = 1 + \frac{n(n - \log(1+n))}{n\log(1+n)} = 1 + \frac{n - \log(1+n)}{\log(1+n)} = \frac{n}{\log(1+n)}.$$

§12 Let us run through several examples and first let $n = 1$; it will be

$$\frac{1}{\log 2} = 1 + \cfrac{1}{1 + \cfrac{4}{1 + \cfrac{9}{1 + \cfrac{16}{1 + \text{etc.}}}}}$$

But for $n = 2$ it will be

$$\frac{2}{\log 3} = 1 + \cfrac{2}{0 + \cfrac{8}{-1 + \cfrac{18}{-2 + \cfrac{32}{-3 + \cfrac{50}{-4 + \text{etc.}}}}}}$$

which expression, because if the negative quantities, is not sufficiently convenient; since this happens, whenever $n > 1$, it will be worth one's while to expand the cases, in which n is taken smaller than 1.

§13 That this can be done more easily, let us return to the expression containing the letters α and β and, having added the head which had been missing, this form results

$$\frac{\beta}{\log \frac{\alpha+\beta}{\alpha}} = \alpha + \cfrac{\alpha\beta}{(2\alpha - \beta) + \cfrac{4\alpha\beta}{(3\alpha - 2\beta) + \cfrac{9\alpha\beta}{(4\alpha - 3\beta) + \text{etc.}}}}$$

Now let us put

$$\alpha = n - m \quad \text{and} \quad \beta = 2m,$$

that we obtain the following form

$$\frac{2m}{\log\frac{n+m}{n-m}} = n-m+\cfrac{2m(n-m)}{2n-4m+\cfrac{8m(n-m)}{3n-7m+\cfrac{18m(n-m)}{4n-10m+\text{etc.}}}}$$

whence the following special cases are deduced.

If $m=1$ and $n=3$, it will be

$$\frac{2}{\log 2} = 2+\cfrac{4}{2+\cfrac{16}{2+\cfrac{36}{2+\cfrac{64}{2+\text{etc.}}}}}$$

which fraction, having divided it by 2 and simplified it, yields this one

$$\frac{1}{\log 2} = 1+\cfrac{1}{1+\cfrac{4}{1+\cfrac{9}{1+\cfrac{16}{1+\text{etc.}}}}}$$

which was found above already.

Let $m=1$ and $n=4$; it will be

$$\frac{2}{\log\frac{5}{3}} = 3+\cfrac{6}{4+\cfrac{24}{5+\cfrac{54}{6+\cfrac{96}{7+\text{etc.}}}}} = 3+\cfrac{6\cdot 1}{4+\cfrac{6\cdot 4}{5+\cfrac{6\cdot 9}{6+\cfrac{6\cdot 16}{7+\text{etc.}}}}}$$

Let $m=1$ and $n=5$; it will be

$$\frac{2}{\log\frac{3}{2}} = 4 + \cfrac{8}{6 + \cfrac{32}{8 + \cfrac{72}{10 + \cfrac{128}{12 + \text{etc.}}}}}$$

or

$$\frac{1}{\log\frac{3}{2}} = 2 + \cfrac{2}{3 + \cfrac{8}{4 + \cfrac{18}{5 + \cfrac{32}{6 + \text{etc.}}}}} = 2 + \cfrac{2 \cdot 1}{3 + \cfrac{2 \cdot 4}{4 + \cfrac{2 \cdot 9}{5 + \cfrac{2 \cdot 16}{6 + \text{etc.}}}}}$$

III. Expansion of the Formula $s = x^n(1 - x^2)$

§14 Therefore, this formula vanishes in the cases $x = 0$ and $x = 1$. But since hence

$$ds = nx^{n-1}dx - (n+2)x^{n+1}dx,$$

reduce this differential to the denominator $\alpha + \beta xx$ and it will be

$$ds = \frac{n\alpha x^{n-1}dx + (n\beta - (n+2)\alpha)x^{n+1}dx - (n+2)\beta x^{n+3}dx}{\alpha + \beta xx}.$$

Hence by integrating again

$$s = n\alpha \int \frac{x^{n-1}dx}{\alpha + \beta xx} + (n\beta - (n+2)\alpha)\int \frac{x^{n+1}dx}{\alpha + \beta xx} - (n+2)\beta \int \frac{x^{n+3}dx}{\alpha + \beta xx}.$$

