On the remarkable Properties of the Coefficients which occur in the Expansion of the binomial raised to an arbitrary power*

Leonhard Euler

Theorem 1

§1 *If for the power of the binomial raised to the exponent n for the sake of brevity we denoted the coefficients by the letters α, β, γ, δ etc. that it is*

$$\alpha = \frac{n}{1}, \quad \beta = \frac{n(n-1)}{1\cdot 2}, \quad \gamma = \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}, \quad \delta = \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4} \quad \textit{etc.},$$

it will always be

$$1+\alpha^2+\beta^2+\gamma^2+\delta^2+\text{etc.} = \frac{2}{1}\cdot\frac{6}{2}\cdot\frac{10}{3}\cdot\frac{14}{4}\cdots\frac{4n-2}{n},$$

which expression for the cases, in which n is a fractional number, can be exhibited by means of an integral in such a way that it is

$$= \frac{2}{\pi}\cdot 2^{2n}\int\frac{x^{2n}dx}{\sqrt{1-xx}}$$

*original title: „De mirabilibus proprietatibus unciarum, quae in evolutione binomii ad potestatem quamcunqua evecti occurrunt", first published in „*Acta Academiae Scientarum Imperialis Petropolitinae* 5, 1784, pp. 74-111", reprinted in „*Opera Omnia*: Series 1, Volume 15, pp. 528 - 568", Eneström-Number E575, translated by: Alexander Aycock for the project „Euler-Kreis Mainz" *having extended this integral from $x = 0$ to $x = 1$, where π denotes the circumference of the circle whose diameter is $= 1$.*

This theorem is even more remarkable, since there is hardly any direct way to demonstrate its truth.

Explanation for the cases in which the exponent n is a positive integer number

§2 In order to see the power of this theorem more clearly, let us expand the simplest cases in the following way:

I. If it is $n = 1$, the coefficients will be $1, 1$ and via the theorem it must be

$$1^2 + 1^2 = 2 = \frac{2}{1}.$$

II. If it is $n = 2$, the coefficients will be $1, 2, 1$ and via the theorem it must be

$$1^2 + 2^2 +^2 = 6 = \frac{2}{1} \cdot \frac{6}{2}.$$

III. If it is $n = 3$, the coefficients will be $1, 3, 3, 1$ and via the theorem it must be

$$1^2 + 3^3 + 3^2 + 1^2 = 20 = \frac{2}{1} \cdot \frac{6}{2} \cdot \frac{10}{3}.$$

IV. If it is $n = 4$, the coefficients will be $1, 4, 6, 4, 1$ and via the theorem it must be

$$1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 70 = \frac{2}{1} \cdot \frac{6}{2} \cdot \frac{10}{3} \cdot \frac{14}{4}.$$

V. If it is $n = 5$, the coefficients will be $1, 5, 10, 10, 5, 1$ and via the theorem it must be

$$1^2 + 5^2 + 10^2 + 10^2 + 5^2 + 1^2 = 252 = \frac{2}{1} \cdot \frac{6}{2} \cdot \frac{10}{3} \cdot \frac{14}{4} \cdot \frac{18}{5}.$$

VI. If it is $n = 6$, the coefficients will be $1, 6, 15, 20, 15, 6, 1$ and via the theorem it must be

$$1^2 + 6^2 + 15^2 + 20^2 + 15^2 + 6^2 + 1^2 = 324 = \frac{2}{1} \cdot \frac{6}{2} \cdot \frac{10}{3} \cdot \frac{14}{4} \cdot \frac{18}{5} \cdot \frac{22}{6}.$$

COROLLARY

§3 Since the formula

$$\frac{2}{1}\cdot\frac{6}{2}\cdot\frac{10}{3}\cdots\frac{4n-2}{n}$$

exhibits the maximal coefficient in the power of the binomial raised to the exponent $2n$, our theorem can also be formulated this way:

If the squares of the coefficients for the power of the exponent n are collected into one single sum, it will always become equal to the largest coefficient occurring in the power of the exponent $2n$.

So for the cases expanded before 2 is the largest coefficient for the exponent 2; further, 6 is the largest coefficient for the exponent 4; then 20 is the largest coefficient for the exponent 6; and in similar manner the following sum 70 is the largest coefficient for the exponent 8 and so forth.

EXPLANATION OF THE THEOREM FOR THE CASES IN WHICH THE EXPONENT n IS A FRACTIONAL NUMBER

§4 Whenever the exponent n is a fractional number, the series of the coefficients is extended to infinity, whence their squares will also constitute an infinite series, whose sum will become known by that integral formula

$$\frac{2}{\pi}\cdot 2^{2n}\int\frac{x^{2n}dx}{\sqrt{1-xx}},$$

if this integral is extended from $x=0$ to $x=1$, which will suffice to have shown it in the one example in which it is $n=\frac{1}{2}$; but then it will be

$$\alpha=\frac{1}{2},\quad \beta=-\frac{1\cdot 1}{2\cdot 4},\quad \gamma=\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6},\quad \delta=-\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8},\quad \varepsilon=\frac{1\cdot 1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10}\quad \text{etc.},$$

the squares of which values will therefore constitute this series

$$1+\frac{1^2}{2^2}+\frac{1^2\cdot 1^2}{2^2\cdot 4^2}+\frac{1^2\cdot 1^2\cdot 3^2}{2^2\cdot 4^2\cdot 6^2}+\frac{1^2\cdot 1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2\cdot 8^2}+\frac{1^2\cdot 1^2\cdot 3^2\cdot 5^2\cdot 7^2}{2^2\cdot 4^2\cdot 6^2\cdot 8^2\cdot 10^2}+\text{etc.},$$

whose sum is therefore to be derived form the integral formula

$$\int\frac{xdx}{\sqrt{1-xx}}\quad \text{extended from}\quad x=0\quad \text{to}\quad x=1.$$

But it is

$$\int \frac{xdx}{\sqrt{1-xx}} = 1 - \sqrt{1-xx};$$

hence having put $x = 1$ now its value becomes $= 1$. Therefore, the sum of the series found will be $= \frac{4}{\pi}$, whose value by means of a decimal fraction is 1.273240; and this value will be continuously better approximated, the more terms of the series

$$1 + \alpha^2 + \beta^2 + \gamma^2 + \text{etc.}$$

are actually added; in order to perform this calculation more easily note that it is

$$\beta\beta = \frac{1}{16}\alpha\alpha, \quad \gamma\gamma = \frac{3^2}{6^2}\beta\beta = \frac{1}{4}\beta\beta, \quad \delta\delta = \frac{25}{64}\gamma\gamma, \quad \varepsilon\varepsilon = \frac{49}{100}\delta\delta,$$
$$\zeta\zeta = \frac{81}{144}\varepsilon\varepsilon, \quad \eta\eta = \frac{121}{196}\zeta\zeta \quad \text{etc.},$$

having observed what the calculation because of $\alpha\alpha = \frac{1}{4}$ will be performed in the following way:

$$\begin{aligned}
1 &= 1.000000\\
\alpha\alpha &= 0.250000\\
\beta\beta &= 0.015625\\
\gamma\gamma &= 0.003906\\
\delta\delta &= 0.001526\\
\varepsilon\varepsilon &= 0.000748\\
\zeta\zeta &= 0.000421\\
\eta\eta &= 0.000260\\
\theta\theta &= 0.000172\\
\hline
\text{Sum} &= 1.272658
\end{aligned}$$

This sum deviates from the true value by the fraction 0.000582, which is therefore to be considered as equal to all following terms, which we left out here, which suffices to confirm this truth of our assertion.

§5 If we wanted to attribute fractional values to the exponent n, that $2n$ is no further an integer number, then the sum of the series would no further depend on the quadrature of the circle, but on higher quadratures. Additionally, it will be helpful to note here, if we would take $n=-\frac{1}{2}$, whence it would be

$$\alpha=-\frac{1}{2}, \quad \beta=\frac{1\cdot 3}{2\cdot 2}, \quad \gamma=-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}, \quad \delta=\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8} \quad \text{etc.},$$

that then the sum of this series

$$1+\frac{1^2}{2^2}+\frac{1^2\cdot 3^2}{2^2\cdot 4^2}+\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}+\frac{1^2\cdot 3^2\cdot 5^2\cdot 7^2}{2^2\cdot 4^2\cdot 6^2\cdot 8^2}+\text{etc.}$$

grows to infinity, as even our integral formula indicates, which will of course be

$$\frac{1}{\pi}\int\frac{dx}{x\sqrt{1-xx}}.$$

But on the other hand one finds

$$\int\frac{dx}{x\sqrt{1-xx}}=\frac{1}{2}\log\frac{1-\sqrt{1-xx}}{1+\sqrt{1-xx}}+C,$$

which constant must be taken in such a way that the integral vanishes for $x=0$, from which it will be $C=-\frac{1}{2}\log 0$ and hence $C=\infty$. Now, let is set $x=1$ and this value will arise as $=\infty$.

