How Sines and Cosines of multiple angles can be expressed by Products *

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§1 Given an arbitrary angle φ, for the sake of brevity, put

$$\cos\varphi + \sqrt{-1}\cdot\sin\varphi = p$$

and

$$\cos\varphi - \sqrt{-1}\cdot\sin\varphi = q,$$

it will be

$$pq = 1;$$

then indeed,

$$p^n = \cos n\varphi + \sqrt{-1}\cdot\sin n\varphi$$

and

$$q^n = \cos n\varphi - \sqrt{-1}\cdot\sin n\varphi,$$

whence

$$p^n + q^n = 2\cos n\varphi$$

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and

$$p^n - q^n = 2\sqrt{-1} \cdot \sin n\varphi.$$

Therefore, matters reduce to this that the formulas $p^n + q^n$ and $p^n - q^n$ are resolved into factors.

§2 First let us consider the formula

$$p^n + q^n = 2\cos n\varphi,$$

which, as often as n is an odd number, has the simple factor $p + q = 2\cos\varphi$, such that in these cases $\cos\varphi$ is a factor of $\cos n\varphi$. But for the remaining factors let us set that a double factor in general is $pp - 2pq\cos\omega + qq$, such that the formula $p^n + q^n$ vanishes for

$$pp - 2pq\cos\omega + qq = 0;$$

but then it will be either

$$p = q(\cos\omega + \sqrt{-1} \cdot \sin\omega)$$

or

$$p = q(\cos\omega - \sqrt{-1} \cdot \sin\omega)$$

and hence

$$p^n = q^n(\cos n\omega \pm \sin n\omega);$$

and so it will have to be

$$q^n(\cos n\omega \pm \sqrt{-1} \cdot \sin n\omega) + q^n = 0$$

or

$$\cos n\omega \pm \sqrt{-1} \cdot \sin n\omega + 1 = 0,$$

whence $\sin n\omega = 0$ and $\cos n\omega = -1$, which immediately implies $\sin n\omega = 0$.

§3 Therefore, since $\cos n\omega = -1$, the angle $n\omega$ will be either π or 3π or 5π or 7π etc. And so, if i denotes an arbitrary integer number, it will be $n\omega = i\pi$ and hence $\omega = \frac{i\pi}{n}$, for which reason the double factor will in general be

$$pp - 2pq\cos\frac{i\pi}{n} + qq.$$

§4 Now, since

$$pp + qq = 2\cos 2\varphi,$$

because of $pq = 1$ that factor will be $2\cos 2\varphi - 2\cos\frac{i\pi}{n}$, which is resolved into two factors without any more effort. For, since

$$\cos A - \cos B = 2\sin\frac{B+A}{2}\sin\frac{B-A}{2},$$

it will be

$$\cos 2\varphi - \cos\frac{i\pi}{n} = 2\sin\left(\frac{i\pi}{2n} + \varphi\right)\sin\left(\frac{i\pi}{n} - \varphi\right)$$

and so one factor in general will be

$$4\sin\left(\frac{i\pi}{2n} + \varphi\right)\sin\left(\frac{i\pi}{2n} - \varphi\right).$$

Hence successively writing the numbers 1, 2, 3, 4 etc. for i it will be

$$2\cos n\varphi = 4\sin\left(\frac{\pi}{2n} + \varphi\right)\sin\left(\frac{\pi}{2n} - \varphi\right)$$

$$\cdot 4\sin\left(\frac{3\pi}{2n} + \varphi\right)\sin\left(\frac{3\pi}{2n} - \varphi\right)\cdot 4\sin\left(\frac{5\pi}{2n} + \varphi\right)\sin\left(\frac{5\pi}{2n} - \varphi\right)\cdot \text{etc.},$$

until one has n factors in total.