If one sets $x = 1$ after the integration, this reduction of integrals will result

$$n\alpha \int \frac{x^{n-1}dx}{\alpha + \beta xx} = ((n+2)\alpha - n\beta)\int \frac{x^{n+1}dx}{\alpha + \beta xx} + (n+2)\beta \int \frac{x^{n+3}dx}{\alpha + \beta xx}.$$

§15 Since here the powers of x are increased by two, let us successively attribute the values 1, 3, 5, 7, 9 etc. to the exponent n and set

$$A = \int \frac{dx}{\alpha + \beta xx}, \quad B = \int \frac{xxdx}{\alpha + \beta xx}, \quad C = \int \frac{x^4dx}{\alpha + \beta xx} \quad \text{etc.}$$

Further, the letters f, g, h etc. with its derivatives will be

$$\begin{array}{lllll} f = \alpha, & f' = 3\alpha, & f'' = 5\alpha, & f''' = 7\alpha & \text{etc.;} \\ g = 3\alpha - \beta, & g' = 5\alpha - 3\beta, & g'' = 7\alpha - 5\beta, & g''' = 9\alpha - 7\beta & \text{etc.;} \\ h = 3\beta, & h' = 5\beta, & h'' = 7\beta, & h''' = 9\beta & \text{etc.;} \end{array}$$

whence the following continued fraction results

$$\frac{\alpha A}{B} = 3\alpha - \beta + \cfrac{9\alpha\beta}{5\alpha - 3\beta + \cfrac{25\alpha\beta}{7\alpha - 5\beta + \cfrac{49\alpha\beta}{9\alpha - 7\beta + \text{etc.}}}}$$

§16 Since

$$B = \int \frac{xxdx}{\alpha + \beta xx},$$

it will be

$$B = \frac{1}{\beta}\int dx - \frac{\alpha}{\beta}\int \frac{dx}{\alpha + \beta xx}$$

and hence

$$B = \frac{1}{\beta} - \frac{\alpha}{\beta}A,$$

having substituted which value we will have

$$\frac{\alpha\beta A}{1 - \alpha A} = 3\alpha - \beta + \cfrac{9\alpha\beta}{5\alpha - 3\beta + \cfrac{25\alpha\beta}{7\alpha - 5\beta + \text{etc.}}}$$

in front of which, since the head is missing, we want to write $\alpha+\beta+\alpha\beta$; but then the sum will be $\beta+\frac{1}{A}$ such that we have

$$\beta+\frac{1}{A}=\alpha+\beta+\cfrac{\alpha\beta}{3\alpha-\beta+\cfrac{9\alpha\beta}{5\alpha-3\beta+\cfrac{25\alpha\beta}{7\alpha-5\beta+\text{etc.}}}}$$

while

$$A=\int\frac{dx}{\alpha+\beta xx}$$

having taken the integral in such a way that it vanishes for $x=0$ but then having taken $x=1$.

§17 First let us expand the simplest cases in which $\alpha=1$ and $\beta=1$, where it will be $A=\frac{\pi}{4}$, whence we will have

$$1+\frac{4}{\pi}=2+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\text{etc.}}}}}$$

or it will be

$$\frac{4}{\pi}=1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\text{etc.}}}}$$

which is the continued fraction once given by Brouncker first, whose investigation, whereas it was found by Wallis through very tedious calculations, was an immediate corollary of our formula here.

§18 But our general form gives infinitely many other similar expressions, depending on how the letters α and β are taken. And first, if α and β were positive numbers, the value of the letter A will always be expressed in terms of circular arcs, otherwise in terms of logarithms. Therefore, first let $\beta = 1$ and it will be

$$A = \int \frac{dx}{\alpha + xx} = \frac{1}{\sqrt{\alpha}} \arctan \frac{x}{\sqrt{\alpha}} = \frac{1}{\sqrt{\alpha}} \arctan \frac{1}{\sqrt{\alpha}},$$

whence this continued fraction arises:

$$1 + \frac{\sqrt{\alpha}}{\arctan \frac{1}{\sqrt{\alpha}}} = \alpha + 1 + \cfrac{\alpha}{3\alpha - 1 + \cfrac{9\alpha}{5\alpha - 3 + \cfrac{25\alpha}{7\alpha - 5 + \text{etc.}}}}$$