§6 But if we set $n=\frac{3}{2}$ that it is

$$\alpha=\frac{3}{2}, \quad \beta=\frac{3\cdot 1}{2\cdot 4}, \quad \gamma=-\frac{3\cdot 1\cdot 1}{2\cdot 4\cdot 6}, \quad \delta=\frac{3\cdot 1\cdot 1\cdot 3}{2\cdot 4\cdot 6\cdot 8}, \quad \varepsilon=-\frac{3\cdot 1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8\cdot 10} \quad \text{etc.},$$

then the sum of the series

$$1+\alpha^2+\beta^2+\gamma^2+\text{etc.}$$

will be

$$\frac{16}{\pi}\int\frac{x^3dx}{\sqrt{1-xx}}.$$

But it is

$$\int \frac{x^3dx}{\sqrt{1-xx}} = C - \sqrt{1-xx} + \frac{1}{3}(1-xx)^{\frac{3}{2}},$$

where one has to take $C = \frac{2}{3}$. Now, having put $x = 1$ the sum of our series will be $= \frac{32}{3\pi}$.

§7 If we wanted to attribute larger values of this kind, e.g. $\frac{5}{2}$, $\frac{7}{2}$, $\frac{9}{2}$, $\frac{11}{2}$ etc., note that it is in general

$$\int \frac{x^{i+2}dx}{\sqrt{1-xx}} = -\frac{x^{i+1}\sqrt{1-xx}}{i+2} + \frac{(i+1)}{(i+2)}\int \frac{x^i dx}{\sqrt{1-xx}}.$$

Hence, if the integrals are extended from $x = 0$ to $x = 1$, it will be

$$\int \frac{x^{i+2}dx}{\sqrt{1-xx}} = \frac{i+1}{i+2}\int \frac{x^i dx}{\sqrt{1-xx}}.$$

Hence, because in the case $i = 1$ it is

$$\int \frac{xdx}{\sqrt{1-xx}} = 1,$$

for the following formulas it will be

$$\int \frac{x^3dx}{\sqrt{1-xx}} = \frac{2}{3},$$
$$\int \frac{x^5dx}{\sqrt{1-xx}} = \frac{2\cdot 4}{3\cdot 5},$$
$$\int \frac{x^7dx}{\sqrt{1-xx}} = \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7},$$
$$\int \frac{x^9dx}{\sqrt{1-xx}} = \frac{2\cdot 4\cdot 6\cdot 8}{3\cdot 5\cdot 7\cdot 9}$$

etc.

Therefore, having mentioned these things in advance, if for the sake of brevity we put

$$1 + \alpha^2 + \beta^2 + \gamma^2 + \text{etc.} = S,$$

it will be as follows:

$$
\begin{array}{llll}
\text{I.} & \text{For the case} \quad n = \frac{1}{2} & \text{it will be} & S = \frac{4}{\pi}, \\
\text{II.} & \text{For the case} \quad n = \frac{3}{2} & \text{it will be} & S = \frac{4}{\pi} \cdot \frac{8}{3}, \\
\text{III.} & \text{For the case} \quad n = \frac{5}{2} & \text{it will be} & S = \frac{4}{\pi} \cdot \frac{8}{3} \cdot \frac{16}{5}, \\
\text{IV.} & \text{For the case} \quad n = \frac{7}{2} & \text{it will be} & S = \frac{4}{\pi} \cdot \frac{8}{3} \cdot \frac{16}{5} \cdot \frac{24}{7}, \\
\text{V.} & \text{For the case} \quad n = \frac{9}{2} & \text{it will be} & S = \frac{4}{\pi} \cdot \frac{8}{3} \cdot \frac{16}{5} \cdot \frac{24}{7} \cdot \frac{32}{9} \\
 & \qquad\qquad\qquad \text{etc.} & & \quad \text{etc.}
\end{array}
$$

§8 From the superior reduction of the integrals even the form of our integral formula given in the theorem can be found; for, because in general it is

$$\int \frac{x^{i+2}dx}{\sqrt{1-xx}} = \frac{i+1}{i+2} \int \frac{x^i dx}{\sqrt{1-xx}},$$

but in the case $i = 0$ it becomes

$$\int \frac{dx}{\sqrt{1-xx}} = \frac{\pi}{2},$$

it will be as follows:

$$
\begin{aligned}
\int \frac{xxdx}{\sqrt{1-xx}} &= \frac{\pi}{2} \cdot \frac{1}{2}, \\
\int \frac{x^4dx}{\sqrt{1-xx}} &= \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{4}, \\
\int \frac{x^6dx}{\sqrt{1-xx}} &= \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6}, \\
\int \frac{x^8dx}{\sqrt{1-xx}} &= \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8}, \\
\int \frac{x^{10}dx}{\sqrt{1-xx}} &= \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{9}{10} \\
&\text{etc.}
\end{aligned}
$$

If we now successively attribute the integer numbers 1, 2, 3, 4 etc. to the exponent n and hence conclude the value of our series

$$1+\alpha^2+\beta^2+\gamma^2+\text{etc.}=S=\frac{2}{\pi}\cdot 2^{2n}\int\frac{x^{2n}dx}{\sqrt{1-xx}},$$

we will find these values for S:

$I.$	For the case	$n=1$	it will be	S	$=2.$
$II.$	For the case	$n=2$	it will be	S	$=\frac{2}{1}\cdot\frac{6}{2}.$
$III.$	For the case	$n=3$	it will be	S	$=\frac{2}{1}\cdot\frac{6}{2}\cdot\frac{10}{3}.$
$IV.$	For the case	$n=4$	it will be	S	$=\frac{2}{1}\cdot\frac{6}{2}\cdot\frac{10}{3}\cdot\frac{14}{4}.$
$V.$	For the case	$n=5$	it will be	S	$=\frac{2}{1}\cdot\frac{6}{2}\cdot\frac{10}{3}\cdot\frac{14}{4}\cdot\frac{18}{5}$
		etc.		etc.	

hence it is plain the for each integer exponent n it will be

$$S=\frac{2}{1}\cdot\frac{6}{2}\cdot\frac{10}{3}\cdots\frac{4n-2}{n},$$

completely as we stated in the theorem.

Scholium

§9 As here for the cases, in which the exponent n is a fractional number, we represented the sum of our series by means of an integral formula, which involves the circumference of the circle π, so one can even in many ways express the value of the same sum S by means of other integral formulas, of which we want to add some here. Of course, the first is

$$I.\quad S=\frac{2}{n\int x^{n-1}dx(1-x)^{n-1}},$$

the second

$$II.\quad S=\frac{1}{n\int x^{n}dx(1-x)^{n-1}},$$

the third

$$III. \quad S = \frac{2}{(2n+1)\int x^n dx(1-x)^n},$$

where as before these integral formulas must be extended from $x = 0$ to $x = 1$. So for the case $n = 1$ the first of these formulas yields

$$S = \frac{2}{\int dx} = \frac{2}{1},$$

the second formula on the other hand gives

$$S = \frac{1}{\int xdx} = \frac{2}{1},$$

further, the third formula gives

$$S = \frac{1}{3\int xdx(1-x)}.$$

On the other hand it is

$$\int xdx(1-x) = \frac{1}{2}xx - \frac{1}{3}x^3 = \frac{1}{6},$$

from which it is $S = \frac{2}{1}$.

Here, let us also consider the case $n = \frac{1}{2}$ and the first of these formulas yields

$$S = \frac{4}{\int \frac{dx}{\sqrt{x-xx}}}.$$

But having put $x = yy$ here it is

$$\int \frac{dx}{\sqrt{x-xx}} = 2\int \frac{dy}{\sqrt{1-yy}} = \frac{2\pi}{2} = \pi$$

and so it will be $S = \frac{4}{\pi}$, completely as it was found above.

Let us also consider the case $n = 3$ and the first of these formulas will give

$$S = \frac{2}{3\int xxdx(1-x)^2}.$$

On the other hand it is

$$\int xxdx(1-x)^2 = \frac{1}{30}$$

and hence it will therefore be $S = 20$, as above.

THEOREM 2

§10 *While the letters α, β, γ, δ etc. remain the coefficients for the power of the exponent n, if in similar manner the letters α', β', γ', δ' etc. denote the coefficients for the power of the exponent n' and hence this series is formed*

$$1 + \alpha\alpha' + \beta\beta' + \gamma\gamma' + \delta\delta' + \text{etc.},$$

it sum will become equal to this product

$$\frac{n+n'}{1} \cdot \frac{n+n'-1}{2} \cdot \frac{n+n'-2}{3} \cdot \frac{n+n'-3}{4} \cdots \frac{n'+1}{n},$$

which same sum can also be expressed by means of the following integral formulas either by means of

$$\frac{1}{n\int x^{n'}dx(1-x)^{n-1}}$$

or by

$$\frac{1}{n'n\int x^{n'-1}dx(1-x)^{n-1}}$$

or by

$$\frac{1}{(n+n'+1)\int x^{n'}dx(1-x)^{n}},$$

where the integrals are to be extended from $x = 0$ to $x = 1$.