§5 Therefore, let us go through this expression according to each value of the number n and it will be

$$\text{if} \quad n=1, \quad 2\cos \quad \varphi = 2\ \sin\left(\frac{\pi}{2}-\varphi\right),$$

$$\text{if} \quad n=2, \quad 2\cos 2\varphi = 2^2 \sin\left(\frac{\pi}{4}-\varphi\right)\ \sin\left(\frac{\pi}{4}+\varphi\right),$$

$$\text{if} \quad n=3, \quad 2\cos 3\varphi = 2^3 \sin\left(\frac{\pi}{6}-\varphi\right)\ \sin\left(\frac{\pi}{6}+\varphi\right)\ \sin\left(\frac{3\pi}{6}-\varphi\right),$$

$$\text{if} \quad n=4, \quad 2\cos 4\varphi = 2^4 \sin\left(\frac{\pi}{8}-\varphi\right)\ \sin\left(\frac{\pi}{8}+\varphi\right)\ \sin\left(\frac{3\pi}{8}-\varphi\right)\ \sin\left(\frac{3\pi}{8}+\varphi\right),$$

$$\text{if} \quad n=5, \quad 2\cos 5\varphi = 2^5 \sin\left(\frac{\pi}{10}-\varphi\right) \sin\left(\frac{\pi}{10}+\varphi\right) \sin\left(\frac{3\pi}{10}-\varphi\right)\ \sin\left(\frac{3\pi}{10}+\varphi\right), \\ \cdot \sin\left(\frac{5\pi}{10}-\varphi\right),$$

$$\text{if} \quad n=6, \quad 2\cos 6\varphi = 2^6 \sin\left(\frac{\pi}{12}-\varphi\right) \sin\left(\frac{\pi}{12}+\varphi\right) \sin\left(\frac{3\pi}{12}-\varphi\right)\ \sin\left(\frac{3\pi}{12}+\varphi\right), \\ \cdot \sin\left(\frac{5\pi}{12}-\varphi\right) \sin\left(\frac{5\pi}{12}+\varphi\right).$$

But in general it will be

$$\cos n\varphi = 2^{n-1} \sin\left(\frac{\pi}{2n}-\varphi\right) \sin\left(\frac{\pi}{2n}+\varphi\right) \sin\left(\frac{3\pi}{2n}-\varphi\right) \sin\left(\frac{3\pi}{2n}+\varphi\right) \cdot \text{etc.},$$

until one has n factors.

§6 Therefore, taking logarithms it will be

$$\log\cos n\varphi = \log 2^{n-1} + \log\sin\left(\frac{\pi}{2n}-\varphi\right) + \log\sin\left(\frac{\pi}{2n}+\varphi\right) \\ + \log\sin\left(\frac{3\pi}{2n}-\varphi\right) + \log\sin\left(\frac{3\pi}{2n}+\varphi\right) + \text{etc.},$$

which equation differentiated yields

$$\frac{nd\varphi \sin n\varphi}{\cos n\varphi} = \frac{d\varphi \cos\left(\frac{\pi}{2n}-\varphi\right)}{\sin\left(\frac{\pi}{2n}-\varphi\right)} - \frac{d\varphi \cos\left(\frac{\pi}{2n}+\varphi\right)}{\sin\left(\frac{\pi}{2n}+\varphi\right)}$$

$$+\frac{d\varphi\cos\left(\frac{3\pi}{2n}-\varphi\right)}{\sin\left(\frac{3\pi}{2n}-\varphi\right)}-\frac{d\varphi\cos\left(\frac{3\pi}{2n}+\varphi\right)}{\sin\left(\frac{3\pi}{2n}+\varphi\right)}+\text{etc.},$$

whence the following remarkable equations are deduced

$$\text{I.}\quad \tan\varphi = \cot\left(\frac{\pi}{2}-\varphi\right),$$

$$\text{II.}\quad 2\tan 2\varphi = \cot\left(\frac{\pi}{4}-\varphi\right) - \cot\left(\frac{\pi}{4}+\varphi\right) = \tan\left(\frac{\pi}{4}+\varphi\right) - \tan\left(\frac{\pi}{4}-\varphi\right),$$

$$\text{III.}\quad 3\tan 3\varphi = \cot\left(\frac{\pi}{6}-\varphi\right) - \cot\left(\frac{\pi}{6}+\varphi\right) + \tan\left(\frac{3\pi}{6}-\varphi\right),$$

or

$$3\tan 3\varphi = \tan\left(\frac{\pi}{3}+\varphi\right) - \tan\left(\frac{\pi}{3}+\varphi\right) + \tan\varphi,$$