Therefore, hence, if one takes $\alpha = 3$, since $\arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6}$, we will have

$$1 + \frac{6\sqrt{3}}{\pi} = 4 + \cfrac{3}{8 + \cfrac{27}{12 + \cfrac{75}{16 + \cfrac{147}{20 + \text{etc.}}}}}$$

or

$$1 + \frac{6\sqrt{3}}{\pi} = 4 + \cfrac{3 \cdot 1}{8 + \cfrac{3 \cdot 9}{12 + \cfrac{3 \cdot 25}{16 + \cfrac{3 \cdot 49}{20 + \text{etc.}}}}}$$

§19 Now let β be any positive number, and since

$$A = \int \frac{dx}{\alpha + \beta xx} = \frac{1}{\beta} \int \frac{dx}{\frac{\alpha}{\beta} + xx},$$

by integration

$$A = \frac{1}{\sqrt{\alpha\beta}} \arctan \sqrt{\frac{\beta}{\alpha}}.$$

Therefore, hence we will have

$$\beta + \frac{\sqrt{\alpha\beta}}{\arctan\sqrt{\frac{\beta}{\alpha}}} = \alpha + \beta + \cfrac{\alpha\beta}{3\alpha - \beta + \cfrac{9\alpha\beta}{5\alpha - 3\beta - \text{etc.}}}$$

Therefore, let us set

$$\alpha + \beta = 2n \quad \text{and} \quad \alpha - \beta = 2m;$$

that

$$\alpha = m + n \quad \text{and} \quad \beta = n - m;$$

having put these values it will be

$$n - m + \frac{\sqrt{nn - mm}}{\arctan\sqrt{\frac{n-m}{n+m}}} = 2n + \cfrac{nn - mm}{2n + 4m + \cfrac{9(nn - mm)}{2n + 8m + \text{etc.}}}$$

§20 Therefore, let us also consider the case in which β is a negative number, and putting

$$\beta = -\gamma,$$

it will be

$$A = \int \frac{dx}{\alpha - \gamma xx} = \frac{1}{\gamma}\int \frac{dx}{\frac{\alpha}{\gamma} - xx},$$

whose integral is

$$A = \frac{1}{2\sqrt{\alpha\gamma}} \log \frac{\sqrt{\frac{\alpha}{\gamma}} + x}{\sqrt{\frac{\alpha}{\gamma}} - x};$$

therefore, having put $x = 1$, it will be

$$A = \frac{1}{2\sqrt{\alpha\gamma}} \log \frac{\sqrt{\alpha} + \sqrt{\gamma}}{\sqrt{\alpha} - \sqrt{\gamma}},$$

whence this continued fraction arises

$$-\gamma+\frac{2\sqrt{\alpha\gamma}}{\log\frac{\sqrt{\alpha}+\sqrt{\gamma}}{\sqrt{\alpha}-\sqrt{\gamma}}}=\alpha-\gamma-\cfrac{\alpha\gamma}{3\alpha+\gamma-\cfrac{9\alpha}{5\alpha+3\gamma-\cfrac{25\alpha\gamma}{7\alpha+5\gamma-\text{etc.}}}}$$

and this way we obtained new continued fractions whose values can also be exhibited in terms of logarithms and which differ completely from those we found before.

§21 Here, one case is more remarkable than all the remaining ones, namely when

$$\gamma=\alpha$$

or, what reduces to the same,

$$\alpha=1 \quad \text{and} \quad \gamma=1;$$

for, since then

$$\log\frac{\sqrt{\alpha}+\sqrt{\gamma}}{\sqrt{\alpha}-\sqrt{\gamma}}=\log\infty=\infty,$$

we will have

$$-1=0-\cfrac{1}{4-\cfrac{9}{8-\cfrac{25}{12-\text{etc.}}}}$$

or, having changed the signs,

$$1=\cfrac{1}{4-\cfrac{9}{8-\cfrac{25}{12-\text{etc.}}}}$$

Hence the first denominator

$$4-\cfrac{9}{8-\cfrac{25}{12-\text{etc.}}}$$

must be $=1$. Therefore, it will be

$$0=3-\cfrac{9}{8-\cfrac{25}{12-\text{etc.}}}$$

or

$$1=\cfrac{3}{8-\cfrac{25}{12-\text{etc.}}}$$

where the denominator must be $=3$; hence

$$0=5-\cfrac{25}{12-\text{etc.}}$$

whose denominator must be $=5$; hence

$$0=7-\cfrac{49}{16-\cfrac{81}{20-\text{etc.}}}$$

from which structure the truth is easily seen.