EXPLANATION FOR THE CASES IN WHICH THE EXPONENTS n AND n' ARE POSITIVE INTEGER NUMBERS

§11 To show the examples of this theorem even more plainly, since the cases, in which it is $n = n'$, were already expanded in the first theorem, let us at first set the difference between these exponents n and n', that it is $n' = n + 1$, and go through the following cases:

$$\begin{array}{rr|llllll}
\text{I.} \quad \text{Let} & n = 1 & 1, & 1 \\
& n' = 2 & 1, & 2, & 1 \\
\hline
\end{array}$$

The sum of the series will be $\quad 1 + 2 + 0 = 3$

Therefore, since it is $n + n' = 3$, the given product becomes $\frac{3}{1}$, as required

$$\begin{array}{rr|llllll}
\text{II.} \quad \text{Let} & n = 2 & 1, & 2, & 1 \\
& n' = 3 & 1, & 3, & 3, & 1 \\
\hline
\end{array}$$

The sum of the series will be $\quad 1 + 6 + 3 + 0 = 10$;

but because of $n + n' = 5$ that product becomes $\frac{5}{1} \cdot \frac{4}{2}$.

$$\begin{array}{rr|llllll}
\text{III.} \quad \text{Let} & n = 3 & 1, & 3, & 3, & 1 \\
& n' = 4 & 1, & 4, & 6, & 4, & 1 \\
\hline
\end{array}$$

The sum of the series will be $\quad 1 + 12 + 18 + 4 + 0 = 35$;

but because of $n + n' = 7$ our product becomes $= \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3}$.

$$\begin{array}{rr|llllll}
\text{IV.} \quad \text{Let} & n = 4 & 1, & 4, & 6, & 4, & 1 \\
& n' = 5 & 1, & 5, & 10, & 10, & 5, & 1, \\
\hline
\end{array}$$

The sum of the series will be $\quad 1 + 20 + 60 + 40 + 5 + 0 = 126$;

but because of $n + n' = 9$ our product will be $= \frac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4}$.

V. Let $n = 5$	1,	5,	10,	10,	5,	1	
$n' = 6$	1,	6,	15,	20,	15,	6,	1

The sum of the series will be $1 + 30 + 150 + 200 + 75 + 6 + 0 = 462$;

hence because of $n + n' = 11$ our product will be $\frac{11\cdot10\cdot9\cdot8\cdot7}{1\cdot2\cdot3\cdot4\cdot5}$

etc.

§12 Now let us set $n' = n + 2$ and the exhibited product will become

$$\frac{2n+2}{1}\cdot\frac{2n+1}{2}\cdot\frac{2n}{3}\cdots\frac{n+3}{n}.$$

Therefore, hence let us go through the following cases:

I. Let $n = 1$	1,	1,		
$n' = 3$	1,	3,	3,	1,

The sum of the series will be $1 + 3 + 0 = 4$;

But our product becomes $\frac{4}{1}$.

II. Let $n = 2$	1,	2,	1,		
$n' = 4$	1,	4,	6,	4,	1

The sum of the series will be $1 + 8 + 6 + 0 = 15$;

but our product $= \frac{6\cdot5}{1\cdot2} = 15$.

$$\begin{array}{ll|llllllll}
\text{III.} \quad \text{Let} & n = 3 & 1, & 3, & 3, & 1, \\
& n' = 5 & 1, & 5, & 10, & 10, & 5, & 1, \\
\hline
\end{array}$$

The sum of the series will be $\quad 1 + 15 + 30 + 10 + 0 = 56$;

but our product becomes $= \frac{8\cdot7\cdot6}{1\cdot2\cdot3} = 56$.

$$\begin{array}{ll|lllllll}
\text{IV.} \quad \text{Let} & n = 4 & 1, & 4, & 6, & 4, & 1, \\
& n' = 6 & 1, & 6, & 15, & 20, & 15, & 6, & 1, \\
\hline
\end{array}$$

The sum of the series will be $\quad 1 + 24 + 90 + 80 + 15 + 0 = 210$;

but our product will be $= \frac{10\cdot9\cdot8\cdot7}{1\cdot2\cdot3\cdot4}$.

$$\begin{array}{ll|llllllll}
\text{V.} \quad \text{Let} & n = 5 & 1, & 5, & 10, & 10, & 5, & 1, \\
& n' = 7 & 1, & 7, & 21, & 35, & 35, & 21, & 7, & 1, \\
\hline
\end{array}$$

The sum of the series will be $\quad 1 + 35 + 210 + 350 + 175 + 21 + 0 = 792$;

but our product will become $= \frac{12\cdot11\cdot10\cdot9\cdot8}{1\cdot2\cdot\cdot3\cdot4\cdot5}$.

etc.

Explanation for the cases in which the one exponent n' is a fractional number

§13 It shall suffice to have taken $n' = \frac{1}{2}$ here, whence the series of coefficients

$$1 + \alpha' + \beta' + \gamma' + \delta' + \text{etc.}$$

it will be

$$1-\frac{1}{2}+\frac{1\cdot 3}{2\cdot 4}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}-\text{etc.}$$

Therefore, having separately multiplied the terms by the series

$$1+\alpha+\beta+\gamma+\delta+\text{etc.}$$

this series will arise

$$1-\frac{1}{2}\alpha+\frac{1\cdot 3}{2\cdot 4}\beta-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\gamma+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\delta-\text{etc.},$$

whose sum will therefore become equal to this product

$$\frac{n-\frac{1}{2}}{1}\cdot\frac{n-\frac{3}{2}}{2}\cdot\frac{n-\frac{5}{2}}{3}\cdots\frac{\frac{1}{2}}{n}$$

or

$$\frac{2n-1}{2}\cdot\frac{2n-3}{4}\cdot\frac{2n-5}{6}\cdot\frac{2n-7}{8}\cdots\frac{1}{2n},$$

how what happens we want to examine in the following cases.

I. Let $n=1$ and hence $\alpha=1$, $\beta=0$, $\gamma=0$ etc., whence our series will be

$$1-\frac{1}{2}=\frac{1}{2};$$

but our product on the other hand will be $=\frac{1}{2}$:

II. Let $n=2$; it will be $\alpha=2$, $\beta=1$, $\gamma=0$ etc., whence our series arises as

$$=1-\frac{2}{2}+\frac{1\cdot 3}{2\cdot 4}=\frac{3}{8};$$

but our product on the other hand becomes $=\frac{3}{8}$.

III. Let $n=3$ and hence $\alpha=3$, $\beta=3$, $\gamma=1$, $\delta=0$ etc., whence our series arises as

$$=1-\frac{3}{2}+\frac{3\cdot 3}{2\cdot 4}-\frac{3\cdot 5}{2\cdot 4\cdot 6}=\frac{5}{16};$$

but our product on the other hand becomes $=\frac{5}{16}$.

IV. Let $n=4$ and hence $\alpha=4$, $\beta=6$, $\gamma=4$, $\delta=1$, $\varepsilon=0$ etc., whence the series will be

$$1 - \frac{1}{2} \cdot 4 + \frac{3}{8} \cdot 6 - \frac{5}{16} \cdot 4 + \frac{35}{128} \cdot 1 = \frac{35}{128};$$

but our product will be

$$\frac{7}{2} \cdot \frac{5}{4} \cdot \frac{3}{6} \cdot \frac{1}{8} = \frac{35}{128}.$$

Explanation for the Cases in which both exponents are fractional numbers

§14 It shall suffice here the have expanded only the case, in which it is $n = \frac{1}{2}$ and $n' = -\frac{1}{2}$. Therefore, here for $n = \frac{1}{2}$ the series of coefficients will be

$$1 + \frac{1}{2} - \frac{1 \cdot 1}{2 \cdot 4} + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} + \text{etc.};$$

but for the exponent $n' = -\frac{1}{2}$ the series of coefficients will be

$$1 - \frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 4} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} - \text{etc.}$$

Therefore, from these series combined the series given in the theorem will arise

$$1 - \frac{1}{2^2} - \frac{1^2 \cdot 3}{2^2 \cdot 4^2} - \frac{1^2 \cdot 3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 6^2} - \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 9}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10^2} - \text{etc.}$$

which series therefore runs to infinity; and since n is not an integer number, the product of the theorem cannot be used; therefore, one has to go back to the integral formulas exhibited in the theorem, the first of which for the sum of this series yields

$$\frac{2}{\int \frac{dx}{\sqrt{x-xx}}},$$

the second form gives

$$\frac{-4 \cdot 0}{\int \frac{dx}{x\sqrt{x-xx}}},$$

but the third form gives

$$\frac{1}{\int \frac{dx\sqrt{1-x}}{\sqrt{x}}}.$$

Here, these integrals are to be extended from the boundary $x = 0$ to the boundary $x = 1$; since they must produce the same value, it will be convenient the exclude the second formula here.

§15 Therefore, let us expand the first formula

$$\frac{2}{\int \frac{dx}{\sqrt{x-xx}}},$$

for which we want to set $x = yy$ that it arises

$$\frac{1}{\int \frac{dy}{\sqrt{1-yy}}}.$$

But it is known that for the assigned boundaries it is

$$\int \frac{dy}{\sqrt{1-yy}} = \frac{\pi}{2},$$

whence the value of our series will be $\frac{2}{\pi}$. But the third formula, which was

$$\frac{1}{\int \frac{dx\sqrt{1-x}}{\sqrt{x}}}.$$

having put $x = yy$ will become

$$\int \frac{dx}{\sqrt{x}}\sqrt{1-x} = 2\int dy\sqrt{1-yy};$$

but this formula $\int \sqrt{1-yy}$ on the other hand expresses the area of the quadrant whose radius is $= 1$; because this is $= \frac{1}{4}\pi$, the sum of our series will be $\frac{2}{\pi}$, as before.