$$\text{IV.}\quad 4\tan 4\varphi = \cot\left(\frac{\pi}{8}-\varphi\right) - \cot\left(\frac{\pi}{8}+\varphi\right) + \tan\left(\frac{3\pi}{8}+\varphi\right) - \tan\left(\frac{3\pi}{4}+\varphi\right),$$

or

$$4\tan 4\varphi = \tan\left(\frac{3\pi}{8}+\varphi\right) - \tan\left(\frac{3\pi}{8}-\varphi\right) + \tan\left(\frac{\pi}{8}+\varphi\right) - \tan\left(\frac{\pi}{8}-\varphi\right).$$

§7 In the same way let us treat the formula

$$p^n - q^n = 2\sqrt{-1}\cdot\sin n\varphi,$$

whose double factor we want to set

$$pp - 2pq\cos\omega + qq,$$

having put which $= 0$ as before we have

$$p = q(\cos\omega \pm \sqrt{-1}\cdot\sin\omega)$$

and hence further

$$p^q = q^n(\cos n\omega \pm \sqrt{-1} \cdot \sin n\omega);$$

and so it must be

$$q^n(\cos n\omega \pm \sqrt{-1} \cdot \sin n\omega) - q^n = 0$$

or

$$\cos n\omega \pm \sqrt{-1} \cdot \sin n\omega - 1 = 0,$$

whence it must be

$$\sin n\omega = 0 \quad \text{and} \quad \cos n\omega = 1,$$

for what reason the angle $n\omega$ will be either 0 or 2π or 4π or 6π or in general $2i\pi$ and hence

$$\omega = \frac{2i\pi}{n}$$

while i denotes all numbers 1, 2, 3, 4 etc. Therefore, hence the double factor will in general be

$$pp - 2pq\cos\frac{2i\pi}{n} + qq = 2\cos 2\varphi - 2\cos\frac{2i\pi}{n},$$

which is resolved into these factors

$$2\sin\left(\frac{i\pi}{n} - \varphi\right) \cdot 2\sin\left(\frac{i\pi}{n} + \varphi\right);$$

but additionally, the formula $p^n - q^n$ has the simple factor

$$p - q = 2\sqrt{-1} \cdot \sin\varphi;$$

as a logical consequence we will have

$$\sin n\varphi = \sin\varphi \cdot 2\sin\left(\frac{i\pi}{n} - \varphi\right) \cdot 2\sin\left(\frac{i\pi}{n} + \varphi\right) \cdot \text{etc.}$$

and hence

$$\sin n\varphi = \sin\varphi \cdot 2\sin\left(\frac{\pi}{n} - \varphi\right) \cdot 2\sin\left(\frac{\pi}{n} + \varphi\right)$$

$$\cdot 2\sin\left(\frac{2\pi}{n}-\varphi\right)\cdot 2\sin\left(\frac{2\pi}{n}+\varphi\right)\cdot \text{etc.},$$

until in total n factors result. Therefore, it will be

$$\sin n\varphi = 2^{n-1}\sin\varphi\sin\left(\frac{\pi}{n}-\varphi\right)\sin\left(\frac{\pi}{n}+\varphi\right)$$
$$\cdot\sin\left(\frac{2\pi}{n}-\varphi\right)\sin\left(\frac{2\pi}{n}+\varphi\right)\cdot\text{etc.}$$

§8 Now from the general formula let us deduce the following special forms

if $n=1$, $\sin\varphi = 2^0\sin\varphi$,

if $n=2$, $\sin 2\varphi = 2\sin\varphi\sin\left(\frac{\pi}{2}-\varphi\right)$,

if $n=3$, $\sin 3\varphi = 4\sin\varphi\sin\left(\frac{\pi}{3}-\varphi\right)\sin\left(\frac{\pi}{3}+\varphi\right)$,

if $n=4$, $\sin 4\varphi = 8\sin\varphi\sin\left(\frac{\pi}{4}-\varphi\right)\sin\left(\frac{\pi}{4}+\varphi\right)\sin\left(\frac{2\pi}{4}-\varphi\right)$,

if $n=5$, $\sin 5\varphi = 16\sin\varphi\sin\left(\frac{\pi}{5}-\varphi\right)\sin\left(\frac{\pi}{5}+\varphi\right)\sin\left(\frac{2\pi}{5}-\varphi\right)\sin\left(\frac{2\pi}{5}+\varphi\right)$,

if $n=6$, $\sin 6\varphi = 32\sin\varphi\sin\left(\frac{\pi}{6}-\varphi\right)\sin\left(\frac{\pi}{6}+\varphi\right)\sin\left(\frac{2\pi}{6}-\varphi\right)\sin\left(\frac{2\pi}{6}+\varphi\right)$
$\sin\left(\frac{3\pi}{6}-\varphi\right)$