§22 Let us take

$$\alpha=4 \quad \text{and} \quad \gamma=1$$

and we will obtain this fraction

$$-1+\frac{4}{\log 3}=3-\cfrac{4\cdot 1}{13-\cfrac{4\cdot 9}{23-\cfrac{4\cdot 25}{33-\cfrac{4\cdot 49}{43-\text{etc.}}}}}$$

But if we take

$$\alpha = 9 \quad \text{and} \quad \gamma = 1,$$

it will be

$$-1 + \frac{6}{\log 2} = 8 - \cfrac{9 \cdot 1}{28 - \cfrac{9 \cdot 9}{48 - \cfrac{9 \cdot 25}{68 - \cfrac{9 \cdot 49}{88 - \text{etc.}}}}}$$

EXPANSION OF THE FORMULA $s = x^n e^{\alpha x}(1-x)$

§22a Here e denotes the number whose hyperbolic logarithm is 1, such that

$$d.e^{\alpha x} = \alpha dx e^{\alpha x}.$$

Therefore, hence it will be

$$ds = nx^{n-1}dxe^{\alpha x} + (\alpha - (n+1))x^n dxe^{\alpha x} - \alpha x^{n+1}dxe^{\alpha x}.$$

Therefore, if after the integration one sets $x = 1$, it will be

$$n\int x^{n-1}dxe^{\alpha x} = (n+1-\alpha)\int x^n dxe^{\alpha x} + \alpha \int x^{n+1}dxe^{\alpha x}.$$

§23 If we now successively write the numbers 1, 2, 3, 4 etc. instead of n and set

$$A = \int e^{\alpha x}dx = \frac{1}{\alpha}(e^{\alpha} - 1) \quad \text{and} \quad B = \int xdxe^{\alpha x} = \frac{\alpha - 1}{\alpha\alpha}e^{\alpha} + \frac{1}{\alpha\alpha},$$

$$\begin{array}{lllll} f = 1, & f' = 2, & f'' = 3, & f''' = 4 & \text{etc.;} \\ g = 2-\alpha, & g' = 3-\alpha, & g'' = 4-\alpha, & g''' = 5-\alpha & \text{etc.;} \\ h = \alpha, & h' = \alpha, & h'' = \alpha, & h''' = \alpha & \text{etc.;} \end{array}$$

this continued fraction will result

$$\frac{A}{B} = 2 - \alpha + \cfrac{2\alpha}{3 - \alpha + \cfrac{3\alpha}{4 - \alpha + \cfrac{4\alpha}{5 - \alpha + \text{etc.}}}}$$

Let us also add $1 - \alpha + \alpha$ at the top; its value will be

$$1 - \alpha + \frac{(\alpha - 1)e^{\alpha} + 1}{e^{\alpha} - 1} = \frac{\alpha}{e^{\alpha} - 1},$$

whence one will have this sufficiently convenient continued fraction

$$\frac{\alpha}{e^{\alpha} - 1} = 1 - \alpha + \cfrac{\alpha}{2 - \alpha + \cfrac{2\alpha}{3 - \alpha + \cfrac{3\alpha}{4 - \alpha - \text{etc.}}}}$$

whence it is plain that, if $\alpha = 0$, because of $e^{\alpha} - 1 = \alpha$, it will be $1 = 1$, of course.

§24 Let us consider some special cases; and first, if $\alpha = 1$, it will be

$$\frac{1}{e - 1} = 0 + \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \text{etc.}}}}}$$

which fraction is easily transformed into this one

$$\frac{1}{e - 1} = \cfrac{1}{1 + \cfrac{\frac{1}{1}}{1 + \cfrac{\frac{1}{2}}{1 + \cfrac{\frac{1}{3}}{1 + \cfrac{\frac{1}{4}}{1 + \text{etc.}}}}}}$$

whence

$$e-1=1+\cfrac{\frac{1}{1}}{1+\cfrac{\frac{1}{2}}{1+\cfrac{\frac{1}{3}}{1+\text{etc.}}}}$$

But this, freed from partial fractions, further gives

$$e-1=1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\cfrac{4}{5+\text{etc.}}}}}}$$

whence it follows

$$\frac{1}{e-2}=1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\cfrac{4}{5+\text{etc.}}}}}$$

which forms seem most remarkable for their simplicity. From the penultimate, by which

$$e=2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\text{etc.}}}}}$$

successively taking 1, 2, 3 and more terms the following approximations will result:

$$e = 2,0000,$$

$$e = 3,0000,$$

$$e = 2,6666,$$

$$e = 2,7272,$$

$$e = 2,7169,$$

which values, alternately too large and too small, converge to the truth sufficiently fast.