§16 Therefore, hence it is plain that the sum of the found series is $= \frac{2}{\pi}$. Hence, if for the sake of brevity we represent the series this way

$$1 - A - C - D - E - F - \text{etc.} = \frac{2}{\pi},$$

it will be

$$A + B + C + D + \text{etc.} = 1 - \frac{2}{\pi}$$

But above we saw that it approximately is $\frac{4}{\pi} = 1.273420$, whence it must be

$$A + B + C + D + \text{etc.} = 0.363380$$

but here it is

$$A = \frac{1}{4}, \quad B = \frac{1 \cdot 3}{16}A, \quad C = \frac{3 \cdot 5}{36}B, \quad D = \frac{5 \cdot}{64}C,$$
$$E = \frac{7 \cdot 9}{100}D, \quad F = \frac{9 \cdot 11}{144}E \quad \text{etc.}$$

Therefore, let us expand these single factors in decimal fractions and it will be

$$\begin{aligned} A &= 0.250000 \\ B &= 0.046875 \\ C &= 0.019531 \\ D &= 0.010681 \\ E &= 0.006729 \\ F &= 0.004626 \\ \hline \text{Sum} &= 0.338442 \end{aligned}$$

which still deviated from the truth by the quantity 0.024938; this is not surprising, since the following terms were left out; for, since they decrease less, they can only cause a slight difference.

SCHOLIUM

§17 What was mentioned until now and was illustrated by several examples seems to confirm the truth of our theorems sufficiently enough, even though no direct proof could be given. But although there seems to be no direct way to explore this truth, it is nevertheless possible in two ways to give a complete proof, the one of which is based on the nature of the coefficients itself, while the other can be taken from the calculus of probabilities. Therefore, we will explain the first way of demonstrating it here carefully, which at the same time will provide us with innumerable other similar theorems.

DEFINITION

§18 By a character of this kind $\left[\frac{p}{q}\right]$ we will denote the product formed from q fractions, whose numerators starting from the superior letter q shall continuously decrease by unity, the denominators on the other hand starting from unity shall increase by unity continuously; hence it is understood that this character $\left[\frac{p}{q}\right]$ denotes this product usually expressed as follows

$$\frac{p}{1}\cdot\frac{p-1}{2}\cdot\frac{p-2}{3}\cdots\frac{p-q+1}{q}.$$

COROLLARY 1

§19 Now this way it will be possible to exhibit the coefficients of the single powers, which by represented by the letters α, β, γ etc. above, sufficiently succinctly and elegantly, which for the exponent n will of course be:

$$\begin{aligned}
&\frac{n}{1}=\left[\frac{n}{1}\right],\\
&\frac{n(n-1)}{1\cdot 2}=\left[\frac{n}{2}\right],\\
&\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}=\left[\frac{n}{3}\right],\\
&\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}=\left[\frac{n}{4}\right],\\
&\frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}=\left[\frac{n}{5}\right]
\end{aligned}$$

etc.

And hence it is understood, since for each power the first of all coefficients is always the unity, that in this new way of representing them it will be

$$\left[\frac{n}{0}\right]=1.$$

And in similar manner, since the last of the coefficients is also the unity, it will even be

$$\left[\frac{n}{n}\right] = 1,$$

since it will be

$$\left[\frac{n}{n}\right] = \frac{n(n-1)(n-2)\cdots 1}{1\cdot 2\cdot 3\cdots n},$$

where the numerator manifestly is equal to the denominator.

Corollary 2

§20 Since, as often as the exponent n is a positive integer, so all coefficients preceding the first as all following the last are equal to zero, according to this way of expressing it will always be

$$\left[\frac{n}{-1}\right] = 0, \quad \left[\frac{n}{-2}\right] = 0, \quad \left[\frac{n}{-3}\right] = 0 \quad \text{etc.},$$

such that while i denotes an arbitrary positive integer it always is

$$\left[\frac{n}{-i}\right] = 0.$$

In similar manner for the coefficients following the last it will always be

$$\left[\frac{n}{n+1}\right], \quad \left[\frac{n}{n+2}\right] = 0, \quad \left[\frac{n}{n+3}\right] = 0, \quad \left[\frac{n}{n+4}\right] = 0 \quad \text{etc.}$$

and hence it will be in general

$$\left[\frac{n}{n+i}\right] = 0.$$

Lemma 1

§21 *In this way of notation it will always be*

$$\left[\frac{p}{q}\right] = \left[\frac{p}{p-q}\right].$$

For, because it is

$$\left[\frac{p}{q}\right] = \frac{p(p-1)(p-2)(p-3)\cdots(p-q+1)}{1\cdot 2\cdot 3\cdot 4\cdots q}$$

and in similar manner

$$\left[\frac{p}{p-q}\right] = \frac{p(p-1)(p-2)(p-3)\cdots(q+1)}{1\cdot 2\cdot 3\cdot 4\cdots(p-q)},$$

these two expressions are manifestly equal to each other; for, by cross multiplying the first denominator multiplied by the second denominator yields the product

$$1\cdot 2\cdot 3\cdots(p-q)(p-q+1)(p-q+2)\cdots p,$$

where the factors without any interruption continuously increase by unity, such that this product is $1\cdot 2\cdot 3\cdot 4\cdots p$. In similar manner the first denominator multiplied by the second numerator gives this product

$$1\cdot 2\cdot 3\cdot 4\cdots q\cdot(q+1)(q+2)\cdots p,$$

which likewise is $1\cdot 2\cdot 3\cdot 4\cdots p$ as before; and so the equality of these two formulas $\left[\frac{p}{q}\right]$ and $\left[\frac{p}{p-q}\right]$ is demonstrated.

Corollary

§22 This lemma manifestly already contains the reason, why the coefficients of all orders, written directly or backwards, proceed according to the same law.

Lemma 2

§23 *Having introduced the same notation for the coefficients it will always be*

$$\left[\frac{p}{q}\right] + \left[\frac{p}{q-1}\right] = \left[\frac{p+1}{q}\right].$$

For, because it is

$$\left[\frac{p}{q}\right] = \frac{p(p-1)(p-2)(p-3)\cdots(p-q+1)}{1\cdot 2\cdot 3\cdot 4\cdots q}$$

and

$$\left[\frac{p}{q-1}\right] = \frac{p(p-1)(p-2)(p-3)\cdots(p-q+2)}{1\cdot 2\cdot 3\cdot 4\cdots(q-1)},$$

the first form becomes equal to the second multiplied by $\frac{p-q+1}{q}$ and it will be

$$\left[\frac{p}{q}\right] + \left[\frac{p}{q-1}\right] = \left[\frac{p}{q-1}\right]\left(1+\frac{p-q+1}{q}\right) = \left[\frac{p}{q-1}\right]\cdot\left(\frac{p+1}{q}\right),$$

whence we will have

$$\left[\frac{p}{q}\right] + \left[\frac{p}{q-1}\right] = \left(\frac{p+1}{q}\right)\cdot\frac{p(p(-1)(p-2)\cdots(p-q+2)}{1\cdot 2\cdot 3\cdots(q-1)},$$

which form manifestly agrees with this one

$$\frac{(p+1)p(p-1)(p-2)\cdots(p-q+2)}{1\cdot 2\cdot 3\cdot 4\cdots q},$$

which therefore in our new notation is represented this way

$$\left[\frac{p+1}{q}\right],$$

such that it is

$$\left[\frac{p}{q}\right] + \left[\frac{p}{q-1}\right] = \left[\frac{p+1}{q}\right].$$

Corollary 1

§24 If instead of q we write $q+1$, having permuted the formulas it will be

$$\left[\frac{p}{q}\right] + \left[\frac{p}{q+1}\right] = \left[\frac{p+1}{q+1}\right]$$

and in similar manner by continuously increasing the number q by unity it will also be as follows:

$$\left[\frac{p}{q+1}\right]+\left[\frac{p}{q+2}\right]=\left[\frac{p+1}{q+2}\right],$$
$$\left[\frac{p}{q+2}\right]+\left[\frac{p}{q+3}\right]=\left[\frac{p+1}{q+3}\right],$$
$$\left[\frac{p}{q+3}\right]+\left[\frac{p}{q+4}\right]=\left[\frac{p+1}{q+4}\right],$$
$$\left[\frac{p}{q+4}\right]+\left[\frac{p}{q+5}\right]=\left[\frac{p+1}{q+5}\right],$$
$$\left[\frac{p}{q+5}\right]+\left[\frac{p}{q+6}\right]=\left[\frac{p+1}{q+6}\right]$$

etc.

COROLLARY 2

§25 If we as two of these equalities following each other, this new equations will arise:

$$\begin{aligned}
\left[\frac{p}{q}\right] &= 2\left[\frac{p}{q+1}\right]+\left[\frac{p}{q+2}\right]=\left[\frac{p+1}{q+1}\right]+\left[\frac{p+1}{q+2}\right]=\left[\frac{p+2}{q+2}\right]\\
\left[\frac{p}{q+1}\right] &= 2\left[\frac{p}{q+2}\right]+\left[\frac{p}{q+3}\right]=\left[\frac{p+1}{q+2}\right]+\left[\frac{p+1}{q+3}\right]=\left[\frac{p+2}{q+3}\right]\\
\left[\frac{p}{q+2}\right] &= 2\left[\frac{p}{q+3}\right]+\left[\frac{p}{q+4}\right]=\left[\frac{p+1}{q+3}\right]+\left[\frac{p+1}{q+4}\right]=\left[\frac{p+2}{q+4}\right]\\
\left[\frac{p}{q+3}\right] &= 2\left[\frac{p}{q+4}\right]+\left[\frac{p}{q+5}\right]=\left[\frac{p+1}{q+4}\right]+\left[\frac{p+1}{q+5}\right]=\left[\frac{p+2}{q+5}\right]\\
\left[\frac{p}{q+4}\right] &= 2\left[\frac{p}{q+5}\right]+\left[\frac{p}{q+6}\right]=\left[\frac{p+1}{q+5}\right]+\left[\frac{p+1}{q+6}\right]=\left[\frac{p+2}{q+6}\right]\\
\left[\frac{p}{q+5}\right] &= 2\left[\frac{p}{q+6}\right]+\left[\frac{p}{q+7}\right]=\left[\frac{p+1}{q+6}\right]+\left[\frac{p+1}{q+7}\right]=\left[\frac{p+2}{q+7}\right]\\
\left[\frac{p}{q+6}\right] &= 2\left[\frac{p}{q+7}\right]+\left[\frac{p}{q+8}\right]=\left[\frac{p+1}{q+7}\right]+\left[\frac{p+1}{q+8}\right]=\left[\frac{p+2}{q+8}\right]
\end{aligned}$$

etc.