§9 Let us, as before, take logarithms here and it will be

$$\log\sin n\varphi = \log 2^{n-1}+\log\sin\varphi+\log\sin\left(\frac{\pi}{n}-\varphi\right)+\log\sin\left(\frac{\pi}{n}+\varphi\right)+\text{etc.},$$

which equation differentiated and divided by $d\varphi$ yields

$$\frac{n\cos n\varphi}{\sin n\varphi}=\frac{\cos\varphi}{\sin\varphi}-\frac{\cos\left(\frac{\pi}{n}-\varphi\right)}{\sin\left(\frac{\pi}{n}-\varphi\right)}+\frac{\cos\left(\frac{\pi}{n}+\varphi\right)}{\sin\left(\frac{\pi}{n}+\varphi\right)}-\frac{\cos\left(\frac{2\pi}{n}-\varphi\right)}{\sin\left(\frac{2\pi}{n}-\varphi\right)}+\text{etc.}$$

or

$$n\cot n\varphi = \cot\varphi - \cot\left(\frac{\pi}{n} - \varphi\right) + \cot\left(\frac{\pi}{n} + \varphi\right) - \cot\left(\frac{2\pi}{n} - \varphi\right) + \text{etc.},$$

until one has n terms.

§10 Therefore, hence we will obtain the following special forms:

if $n = 1$, $\cot\varphi = \cot\varphi$,

if $n = 2$, $2\cot 2\varphi = \cot\varphi - \cot\left(\frac{\pi}{2} - \varphi\right)$,

if $n = 3$, $3\cot 3\varphi = \cot\varphi - \cot\left(\frac{\pi}{3} - \varphi\right) + \cot\left(\frac{\pi}{3} + \varphi\right)$,

if $n = 4$, $4\cot 4\varphi = \cot\varphi - \cot\left(\frac{\pi}{4} - \varphi\right) + \cot\left(\frac{\pi}{4} + \varphi\right) - \cot\left(\frac{2\pi}{4} - \varphi\right)$.

§11 If we differentiate the formula found for $\cot n\varphi$ again, because of

$$d\cot\theta = \frac{-d\theta}{\sin^2\theta},$$

by dividing by $-d\varphi$ we will have

$$\frac{nn}{\sin^2 n\varphi} = \frac{1}{\sin^2\varphi} + \frac{1}{\sin^2\left(\frac{\pi}{n} - \varphi\right)} + \frac{1}{\sin^2\left(\frac{\pi}{n} + \varphi\right)} + \frac{1}{\sin^2\left(\frac{2\pi}{n} - \varphi\right)} + \text{etc.},$$

until one has n terms, whence the following cases should be noted:

$$\text{if} \quad n = 1, \quad \frac{1}{\sin^2 \varphi} = \frac{1}{\sin^2 \varphi},$$

$$\text{if} \quad n = 2, \quad \frac{4}{\sin^2 2\varphi} = \frac{1}{\sin^2 \varphi} + \frac{1}{\sin^2 \left(\frac{\pi}{2} - \varphi\right)},$$

$$\text{if} \quad n = 3, \quad \frac{9}{\sin^2 3\varphi} = \frac{1}{\sin^2 \varphi} + \frac{1}{\sin^2 \left(\frac{\pi}{3} - \varphi\right)} + \frac{1}{\sin^2 \left(\frac{\pi}{3} + \varphi\right)},$$

$$\text{if} \quad n = 4, \quad \frac{16}{\sin^2 4\varphi} = \frac{1}{\sin^2 \varphi} + \frac{1}{\sin^2 \left(\frac{\pi}{4} - \varphi\right)} + \frac{1}{\sin^2 \left(\frac{\pi}{4} + \varphi\right)} + \frac{1}{\sin^2 \left(\frac{2\pi}{4} - \varphi\right)},$$

etc.