§25 Let us take $\alpha = 2$; it will be

$$\frac{2}{ee-1} = -1 + \cfrac{2}{0 + \cfrac{4}{1 + \cfrac{6}{2 + \cfrac{8}{3 + \text{etc.}}}}}$$

From this fraction further this one is deduced

$$\frac{2(ee-1)}{ee+1} = 0 + \cfrac{4}{1 + \cfrac{6}{2 + \cfrac{8}{3 + \text{etc.}}}}$$

and in like manner, if greater numbers are taken for α, one will be able to do the reduction.

§26 One can also take negative numbers for α. So, if $\alpha = -1$, it will be

$$\frac{e}{e-1} = 2 - \cfrac{1}{3 - \cfrac{2}{4 - \cfrac{3}{5 - \cfrac{4}{6 - \text{etc.}}}}}$$

which is reduced to this form

$$\frac{e}{e-1} = 2 + \cfrac{1}{-3 + \cfrac{2}{4 + \cfrac{3}{-5 + \cfrac{4}{6 + \text{etc.}}}}}$$

and in like manner, larger values can be handled.

§27 Let us also set $\alpha = \frac{1}{2}$ and one will find this expression

$$\frac{1}{2(\sqrt{e}-1)} = \frac{1}{2} + \cfrac{\frac{1}{2}}{\frac{3}{2} + \cfrac{1}{\frac{5}{2} + \cfrac{\frac{3}{2}}{\frac{7}{2} + \cfrac{\frac{4}{2}}{\frac{9}{2} + \text{etc.}}}}}$$

which, freed from partial fractions, becomes

$$\frac{1}{-1+\sqrt{e}} = 1 + \cfrac{2}{3 + \cfrac{4}{5 + \cfrac{6}{7 + \cfrac{8}{9 + \text{etc.}}}}}$$

In like manner, if we take $\alpha = \frac{1}{3}$, it will be

$$\frac{1}{3(\sqrt[3]{e}-1)} = 2:3 + \cfrac{1:3}{5:3 + \cfrac{2:3}{8:3 + \cfrac{3:3}{11:3 + \cfrac{4:3}{14:3 + \text{etc.}}}}}$$

which, freed from partial fractions, gives

$$\frac{1}{-1+\sqrt[3]{e}} = 2 + \cfrac{3}{5 + \cfrac{6}{8 + \cfrac{9}{11 + \cfrac{12}{14 + \text{etc.}}}}}$$

but if one sets $\alpha = \frac{2}{3}$, this continued fraction results

$$\frac{2}{3(\sqrt[3]{ee}-1)} = 1:3 + \cfrac{2:3}{4:3 + \cfrac{4:3}{7:3 + \cfrac{6:3}{10:3 + \cfrac{8:3}{13:3 + \text{etc.}}}}}$$

which, freed from partial fractions, becomes

$$\frac{2}{\sqrt[3]{ee}-1} = 1 + \cfrac{6}{4 + \cfrac{12}{7 + \cfrac{18}{10 + \cfrac{24}{13 + \text{etc.}}}}}$$

§28 Having expanded these formulas as the principal and simpler ones, in like manner, it will be possible to treat other a lot more general ones which will lead to more complicated continued fractions, as it will become clear from the cases which follow.

V. Expansion of the Formula $s = x^n(a - bx^\theta - cx^{2\theta})^\lambda$

§29 Therefore, it will hence be

$$ds = (a - bx^\theta - cx^{2\theta})^{\lambda-1} \begin{Bmatrix} (nax^{n-1}dx - b(n+\lambda\theta)x^{n+\theta-1}dx \\ -c(n+2\lambda\theta)x^{n+2\theta-1}dx \end{Bmatrix},$$

whence integrating term by term, but then setting

$$a - bx^\theta - cx^{2\theta} = 0,$$

which happens, if it was

$$x^\theta = \frac{-b+\sqrt{bb+4ac}}{2c},$$

one will have this general reduction

$$na\int x^{n-1}dx(a-bx^\theta-cx^{2\theta})^{\lambda-1}$$

$$=(n+\lambda\theta)b\int x^{n+\theta-1}dx(a-bx^\theta-cx^{2\theta})^{\lambda-1}$$

$$+(n+2\lambda\theta)c\int x^{n+2\theta-1}dx(a-bx^\theta-cx^{2\theta})^{\lambda-1}.$$