COROLLARY 3

§26 If we again add two of these equalities following each other, we will at first find

$$\left[\frac{p}{q}\right]+3\left[\frac{p}{q+1}\right]+3\left[\frac{p}{q+2}\right]+\left[\frac{p}{q+3}\right]=\left[\frac{p+2}{q+2}\right]+\left[\frac{p+2}{q+3}\right]=\left[\frac{p+3}{q+3}\right].$$

In similar manner the following equations will arise

$$ebinompq+1+3\left[\frac{p}{q+2}\right]+3\left[\frac{p}{q+3}\right]+\left[\frac{p}{q+4}\right]=\left[\frac{p+3}{q+4}\right]$$

and in the same way further

$$ebinompq+2+3\left[\frac{p}{q+3}\right]+3\left[\frac{p}{q+4}\right]+\left[\frac{p}{q+5}\right]=\left[\frac{p+3}{q+5}\right];$$

and it the same manner it will be possible to proceed as far as one desires; and hence we will be able to solve the following problem.

PROBLEM

§27 *Having taken arbitrary integer numbers for p and q, if furthermore the letter n also denotes an arbitrary number of this kind, to investigate the sum of this series*

$$\left[\frac{n}{0}\right]\cdot\left[\frac{p}{q}\right]+\left[\frac{n}{q}\right]\cdot\left[\frac{p}{q+1}\right]+\left[\frac{n}{2}\right]\cdot\left[\frac{p}{q+2}\right]+\left[\frac{n}{3}\right]\cdot\left[\frac{p}{q+3}\right]+etc.$$

SOLUTION

Because it is $\left[\frac{n}{0}\right]=1$ and these formulas

$$\left[\frac{n}{0}\right],\quad\left[\frac{n}{1}\right],\quad\left[\frac{n}{2}\right],\quad\left[\frac{n}{3}\right]\quad\text{etc.}$$

exhibit the coefficients for the power of the exponent n, in the preceding Corollaries we saw that it will be for the case $n=1$

$$\text{I.}\quad\left[\frac{p}{q}\right]+\left[\frac{p}{q+1}\right]=\left[\frac{p+1}{q+1}\right];$$

but then for the case $n = 2$ the second Corollary gave

$$\text{II.} \quad \left[\frac{p}{q}\right] + 2\left[\frac{p}{q+1}\right] + \left[\frac{p}{q+2}\right] = \left[\frac{p+2}{q+2}\right];$$

further, for the case $n = 3$ we found in Corollary 3

$$\text{III.} \quad \left[\frac{p}{q}\right] + 3\left[\frac{p}{q+1}\right] + 3\left[\frac{p}{q+2}\right] + \left[\frac{p}{q+3}\right] = \left[\frac{p+3}{q+3}\right];$$

and if in the same Corollary we add the first two equations, for the case $n = 4$ this equation will arise

$$\text{IV.} \quad \left[\frac{p}{q}\right] + 4\left[\frac{p}{q+1}\right] + 6\left[\frac{p}{q+2}\right] + 4\left[\frac{p}{q+3}\right] + \left[\frac{p}{q+4}\right] = \left[\frac{p+4}{q+4}\right];$$

hence it is now seen clearly enough that for the case $n = 5$ it will be

$$\text{V.} \quad \left[\frac{p}{q}\right] + 5\left[\frac{p}{q+1}\right] + 10\left[\frac{p}{q+2}\right] + 10\left[\frac{p}{q+3}\right] + 5\left[\frac{p}{q+4}\right] + \left[\frac{p}{q+5}\right] = \left[\frac{p+5}{q+5}\right],$$

and hence it is now possible to enunciate that the sum of the series propounded in the problem

$$\left[\frac{n}{0}\right] \cdot \left[\frac{p}{q}\right] + \left[\frac{n}{1}\right] \cdot \left[\frac{p}{q+1}\right] + \left[\frac{n}{2}\right] \cdot \left[\frac{p}{q+2}\right] + \left[\frac{n}{3}\right] \cdot \left[\frac{p}{q+3}\right] + \text{etc.}$$

is

$$= \left[\frac{p+n}{q+n}\right],$$

by which formula this product is indicated

$$\frac{p+n}{1} \cdot \frac{p+n-1}{2} \cdot \frac{p+n-2}{3} \cdot \frac{p+n-3}{4} \cdots \frac{p-q+1}{q+n}.$$

COROLLARY 1

§28 Because by means of Lemma 1 it is in general

$$\left[\frac{p}{q}\right]=\left[\frac{p}{p-q}\right],$$

the sum of our propounded series will also be $=\left[\frac{p+n}{p-q}\right]$, by which form this product is indicated

$$\frac{p+n}{1}\cdot\frac{p+n-1}{2}\cdot\frac{p+n-2}{3}\cdot\frac{p+n-3}{4}\cdots\frac{q+n+1}{p-q}.$$

COROLLARY 2

§29 If we take $q=0$, these formulas

$$\left[\frac{p}{0}\right]+\left[\frac{p}{1}\right]+\left[\frac{p}{2}\right]+\left[\frac{p}{3}\right]+\text{etc.}$$

will exhibit the coefficients for the power of the exponent p; therefore, if they are each separately multiplied by the coefficients for the power of the exponent n, this series will result

$$\left[\frac{n}{0}\right]\cdot\left[\frac{p}{0}\right]+\left[\frac{n}{1}\right]\cdot p1+\left[\frac{n}{2}\right]\cdot p2+\left[\frac{n}{3}\right]\cdot p3+\text{etc.},$$

whose sum will therefore be $=\left[\frac{p+n}{n}\right]$ or even $\left[\frac{p+n}{p}\right]$, the latter of which formulas gives this product

$$\frac{p+n}{1}\cdot\frac{p+n-1}{2}\cdot\frac{p+n-2}{3}\cdot\frac{p+n-3}{4}\cdots\frac{p+1}{n},$$

the other formula expanded on the other hand becomes equal to this product

$$\frac{p+n}{1}\cdot\frac{p+n-1}{2}\cdot\frac{p+n-2}{3}\cdot\frac{p+n-3}{4}\cdots\frac{n+1}{p},$$

and so the truth of the second theorem mentioned above is demonstrated, since the letters α, β, γ etc. there denote the coefficients for the exponent n, the others α', β', γ' etc. the coefficients for the exponent n', instead of which we have p here.

Corollary 3

§30 if we furthermore take $p = n$, our series will go over into the one which we contemplated in theorem 1, namely

$$1 + \left[\frac{n}{1}\right]^2 + \left[\frac{n}{2}\right]^2 + \left[\frac{n}{3}\right]^2 + \text{etc.},$$

instead of which we there had

$$1 + \alpha^2 + \beta^2 + \gamma^2 + \text{etc.};$$

therefore, its sum from the thing mentioned here will be

$$\left[\frac{2n}{n}\right] = \frac{2n}{1} \cdot \frac{2n-1}{2} \cdot \frac{2n-2}{3} \cdot \frac{2n-3}{4} \cdots \frac{n+1}{n};$$

and so also Theorem 1 is rigorously proved.

Scholium

§31 In the first Theorem we expressed the sum of the series by means of another product, namely,

$$\frac{2}{1} \cdot \frac{6}{2} \cdot \frac{10}{3} \cdot \frac{14}{4} \cdot \frac{18}{5} \cdots \frac{4n-2}{n};$$

but that this formula agrees with the one exhibited here completely can be shown easily. For, because the denominators of both are the same, it is to be demonstrated that this product

$$2n(2n-1)(2n-2)\cdots(n+1)$$

is always equal to this product

$$2 \cdot 6 \cdot 10 \cdot 14 \cdots (4n-2).$$

For this aim let us put the first product $= P$, but let us denote the following, which arises, if instead of n we write $n+1$, on the other hand by the letter Q, such that it is

$$Q = (2n+2)(2n+1)2n(2n-1)(2n-2)\cdots(n+2),$$

whence it is plain that it will be

$$\frac{Q}{P} = \frac{(2n+2)(2n+1)}{n+1} = 4n+2$$

and hence

$$Q = (4n+2)P;$$

hence it is plain, how from each value for n the following value for $n+1$ is defined. hence, because for the case $n=1$ that product P is $=2$, the following product will be $Q = 2 \cdot 6$, which therefore holds for $n=2$; if this is now again called $=P$, the following product Q will be $= 2 \cdot 6 \cdot 10$ for the case $n=3$; if this is again denoted by P, the following product will be $Q = 2 \cdot 6 \cdot 10 \cdot 14$ for the case $n=4$; hence the truth of this identity becomes manifest. Additionally, the problem we just treated extends a lot further than the theorem mentioned in the beginning, whence it will be worth one's while to formulate the theorem arising from this here in all clarity.