Expansion of the formula $p^{2n} - 2p^n q^n \cos\theta + q^{2n}$

§12 Let us as before take

$$p = \cos\varphi + \sqrt{-1} \cdot \sin\varphi$$

and

$$q = \cos\varphi - \sqrt{-1} \cdot \sin\varphi,$$

such that that formula involves this value

$$2\cos 2n\varphi - 2\cos\theta = 4\sin\left(n\varphi + \frac{1}{2}\theta\right)\sin\left(\frac{1}{2}\theta - n\varphi\right).$$

Now let $pp - 2pq\cos\omega + qq$ be a double factor of this formula, which must therefore vanish for

$$p = q(\cos\omega \pm \sqrt{-1} \cdot \sin\omega),$$

whence after the substitution it will result

$$q^{2n}(\cos 2n\omega \pm \sqrt{-1} \cdot \sin 2n\omega) - 2q^{2n}\cos\theta(\cos n\omega \pm \sqrt{-1} \cdot \sin n\omega) + q^{2n} = 0,$$

this means

$$\cos 2n\omega - 2\cos\theta\cos n\omega + 1 \pm \sqrt{-1}\cdot\sin 2n\omega \mp 2\cos\theta\sqrt{-1}\cdot\sin n\omega = 0,$$

whence these two equations result

$$\cos 2n\omega - 2\cos\theta\cos n\omega + 1 = 0$$

and

$$\sin 2n\omega - 2\cos\theta\sin n\omega = 0.$$

Now since

$$\cos 2n\omega = 2\cos^2 n\omega - 1 \quad \text{and} \quad \sin 2n\omega = 2\sin n\omega\cos n\omega,$$

these two equations will be

$$2\cos^2 n\omega - 2\cos\theta\cos n\omega = 0 \quad \text{and} \quad 2\sin n\omega\cos n\omega - 2\cos\theta\sin n\omega = 0$$

or

$$\cos n\omega - \cos\theta = 0 \quad \text{and} \quad \cos n\omega - \cos\theta = 0,$$

whence $\cos n\omega = \cos\theta$ follows. Therefore, it will be either $n\omega = \theta$ or $n\omega = 2\pi + \theta$ or $4\pi + \theta$ or $6\pi + \theta$ or in general $n\omega = 2i\pi + \theta$, whence in general

$$\omega = \frac{2i\pi + \theta}{n},$$

such that i denotes the numbers 0, 1, 2, 3, 4 etc.

§**13** Therefore, the double factor of our formula will in general be

$$pp + qq - 2pq\cos\left(\frac{2i\pi + \theta}{n}\right).$$

On the other hand

$$pp + qq = 2\cos 2\varphi \quad \text{and} \quad pq = 1,$$

whence this factor will be

$$2\left(\cos 2\varphi - \cos\left(\frac{2i\pi+\theta}{n}\right)\right),$$

which is reduced to these simple factors

$$4\sin\frac{2i\pi+2n\varphi+\theta}{2n}\sin\frac{2i\pi+\theta-2n\varphi}{2n};$$

hence writing the numbers 1, 2, 3, 4 etc. instead of i our formulas will be

$$4\sin\frac{2n\varphi+\theta}{2n}\sin\frac{\theta-2n\varphi}{2n}\cdot 4\sin\frac{2\pi+2n\varphi+\theta}{2n}\sin\frac{2\pi+\theta-2n\varphi}{2n}$$
$$\cdot 4\sin\frac{4\pi+2n\varphi+\theta}{2n}\sin\frac{4\pi+\theta-2n\varphi}{2n}\cdot 4\sin\frac{6\pi+2n\varphi+\theta}{2n}\sin\frac{6\pi+\theta-2n\varphi}{2n}\cdot \text{etc.},$$

which factors must be continued until their number becomes $= n$.