§30 If we now want to compare this form to our general one given initially, the values to be assumed successively for the letter n must be increased by the difference θ. Further, it is not necessary that the first value of n, as we have done up to this point, is taken $=1$; therefore, let us set its first value $=\alpha$ and find the values of the two following integral formulas, i.e.

$$A=\int x^{\alpha-1}dx(a-bx^\theta-cx^{2\theta})^{\lambda-1}$$

and

$$B=\int x^{\alpha+\theta-1}dx(a-bx^\theta-cx^{2\theta})^{\lambda},$$

which integrals are to be taken in such a way that they vanish for $x=0$, having done which that value must be attributed to x which renders the formula $a-bx^\theta-cx^{2\theta}=0$. But since this can not be achieved in general, we have to be content to indicate those values by the letters A and B, which we will therefore consider to be known.

§31 Furthermore, the letters f, g, h with its derivatives will have the following values

$$\begin{array}{llll} f=\alpha a, & f'=(\alpha+\theta)a, & f''=(\alpha+2\theta)a, & f'''=(\alpha+3\theta)a \quad \text{etc.} \\ g=(\alpha+\lambda\theta)b, & g'=(\alpha+\theta+\lambda\theta)b, & g''=(\alpha+2\theta+\lambda\theta)b & \text{etc.} \\ h=(\alpha+2\lambda\theta)c, & h'=(\alpha+\theta+2\lambda\theta)c, & h''=(\alpha+2\theta+2\lambda\theta)c & \text{etc.} \end{array}$$

From these the following continued fraction will be formed

$$\frac{\alpha a A}{B} = (\alpha+\lambda\theta)b + \cfrac{(\alpha+\theta)(\alpha+2\lambda\theta)ac}{(\alpha+\theta+\lambda\theta)b + \cfrac{(\alpha+2\theta)(\alpha+\theta+2\lambda\theta)ac}{(\alpha+2\theta+\lambda\theta)b + \cfrac{(\alpha+3\theta)(\alpha+2\theta+2\lambda\theta)ac}{(\alpha+3\theta+\lambda\theta)b + \text{etc.}}}}$$

which form is obviously very general; but we will not spend more time on its further expansion.

VI. Expansion of the Formula $s = x^n(1-x^\theta)^\lambda$

§32 Therefore, hence

$$ds = nx^{n-1}dx(1-x^\theta)^\lambda - \lambda\theta x^{n+\theta-1}(1-x^\theta)^{\lambda-1},$$

whence only two integral formulas would result; therefore, let us attribute the arbitrary denominator $a+bx^\theta$ to this differential that we have

$$ds = \frac{(1-x^\theta)^{\lambda-1}}{a+bx^\theta}\left(nax^{n-1}dx - (a(n+\lambda\theta)-bn)x^{n+\theta-1}dx - b(n+\lambda\theta)x^{n+2\theta-1}dx\right).$$

Therefore, now putting $x=1$ after the integration we deduce this reduction

$$na\int\frac{x^{n-1}dx(1-x^\theta)^{\lambda-1}}{a+bx^\theta} = (a(n+\lambda\theta)-bn)\int\frac{x^{n+\theta-1}dx(1-x^\theta)^{\lambda-1}}{a+bx^\theta}$$
$$+b(n+\lambda\theta)\int\frac{x^{n+2\theta-1}dx(1-x^\theta)^{\lambda-1}}{a+bx^\theta}.$$