General Theorem

§32 *If the letters p, n and q denote arbitrary integer numbers, the sum of the series formed from this*

$$\left[\frac{n}{0}\right] \cdot \left[\frac{p}{q}\right] + \left[\frac{n}{1}\right] \cdot \left[\frac{p}{q+1}\right] + \left[\frac{n}{2}\right] \cdot \left[\frac{p}{q+2}\right] + \text{etc.}$$

is always equal to this formula

$$\left[\frac{p+n}{q+n}\right]$$

or also to this one

$$\left[\frac{p+n}{p-q}\right],$$

of which the first yields this product

$$\frac{p+n}{1} \cdot \frac{p+n-1}{2} \cdot \frac{p+n-2}{3} \cdots \frac{p-q+1}{q+n},$$

the latter on the other hand this one

$$\frac{p+n}{1}\cdot\frac{p+n-1}{2}\cdot\frac{p+n-2}{3}\cdots\frac{q+n+1}{p-q}.$$

COROLLARY 1

§33 Therefore, if we denote this series by the letter S, such that it is

$$S=\left[\frac{n}{0}\right]\cdot\left[\frac{p}{q}\right]+\left[\frac{n}{1}\right]\cdot\left[\frac{p}{q+1}\right]+\text{etc.},$$

it will be

$$S=\left[\frac{p+n}{q+n}\right].$$

Now instead of p let us write $p+1$ and the refer to the series arising from this by S' such that it is

$$S'=\left[\frac{n}{0}\right]\cdot\left[\frac{p+1}{q}\right]+\left[\frac{n}{1}\right]\cdot\left[\frac{p+1}{q+1}\right]+\left[\frac{n}{2}\right]\cdot\left[\frac{p+1}{q+2}\right]+\text{etc.};$$

it will be

$$S'=\left[\frac{p+n+1}{q+n}\right]$$

and having done the expansion it will be

$$S'=\frac{p+n+1}{1}\cdot\frac{p+n}{2}\cdot\frac{p+n-1}{3}\cdots\frac{p-q+2}{q+n},$$

whence it is concluded that it will be

$$\frac{S'}{S}=\frac{p+n+1}{p-q+1},$$

such that it is

$$S'=\frac{p+n+1}{p-q+1}S.$$

COROLLARY 2

§34 In similar manner, if in our series S instead of p we write $p+2$ and set

$$S'' = \left[\frac{n}{0}\right] \cdot \left[\frac{p+2}{q}\right] + \left[\frac{n}{1}\right] \cdot \left[\frac{p+2}{q+1}\right] + \left[\frac{n}{2}\right] \cdot \left[\frac{p+2}{q+2}\right] + \left[\frac{n}{3}\right] \cdot \left[\frac{p+2}{q+3}\right] + \text{etc.},$$

it will be

$$S'' = \frac{p+n+2}{p-q+2} S',$$

whence we will now have

$$S'' = \frac{p+n+1}{p-q+1} \cdot \frac{p+n+2}{p-q+2} S.$$

In similar manner, if we again augment the letter p by the unity and set

$$S''' = \left[\frac{n}{0}\right] \cdot \left[\frac{p+3}{q}\right] + \left[\frac{n}{1}\right] \cdot \left[\frac{p+3}{q+1}\right] + \left[\frac{n}{2}\right] \cdot \left[\frac{p+3}{q+2}\right] + \text{etc.},$$

it will be

$$S''' = \frac{p+n+3}{p-q+3} S''$$

and hence we will now have

$$S''' = \frac{p+n+1}{p-q+1} \cdot \frac{p+n+2}{p-q+2} \cdot \frac{p+n+3}{p-q+3} S.$$

And so one can proceed further as far as one desires, such that by means of the value S all the following ones S', S'', S''', S'''' etc. can easily be exhibited, what will have an immense use in the following.

PROBLEM

§35 *If either two or even all of the letters p, q and n were fractions, to explore the true value of the propounded series*

$$S = ebinomn0 \cdot ebinompq + ebinomn1 \cdot ebinompq + 1 + ebinomn2 \cdot ebinompq + 2 + \text{etc.}$$

by means of an integral formula.

Solution

Above in Theorem 2 we observed that the value of this product

$$\frac{n+n'}{1} \cdot \frac{n+n'-1}{2} \cdot \frac{n+n'-2}{3} \cdot \frac{n'+1}{n}$$

becomes equal either to this formula

$$\text{I.} \quad \frac{1}{n \int x^{n'} dx (1-x)^{n-1}}$$

or to this formula

$$\text{II.} \quad \frac{1}{nn' \int x^{n'-1} dx (1-x)^{n-1}}$$

or even to this one

$$\text{III.} \quad \frac{1}{(n+n'+1) \int x^{n'} dx (1-x)^{n}}.$$

Therefore, because in our case it is

$$S = \frac{p+n}{1} \cdot \frac{p+n-1}{2} \cdot \frac{p+n-2}{3} \cdots \frac{p-q+1}{q+n},$$

having done the comparison correctly it is plain that, what was n there, is $q+n$ here, and what was $n+n'$ there, is $p+n$ here; and hence what was n' here, will be $p-q$ here; having noted these things we will be able to express the value of S by means of integral formulas in three ways, which are

$$\begin{aligned}
&\text{I.} \quad S = \frac{1}{(q+n) \int x^{p-q} dx (1-x)^{q+n-1}}, \\
&\text{II.} \quad S = \frac{p+n}{(p-q)(q+n) \int x^{p-q-1} dx (1-x)^{q+n-1}}, \\
&\text{III.} \quad S = \frac{1}{(p+n+1) \int x^{p-q} dx (1-x)^{q+n}},
\end{aligned}$$

if these single integrals are extended from the boundary $x=0$ to the boundary $x=1$. But it will always do not matter, which of these three formulas we want to use, since they agree perfectly, as it is understood from the well known reduction of integrals. But it is manifest, whatever fractions are denoted by the letters p, q and n, that the value of the sum S is reduced to a certain integral formula or quadrature.

COROLLARY 1

§36 Therefore, it its hence plain that innumerable series of that kind can have a common sum. As if one has another arbitrary series of this form

$$\left[\frac{N}{0}\right]\cdot PQ+\left[\frac{N}{1}\right]\cdot\left[\frac{P}{Q+1}\right]+\left[\frac{N}{2}\right]\cdot\left[\frac{P}{Q+2}\right]+\text{etc.},$$

that its sum becomes equal to the preceding one, first it is required that it is $Q+N=q+n$, secondly that $P-Q=p-q$ and hence

$$Q=q+n-N \quad \text{and} \quad P=p+n-N,$$

where therefore N is arbitrary; and as long as only by the letters P and Q these values are assigned, the series resulting from this will always be equal to the series summed here.

COROLLARY 2

§37 If, as we did it before, we instead of p successively write $p+1$, $p+2$, $p+3$ etc. and denote the series arising from this by S', S'', S''' etc., these will be expressed by means of integrals formulas the following way:

$$\begin{aligned}
&\text{I.} && \frac{p+n+1}{(p-q+1)(q+n)\int x^{p-q}dx(1-x)^{q+n-1}},\\
&\text{II.} && \frac{(p+n+1)(p+n)}{(p-q+1)(p-q)(q+n)\int x^{p-q-1}dx(1-x)^{q+n-1}},\\
&\text{III.} && \frac{1}{(p-q+1)\int x^{p-q}dx(1-x)^{q+n}}.
\end{aligned}$$

Hence it is sufficiently clear, how even the following values S'', S''' etc. must be expressed.

SCHOLIUM

§38 Therefore, if the letters p, q and n denote integer numbers, it is evident that these single formulas become actually integrable and hence the same arise we found above for the sum S; but if among these letters fractions occur, the summation will be reduced to the higher quadratures the more complicated the fractions were; among these especially those remarkable cases occurs which can be reduced to circular arcs, what happens in this formula

$$\int \frac{x^{\lambda}}{(1-x)^{\lambda}} \cdot \frac{dx}{x}$$

or

$$\int x^{\lambda-1} dx (1-x)^{-\lambda},$$

such that having made the comparison to the first of our formulas it is

$$p - q = \lambda - 1 \quad \text{and} \quad q + n - 1 = -\lambda$$

and hence

$$p = -n \quad \text{and} \quad q = 1 - n - \lambda.$$

But for integrating the formula

$$\int \frac{x^{\lambda}}{(1-x)^{\lambda}} \cdot \frac{dx}{x}$$

let us set

$$\frac{x}{1-x} = y$$

and hence it will be

$$x = \frac{y}{1+y} \quad \text{and} \quad \frac{dx}{x} = \frac{dy}{y(1+y)}$$

and hence our formula will become

$$\int \frac{y^{\lambda-1} dy}{1+y},$$

where it is to noted that in the case $x = 0$ it will be $y = 0$, but in the case $x = 1$ it will be $y = \infty$, such that this integral must be taken from the boundary $y = 0$ to the boundary $y = \infty$. Since now we consider the exponent λ as a fractions, let us put $\lambda = \frac{\mu}{\nu}$ and the integral formula will be

$$\int \frac{y^{\frac{\mu-\nu}{\nu}}}{1+y}.$$

Here, let us further set $y = z^\nu$ that it is

$$y^{\frac{\mu-\nu}{\nu}} = z^{\mu-\nu} \quad \text{and} \quad dy = \nu z^{\nu-1} dz,$$

whence the integral formula will be

$$\nu \int \frac{z^{\mu-1} dz}{1+z^\nu}.$$

But about this formula it is known that its integral from the boundary $z = 0$ to the boundary $z = \infty$ is

$$= \frac{\pi}{\sin \frac{\mu\pi}{\nu}};$$

therefore, as often as it was

$$p = -n \quad \text{and} \quad q = 1 - n - \frac{\mu}{\nu},$$

then the value of our integral formula will be

$$\int x^{p-q} dx (1-x)^{q+n-1} = \frac{\pi}{\sin \frac{\mu\pi}{\nu}},$$

whence we will be able to solve the following problem.