§**14** Therefore, since this product is equal to the formula

$$4\sin\left(n\varphi+\frac{1}{2}\theta\right)\sin\left(\frac{1}{2}\theta-n\varphi\right)$$

and in our product the numerical factor is $4^n = 2^{2n}$, diving by 4 we will have this equation

$$\sin\left(n\varphi+\frac{1}{2}\theta\right)\sin\left(\frac{1}{2}\theta-n\varphi\right) = 2^{2n-2}\sin\left(\frac{2n\varphi+\theta}{2n}\right)\sin\left(\frac{\theta-2n\varphi}{2n}\right)$$
$$\cdot\sin\left(\frac{2\pi+2n\varphi+\theta}{2n}\right)\sin\left(\frac{2\pi+\theta-2n\varphi}{2n}\right)\sin\left(\frac{4\pi+2n\varphi+\theta}{2n}\right)\sin\left(\frac{4\pi+\theta-2n\varphi}{2n}\right)\cdot \text{etc.},$$

to simplify which equation, let us put $\theta = 2n\alpha$ and it will be

$$\sin n(\alpha+\varphi)\sin n(\alpha-\varphi) = 2^{2n-2}\sin(\alpha+\varphi)\sin(\alpha-\varphi)$$
$$\cdot\sin\left(\frac{\pi}{n}+\alpha+\varphi\right)\sin\left(\frac{\pi}{n}+\alpha-\varphi\right)\cdot\sin\left(\frac{2\pi}{n}+\alpha+\varphi\right)\sin\left(\frac{2\pi}{n}+\alpha-\varphi\right)\cdot \text{etc.}$$

§**15** But this expression is not new, but already contained in the preceding, which was

$$\sin n\varphi = 2^{n-1}\sin\varphi\sin\left(\frac{\pi}{n}-\varphi\right)\sin\left(\frac{\pi}{n}+\varphi\right)\sin\left(\frac{2\pi}{n}-\varphi\right)\sin\left(\frac{2\pi}{n}+\varphi\right)\text{ etc.,}$$

and since

$$\sin(o-\varphi)=\sin(\pi-o+\varphi),$$

it will be

$$\sin\left(\frac{\pi}{n}-\varphi\right)=\sin\left(\frac{(n-1)\pi}{n}+\varphi\right),$$
$$\sin\left(\frac{2\pi}{n}-\varphi\right)=\sin\left(\frac{(n-2)\pi}{n}+\varphi\right),$$
$$\sin\left(\frac{3\pi}{n}-\varphi\right)=\sin\left(\frac{(n-3)\pi}{n}+\varphi\right);$$

hence that expression will be reduced to this form

$$\sin n\varphi = 2^{n-1}\sin\varphi\cdot\sin\left(\frac{\pi}{n}+\varphi\right)\sin\left(\frac{2\pi}{n}+\varphi\right)$$
$$\cdot\sin\left(\frac{3\pi}{n}+\varphi\right)\cdots\sin\left(\frac{(n-1)\pi}{n}+\varphi\right),$$

where the arcs proceed in an arithmetic progression. But if here instead of φ we write $\alpha+\varphi$ first, then $\alpha-\varphi$, hence the following two forms will arise:

$$\sin n(\alpha+\varphi) = 2^{n-1}\sin(\alpha+\varphi)\sin\left(\frac{\pi}{n}+\alpha+\varphi\right)$$
$$\cdot\sin\left(\frac{2\pi}{n}+\alpha+\varphi\right)\sin\left(\frac{3\pi}{n}+\alpha+\varphi\right)\cdot\text{etc.,}$$
$$\sin n(\alpha-\varphi) = 2^{n-1}\sin(\alpha-\varphi)\sin\left(\frac{\pi}{n}+\alpha-\varphi\right)$$
$$\cdot\sin\left(\frac{2\pi}{n}+\alpha-\varphi\right)\sin\left(\frac{3\pi}{n}+\alpha-\varphi\right)\cdot\text{etc.,}$$

which two equations multiplied by each other yield

$$\sin n(\alpha+\varphi)\sin n(\alpha-\varphi) = 2^{2n-2}\sin(\alpha+\varphi)\sin(\alpha-\varphi)$$

$$\cdot\sin\left(\frac{\pi}{n}+\alpha+\varphi\right)\sin\left(\frac{\pi}{n}+\alpha-\varphi\right)$$