§33 But here it is evident again that the values of n must by increased by the difference θ. But set the first value of $n=\alpha$ and for each case find the two following integral formulas

$$A = \int\frac{x^{\alpha-1}dx(1-x^\theta)^{\lambda-1}}{a+bx^\theta} \quad \text{and} \quad B = \int\frac{x^{\alpha+\theta-1}dx(1-x^\theta)^{\lambda-1}}{a+bx^\theta},$$

where x was put $=1$ after the integration, of course. Since, having found these,

$$f = \alpha a, \quad f' = (\alpha+\theta)a, \quad f'' = (\alpha+2\theta)a, \quad f''' = (\alpha+3\theta)a \quad \text{etc.};$$

$$g = (\alpha+\lambda\theta)a - \alpha b, \quad g' = (\alpha+\theta+\lambda\theta)a - (\alpha+\theta)b,$$

$$g'' = (\alpha+2\theta+\lambda\theta)a - (\alpha+2\theta)b \quad \text{etc.};$$

$$h = (\alpha+\lambda\theta)b, \quad h' = (\alpha+\theta+\lambda\theta)b, \quad h'' = (\alpha+2\theta+\lambda\theta)b \quad \text{etc.};$$

Hence the following continued fraction will be formed

$$\frac{\alpha a A}{B} = (\alpha+\lambda\theta)a - \alpha b$$

$$+\cfrac{(\alpha+\theta)(\alpha+\lambda\theta)ab}{(\alpha+\theta+\lambda\theta)a-(\alpha+\theta)b+\cfrac{(\alpha+2\theta)(\alpha+\theta+\lambda\theta)ab}{(\alpha+2\theta+\lambda\theta)a-(\alpha+2\theta)b+\cfrac{(\alpha+3\theta)(\alpha+2\theta+\lambda\theta)ab}{(\alpha+3\theta+\lambda\theta)a-(\alpha+3\theta)b+\text{etc.}}}}$$

the further expansion of which formula we want to omit here.

VII. Expansion of the Formula $s = x^n e^{\alpha x}(1-x)^\lambda$

§34 Therefore, hence

$$ds = e^{\alpha x}(1-x)^{\lambda-1}\left(nx^{n-1}dx - (n+\lambda-\alpha)x^n dx - \alpha x^{n+1}dx\right);$$

therefore, hence, if one sets $x=1$ after the integration everywhere, in which case $s=0$, we will have this reduction

$$n\int x^{n-1}dxe^{\alpha x}(1-x)^{\lambda-1}$$
$$= (n+\lambda-\alpha)\int x^n dxe^{\alpha x}(1-x)^{\lambda-1} + \alpha\int x^{n+1}dxe^{\alpha x}(1-x)^{\lambda-1}.$$

§35 Therefore, in these formulas values increasing by 1 must be attributed to the exponent n, but then let us take $n=\delta$ as the smallest value here, and the values of the letters A and B must be found from these formulas, putting $x=1$ after the integration:

$$A = \int x^{\delta-1}dxe^{\alpha x}(1-x)^{\lambda-1}, \quad B = \int x^{\delta}dxe^{\alpha x}(1-x)^{\lambda-1};$$

further, because of these values

$$
\begin{array}{llll}
f = \delta, & f' = \delta + 1, & f'' = \delta + 2, & f''' = \delta + 3 \quad \text{etc.;} \\
g = \delta + \lambda - \alpha, & g' = \delta + 1 + \lambda - \alpha, & g'' = \delta + 2 + \lambda - \alpha & \text{etc.} \\
h = \alpha, & h' = \alpha, & h'' = \alpha & \text{etc.}
\end{array}
$$

this continued fraction follows

$$\frac{\delta A}{B} = \delta + \lambda - \alpha + \cfrac{(\delta + 1)\alpha}{\delta + 1 + \lambda - \alpha + \cfrac{(\delta + 2)\alpha}{\delta + 2 + \lambda - \alpha + \cfrac{(\delta + 3)\alpha}{\delta + 3 + \lambda - \alpha + \text{etc.}}}}$$

where it must especially be noted that the exponents λ and θ have to taken greater than zero, since otherwise the principal formula $x^n e^{\alpha x}(1 - x)^\lambda$ would not vanish in the case $x = 1$.

§36 If the value 1 is attributed to the letters δ and λ, the case treated above already [§ 23] will result; and if integer numbers are assigned to these letters, continued fractions of such a kind will result, which can be reduced to the first by certain operations. But if we want to assign fractions to these letters δ and λ, either to one or both of them, then forms completely irreducible to the first will result and whose values can only be expressed by most transcendental quantities. As if it was $\delta = \frac{1}{2}$ and $\lambda = \frac{1}{2}$, the value of the the letter A must be found from this integral formula

$$A = \int \frac{e^{\alpha x} dx}{\sqrt{x - xx}},$$

whose integration leads to most transcendental quantities, such that the value of such continued fraction results as highly intricate.