Problem

§39 *Having propounded this series*

$$S = \begin{bmatrix} n \\ 0 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} + \begin{bmatrix} n \\ 1 \end{bmatrix} \begin{bmatrix} p \\ q+1 \end{bmatrix} + \begin{bmatrix} n \\ 2 \end{bmatrix} \begin{bmatrix} p \\ q+2 \end{bmatrix} + etc.$$

to find the relation between the numbers p, q and n that its sum S can be expressed by means of the quadrature of the circle.

Solution

Among the three integral formulas given for the sum of this series S given above [§ 35] the first was

$$S = \frac{1}{(q+n)\int x^{p-q} dx (1-x)^{q+n-1}};$$

but we just saw, as often as it was

$$p = -n \quad \text{and} \quad q = 1 - n - \frac{\mu}{\nu},$$

that so often it will be

$$\int x^{p-q} dx (1-x)^{q+n-1} = \frac{\pi}{\sin \frac{\mu\pi}{\nu}};$$

having substituted this value it will be

$$S = \frac{\sin \frac{\mu\pi}{\nu}}{(q+n)\pi} = \frac{\nu \sin \frac{\mu\pi}{\nu}}{(\nu - \mu)\pi}.$$

Therefore, that this value holds, two conditions are required, the first of which postulates that it is

$$p = -n \quad \text{or} \quad p + n = 0,$$

the second on the other hand that it is

$$q = 1 - n - \frac{\mu}{\nu} \quad \text{or} \quad p - q = \frac{\mu}{\nu} - 1.$$

Corollary

§40 Therefore, if these conditions hold, if instead of p we successively write $p+1$, $p+2$, $p+3$ etc., but denote the sum of our series arising from this by S', S'', S''' etc., since we found

$$S = \frac{\nu \sin \frac{\mu\pi}{\nu}}{(\nu - \mu)\pi},$$

it will be

$$S' = \frac{p+n+1}{p-q+1} \cdot \frac{\nu \sin \frac{\mu\pi}{\nu}}{(\nu-\mu)\pi} = \frac{\nu^2}{\mu(\nu-\mu)\pi} \cdot \sin\frac{\mu\pi}{\nu},$$

$$S'' = \frac{p+n+2}{p-q+2} S' = \frac{1 \cdot 2\nu^3}{\mu(\mu+\nu)(\nu-\mu)\pi} \cdot \sin\frac{\mu\pi}{\nu},$$

$$S''' = \frac{p+n+3}{p-q+3} S'' = \frac{1 \cdot 2 \cdot 3\nu^4}{\mu(\mu+\nu)(\mu+2\nu)(\nu-\mu)\pi} \cdot \sin\frac{\mu\pi}{\nu},$$

$$S'''' = \frac{p+n+4}{p-q+4} S''' = \frac{1 \cdot 2 \cdot 3 \cdot 4\nu^5}{\mu(\mu+\nu)(\mu+2\nu)(\mu+3\nu)(\nu-\mu)\pi} \cdot \sin\frac{\mu\pi}{\nu}$$

etc.

EXAMPLE

§41 Let us accommodate these to the case of our second theorem, for which one must set $q = 0$ that we obtain this series

$$S = \left[\frac{n}{0}\right] \cdot \left[\frac{p}{0}\right] + \left[\frac{n}{1}\right] \cdot \left[\frac{p}{1}\right] + \left[\frac{n}{2}\right] \cdot \left[\frac{p}{2}\right] + \left[\frac{n}{3}\right] \cdot \left[\frac{p}{3}\right] + \text{etc.}$$

But because it was $q = 1 - n - \frac{\mu}{\nu}$, here it will be

$$n = \frac{\nu - \mu}{\nu} \quad \text{and hence} \quad p = \frac{\mu - \nu}{\nu}.$$

But here we will conveniently be able to retain the number n itself in the computation, such that it is $p = -n$; but then it will be

$$\frac{\mu}{\nu} = 1 - n \quad \text{or} \quad \mu = \nu(1-n),$$

from what our sum will be

$$S = \frac{\sin(1-n)\pi}{n\pi},$$

which therefore is the sum of this series

$$S = 1 + \left[\frac{n}{1}\right] \cdot \left[\frac{-n}{1}\right] + \left[\frac{n}{2}\right] \cdot \left[\frac{-n}{2}\right] + \left[\frac{n}{3}\right] \cdot \left[\frac{-n}{3}\right] + \text{etc.}$$

§42 If we now augment the number p by the unity, because of $p+1=1-n$ our series will be

$$S' = 1 + \left[\frac{n}{1}\right]\left[\frac{1-n}{1}\right] + \left[\frac{n}{2}\right]\left[\frac{1-n}{2}\right] + \left[\frac{n}{3}\right]\left[\frac{1-n}{3}\right] + \text{etc.},$$

whence because of $\mu = \nu(1-n)$ this sum will be

$$S' = \frac{\sin(1-n)\pi}{n(1-n)\pi};$$

or, because it is $\sin(1-n)\pi = \sin n\pi$, we will even more nicely have

$$S = \frac{\sin n\pi}{n\pi} \quad \text{and} \quad S' = \frac{\sin n\pi}{n(1-n)\pi}.$$

§43 If we now further set

$$S'' = 1 + \left[\frac{n}{1}\right]\left[\frac{2-n}{1}\right] + \left[\frac{n}{2}\right]\left[\frac{2-n}{2}\right] + \left[\frac{n}{3}\right]\left[\frac{2-n}{3}\right] + \text{etc.},$$

this sum will be found

$$S'' = \frac{1 \cdot 2 \sin n\pi}{n(1-n)(2-n)\pi}.$$

In similar manner, if we further set

$$S''' = 1 + \left[\frac{n}{1}\right]\left[\frac{3-n}{1}\right] + \left[\frac{n}{2}\right]\left[\frac{3-n}{2}\right] + \left[\frac{n}{3}\right]\left[\frac{3-n}{3}\right] + \text{etc.},$$

this sum will arise

$$S''' = \frac{1 \cdot 2 \cdot 3}{n(1-n)(2-n)(3-n)\pi}.$$

And this way it is possible to continue these series arbitrarily far.

§44 Now if we expand the characters introduced here for the sake of brevity in usual manner, the first series will obtain this form

$$S = 1 - \frac{nn}{1} + \frac{nn(nn-1)}{1 \cdot 4} - \frac{nn(nn-1)(nn-4)}{1 \cdot 4 \cdot 9} + \frac{nn(nn-1)(nn-4)(nn-9)}{1 \cdot 4 \cdot 9 \cdot 16} - \text{etc.}$$

$$= \frac{\sin n\pi}{n\pi},$$

the reason of which summation can be shown from elsewhere this way: Divide by $nn-1$ on both sides and it will be

$$\frac{S}{nn-1} = -1 + \frac{nn}{4} - \frac{nn(nn-4)}{1\cdot 4\cdot 9} + \frac{nn(nn-4)(nn-9)}{1\cdot 4\cdot 9\cdot 16} - \frac{nn(nn-4)(nn-9)(nn-16)}{1\cdot 4\cdot 9\cdot 16\cdot 25} + \text{etc.}$$

Further, let us divide by $\frac{nn-4}{4}$ on both sides and it will be

$$\frac{4S}{(nn-1)(nn-4)} = 1 - \frac{nn}{9} + \frac{nn(nn-9)}{9\cdot 16} - \frac{nn(nn-9)(n-16)}{9\cdot 16\cdot 25} + \text{etc.}$$

Further, let us divide by $\frac{nn-9}{9}$ and it will arise

$$\frac{4\cdot 9S}{(nn-1)(nn-4)(nn-9)} = -1 + \frac{nn}{16} - \frac{nn(nn-16)}{16\cdot 25} + \frac{nn(nn-16)(nn-25)}{16\cdot 25\cdot 36} - \text{etc.}$$

Further, divide us by $\frac{nn-16}{16}$ and it will arise

$$\frac{4\cdot 9\cdot 16S}{(nn-1)(nn-4)(nn-9)(nn-16)} = 1 - \frac{nn}{25} + \text{etc.}$$

Hence if these operations are continued to infinity, finally this equation will arise

$$\frac{1\cdot 4\cdot 9\cdot 16\cdots S}{(nn-1)(nn-4)(nn-9)(n-16)\cdot\text{etc.}} = \pm 1;$$

to get rid of the ambiguity of the sign, let us change the signs in the single denominators and we will have

$$\frac{1\cdot 4\cdot 9\cdot 16\cdots S}{(1-nn)(4-nn)(9-nn)(16-nn)\cdot\text{etc.}} = +1;$$

therefore, hence it follows that it will be

$$S = \frac{1-nn}{1}\cdot\frac{4-nn}{4}\cdot\frac{9-nn}{9}\cdot\frac{16-nn}{16}\cdot\text{etc. to infinity},$$

the value of which infinite product is already well known to be $= \frac{\sin n\pi}{n\pi}$. For, if we take $n = \frac{1}{2}$ here, hence it will be

$$\frac{2}{\pi} = \frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdot\frac{7\cdot 9}{8\cdot 8}\cdot\text{etc.},$$

which is the all known expression found by Wallis.