$$\cdot\sin\left(\frac{2\pi}{n}+\alpha+\varphi\right)\sin\left(\frac{2\pi}{n}+\alpha-\varphi\right)$$

$$\cdot\sin\left(\frac{3\pi}{n}+\alpha+\varphi\right)\text{ etc.}$$

§16 Let us now consider the origin of these formulas, since from our formula

$$p^{2n}-2p^nq^n\cos 2n\alpha+q^{2n}$$

this one originated

$$4\sin n(\alpha+\varphi)\sin n(\alpha-\varphi)$$

while

$$p=\cos\varphi+\sqrt{-1}\cdot\sin\varphi$$

and

$$q=\cos\varphi-\sqrt{-1}\cdot\sin\varphi,$$

if we put

$$f=\cos(\alpha+\varphi)+\sqrt{-1}\cdot\sin(\alpha+\varphi)$$

and

$$g=\cos(\alpha+\varphi)-\sqrt{-1}\cdot\sin(\alpha+\varphi),$$

then it will be

$$f^n-g^n=2\sqrt{-1}\cdot\sin n(\alpha+\varphi).$$

Further, if we put

$$h=\cos(\alpha-\varphi)+\sqrt{-1}\cdot\sin(\alpha-\varphi)$$

and

$$k = \cos(\alpha - \varphi) - \sqrt{-1} \cdot \sin(\alpha - \varphi),$$

in like manner, it will be

$$h^n - k^n = 2\sqrt{-1} \cdot \sin n(\alpha - \varphi),$$

whence it will be

$$\begin{aligned}(f^n - g^n)(h^n - k^n) &= -4 \sin n(\alpha + \varphi) \sin n(\alpha - \varphi) \\ &= -p^{2n} + 2p^n q^n \cos 2n\alpha - q^{2n}.\end{aligned}$$

To prove this note that

$$\begin{aligned} f &= p(\cos \alpha + \sqrt{-1} \cdot \sin \alpha), \\ g &= q(\cos \alpha - \sqrt{-1} \cdot \sin \alpha), \\ h &= q(\cos \alpha + \sqrt{-1} \cdot \sin \alpha), \\ k &= p(\cos \alpha - \sqrt{-1} \cdot \sin \alpha) \end{aligned}$$

whence

$$\begin{aligned} f^n &= p^n(\cos n\alpha + \sqrt{-1} \cdot \sin n\alpha), \\ g^n &= q^n(\cos n\alpha - \sqrt{-1} \cdot \sin n\alpha), \\ h^n &= q^n(\cos n\alpha + \sqrt{-1} \cdot \sin n\alpha), \\ k^n &= p^n(\cos n\alpha - \sqrt{-1} \cdot \sin n\alpha). \end{aligned}$$

Therefore, for the sake of brevity, let us set

$$\cos n\alpha + \sqrt{-1} \cdot \sin n\alpha = A, \quad \cos n\alpha - \sqrt{-1} \cdot \sin n\alpha = B,$$

that

$$f^n = Ap^n, \quad g^n = Bq^n, \quad h^n = Aq^n \quad \text{and} \quad k^n = Bp^n$$

and hence further

$$f^n - g^n = Ap^n - Bq^n$$

and

$$h^n - k^n = Aq^n - Bp^n,$$

which two formulas multiplied by each other yield

$$(f^n - g^n)(h^n - k^n) = (A^2 + B^2)p^n q^n - AB(p^{2n} + q^{2n});$$

since here

$$AB = 1 \quad \text{and} \quad AA + BB = 2\cos 2n\alpha,$$

this product will be

$$-p^{2n} + 2p^n q^n \cos 2n\alpha - q^{2n},$$

which is the same as we found.

COROLLARY

Therefore, hence we understand that the formula

$$p^{2n} - 2p^n q^n \cos 2n\alpha + q^{2n}$$

is resolved into these factors

$$(Ap^n - Bq^n) \quad \text{and} \quad (Bp^n - Aq^n)$$

while

$$A = \cos n\alpha + \sqrt{-1} \cdot \sin n\alpha,$$
$$B = \cos n\alpha - \sqrt{-1} \cdot \sin n\alpha,$$