Scholium

§45 Therefore, until now we demonstrated the truth of our theorems, which at first sight justifiably seemed to be most difficult, from the most plain principles of analysis most rigorously. But still another way is given leading to the same goal derived from the theory of combinations, which, although it might to seem rather alien to our undertaking, we wan to explain here more clearly.

Problem

If one has several schedules or slips of paper, whose number shall be $= s$, among which one finds n slips of paper marked with certain signs and from the total amount k are taken, to investigate the number of cases, in which either none of those marked slips of paper among these k taken charts are found or only one or two or three or four etc. or even all n, if the number n does not exceed the number k, of course.

Solution

§46 Since the number of all charts is $= s$, if hence one slip of paper is extracted, the amount of varieties would be $= s$; but if only two would be taken, the number of possibilities would be

$$\frac{s(s-1)}{1\cdot 2},$$

which number expressed by means of our characters will be $\left[\frac{s}{2}\right]$; but if three slips of paper are extracted, the number of varieties is calculated to be

$$= \frac{s(s-1)(s-2)}{1\cdot 2\cdot 3} = s3$$

and hence we conclude, if the number of the extracted slips of paper was $= k$, that the number of all possible varieties will be $\left[\frac{s}{k}\right]$.

§47 Since now the number of marked slips of paper is $= n$, first let us ask, in how many ways it can happen that none of them occurs among the k taken slips of paper; to find this let us exclude all signed slips of paper from our total amount and the number of the remaining slips of paper will be $= s - n$; hence, if k slips are extracted, the number of all varieties will be

$$= \left[\frac{s-n}{k}\right],$$

which number contains all cases, in which non of the signed slips of paper will be found among the extracted ones.

§48 Now, let us investigate, in how many ways it can happen that one single marked slip of paper is found among the extracted ones; therefore, it is necessary that the remaining extracted ones, whose number is $k - 1$, are not marked; since their number is $s - n$, if hence only $k - 1$ slips of paper are extracted, the number of all varieties will be

$$= \left[\frac{s-n}{k-1}\right].$$

Hence, if in these single case we add only one marked slip of paper, what can be done in n different ways, the number of all these cases will be

$$= n\left[\frac{s-n}{k-1}\right] = \left[\frac{n}{1}\right] \cdot \left[\frac{s-n}{k-1}\right].$$

§49 In similar manner, let us investigate all cases, in which two marked slips of paper will be found among the k extracted ones; therefore, the remaining of these slips of papers, whose number is $= k - 2$, must not be marked and be subtracted from the number of slips of paper $s - n$m whence the number of all varieties will be

$$= \left[\frac{s-n}{k-2}\right];$$

therefore, it will be necessary to ass two marked slips of papers to these single ones, which can be done in

$$\frac{n(n-1)}{1 \cdot 2} = \left[\frac{n}{2}\right]$$

cases; hence the number of all cases, in which only two marked slips of paper will be found among the extracted ones, will be

$$\left[\frac{n}{2}\right] \cdot \left[\frac{s-n}{k-2}\right].$$

In the same way it will easily become clear, that only three marked slips of paper occur among the extracted ones, the number of all possible cases will be

$$\left[\frac{n}{3}\right] \cdot \left[\frac{s-n}{k-3}\right].$$

Therefore, further, that four marked slips of paper are found among the extracted ones, the number of all possible varieties will be

$$= \left[\frac{n}{4}\right] \cdot \left[\frac{s-n}{k-4}\right].$$

§50 Therefore, hence we conclude that in general, if λ marked slips of paper are found among the extracted ones, the number of all possible varieties will be

$$= \left[\frac{n}{\lambda}\right] \cdot \left[\frac{s-n}{k-\lambda}\right],$$

which number will become zero in two ways, first of course, as we initially observed, when it was $\lambda > n$; but then also, if it was $\lambda > k$; for, in these two cases such an extraction, as it is desired, is not possible. Hence, if the number of marked slips of papers n was nit larger than the number of extracted ones k, the number of all possible cases, in which all n marked slips of paper will occur among the extracted ones, where it is $\lambda = n$, will be

$$\left[\frac{n}{n}\right] \cdot \left[\frac{s-n}{k-n}\right] = \left[\frac{s-n}{k-n}\right] \quad \text{because of} \quad \left[\frac{n}{n}\right] = 1.$$

§51 But if the number of marked slips of paper n war larger than the number of extracted ones k, the last case will be the one, in which all k extracted slips of paper will be marked at the same time, whence here one must take $\lambda = k$ and the number of all these possible cases will be

$$\left[\frac{n}{k}\right] \cdot \left[\frac{s-n}{0}\right] = \left[\frac{n}{k}\right] \quad \text{and} \quad \left[\frac{s-n}{0}\right] = 1.$$

§52 To show all these different cases quite plainly, we want to add the following table, whose first column indicates, how many marked slips of papers must occur among all extracted ones; the second column on the other hand indicated the number of all cases, in which this can happen.

Number of marked slips of paper occuring among the extracted ones	Number of all possible cases, in which this can happen
0	$\left[\frac{n}{0}\right] \cdot \left[\frac{s-n}{k}\right]$
1	$\left[\frac{n}{1}\right] \cdot \left[\frac{s-n}{k-1}\right]$
2	$\left[\frac{n}{2}\right] \cdot \left[\frac{s-n}{k-2}\right]$
3	$\left[\frac{n}{3}\right] \cdot \left[\frac{s-n}{k-3}\right]$
4	$\left[\frac{n}{4}\right] \cdot \left[\frac{s-n}{k-4}\right]$
5	$\left[\frac{n}{5}\right] \cdot \left[\frac{s-n}{k-5}\right]$
$\vdots$	$\vdots$
in general λ	$\left[\frac{n}{\lambda}\right] \cdot \left[\frac{s-n}{k-\lambda}\right]$

which formula must be continued until they vanish; and hence another proof the theorems mentioned above and especially of the general theorem given in § 32 immediately follows, which we want to develop here.

Proof of the Theorem stated in § 32

§53 If we continue the number of the cases assigned in the table of the superior paragraph until they vanish and collect them all into one sum, the number of completely all cases will arise, in which either none of the marked slips of paper will be found among the extracted ones or only one or two or three or four etc. till the end; therefore, the sum will have to be equal to the number of all varieties, which can occur in k extracted slips of papers, which number we saw to be $= \left[\frac{s}{k}\right]$; hence, if we put

$$S = \left[\frac{n}{0}\right] \cdot \left[\frac{s-n}{k}\right] + \left[\frac{n}{1}\right] \cdot \left[\frac{s-n}{k-1}\right] + \left[\frac{n}{2}\right] \cdot \left[\frac{s-n}{k-2}\right] + \left[\frac{n}{3}\right] \cdot s - nk - 3 + \text{etc.},$$

it will be

$$S = \left[\frac{s}{k}\right].$$

§54 This series certainly deviates from that one which is treated in the theorem, but can easily be reduced to this form by means of our Lemma 1 [§ 21], by which it was

$$\left[\frac{p}{q}\right] = \left[\frac{p}{p-q}\right].$$

For, having done this reduction the superior series will obtain the following form

$$S = \left[\frac{n}{0}\right] \cdot \left[\frac{s-n}{s-n-k}\right] + \left[\frac{n}{1}\right] \cdot \left[\frac{s-n}{s-n-k+1}\right] + \left[\frac{n}{2}\right] \cdot \left[\frac{s-n}{s-n-k+2}\right]$$
$$+ \left[\frac{n}{3}\right] \cdot \left[\frac{s-n}{s-n-k+3}\right] + \text{etc.},$$

which sum will therefore be

$$S = \left[\frac{s}{k}\right]$$

or even

$$S = \left[\frac{s}{s-k}\right].$$

§55 Now on the other hand the series summed in the superior Theorem was this one

$$S = \left[\frac{n}{0}\right] \cdot \left[\frac{p}{q}\right] + \left[\frac{n}{1}\right] \cdot \left[\frac{p}{q+1}\right] + \left[\frac{n}{2}\right] \cdot \left[\frac{p}{q+2}\right] + \left[\frac{n}{3}\right] \cdot \left[\frac{p}{q+3}\right] + \text{etc.},$$

to which form we will reduce the series found here, if we set $s - n = p$ and $s - n - k = q$, whence the letters s and k are determined in such a way that it is $s = p + n$ and $k = p - q$; therefore, by means of the things we explained here, the sum of the propounded series will be

$$S = \left[\frac{p+n}{p-q}\right]$$

or even

$$S = \left[\frac{p+n}{q+n}\right],$$

which same sum we assigned to this series in the superior Theorem.