On the transformation of the divergent series $1 - mx + m(m+n)x^2 - m(m+n)(m+2n)x^3 +$ etc. into a continued fraction *

Leonhard Euler

§1 After I had once investigated the nature of divergent series of this kind and had assigned the true sum of the hypergeometric series

$$1 - 1 + 2 - 6 + 24 - 120 + 720 - \text{etc.}$$

by means of a transformation into a continued fraction, I also mentioned this series extending a lot further

$$\begin{gathered} 1 - mx + m(m+n)x^2 - m(m+n)(m+2n)x^3 \\ + m(m+n)(m+2n)(m+3n)x^4 - \text{etc.}, \end{gathered}$$

whose sum I had found to be equal to this continued fraction

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$$\cfrac{1}{1+\cfrac{mx}{1+\cfrac{nx}{1+\cfrac{(m+n)x}{1+\cfrac{2nx}{1+\cfrac{(m+2n)x}{1+\text{etc.}}}}}}}$$

the truth of which statement I had deduced from the conversion of the Riccati equation into a continued fraction. But since this proof can seem to be too non straight-forward, I will give the same reduction from simpler principles here.

§2 But first it will be convenient to contract that general series into a more pleasant form by putting

$$mx = a \quad \text{and} \quad nx = b,$$

that this infinite series is propounded

$$1 - a + a(a+b) - a(a+b)(a+2b) + a(a+b)(a+2b)(a+3b) - \text{etc.}$$

But furthermore, that the following resolutions can be done more conveniently and not that long calculations are necessary, I set as follows

$$a = A, \quad a+b = B, \quad a+2b = C, \quad a+3b = D \quad \text{etc.}$$

and so one will have this series

$$1 - A + AB - ABC + ABCD - \text{etc.},$$

the sum in question of which we want to denote by the letter S such that

$$S = 1 - A + AB - ABC + ABCD - \text{etc.};$$

and hence further,

$$\frac{1}{S} = \frac{1}{1 - A + AB - ABC + ABCD - \text{etc.}}$$

§**3** Therefore, since $\frac{1}{S} > 1$, reduce the last equation to this form

$$\frac{1}{S} = 1 + \frac{A - AB + ABC - ABCD + \text{etc.}}{1 - A + AB - ABC + ABCD - \text{etc.}}.$$

But now let us put $\frac{1}{S} = 1 + \frac{1}{P}$ and it will be

$$P = \frac{1 - A + AB - ABC + ABCD - \text{etc.}}{1 - B + BC - BCD + BCDE - \text{etc.}};$$

since this expression exceeds 1 again, because of

$$B - A = b, \quad C - A = 2b, \quad D - A = 3b \quad \text{etc.},$$

it will give

$$P = 1 + \frac{b - 2bB + 3bBC - 4bBCD + \text{etc.}}{1 - B + BC - BCD + BCDE - \text{etc.}}.$$

Therefore, put $P = 1 + \frac{b}{Q}$ and it will be

$$Q = \frac{1 - B + BC - BCD + BCDE - \text{etc.}}{1 - 2B + 3BC - 4BCD + \text{etc.}},$$

whence we deduce

$$Q = 1 + \frac{B - 2BC + 3BCD - 4BCDE + \text{etc.}}{1 - 2B + 3BC - 4BCD + \text{etc.}}.$$

For this reason let us now put $Q = 1 + \frac{B}{R}$ and it will result

$$R = \frac{1 - 2B + 3BC - 4BCD + \text{etc.}}{1 - 2C + 3CD - 4CDE + \text{etc.}}$$

§**4** Therefore, here so in the numerator as in the denominator the same coefficients occur, but the capital letters in the denominator are moved forward by one step. Therefore, since

$$C - B = b, \quad D - B = 2b, \quad E - B = 3b \quad \text{etc.},$$

it will be

$$R = 1 + \frac{2b - 2 \cdot 3bC + 3 \cdot 4bCD - 4 \cdot 5CDE + \text{etc.}}{1 - 2C + 3CD - 4CDE + 5CDEF - \text{etc.}}.$$

Therefore, if we put $R = 1 + \frac{2b}{S}$, it will be

$$S = \frac{1 - 2C + 3CD - 4CDE + \text{etc.}}{1 - 3C + 6CD - 10CDE + \text{etc.}},$$

where in the denominator all triangular numbers occur, of course; that expression is reduced to this one

$$S = 1 + \frac{C - 3CD + 6CDE - 10CDEF + \text{etc.}}{1 - 3C + 6CD - 10CDE + \text{etc.}}$$

Therefore, if we set $S = 1 + \frac{C}{T}$, it will be

$$T = \frac{1 - 3C + 6CD - 10CDE + 15CDEF - \text{etc.}}{1 - 3D + 6DE - 10DEF + 15DEFG - \text{etc.}}$$

§**5** This form, because of

$$D - C = b, \quad E - C = 2b, \quad F - C = 3b \quad \text{etc.},$$

goes over into this one

$$T = 1 + \frac{3b - 2 \cdot 6bD + 3 \cdot 10bDE - 4 \cdot 15bDEF + \text{etc.}}{1 - 3D + 6DE - 10DEF + 15DEFG - \text{etc.}}.$$

Let us put $T = 1 + \frac{3b}{U}$, that

$$U = \frac{1 - 3D + 6DE - 10DEF + 15DEFG - \text{etc.}}{1 - 4D + 10DE - 20DEF + 35DEFG - \text{etc.}},$$

where in the denominator one finds the first pyramidal numbers or the sum of the triangular numbers, and hence we obtain

$$U = 1 + \frac{D - 4DE + 10DEF - 20DEFG + \text{etc.}}{1 - 4D + 10DE - 20DEF + 35DEFG - \text{etc.}},$$

where now in the numerator and the denominator the pyramidal numbers occur. Further, set

$$U = 1 + \frac{D}{V}$$

and it will be

$$V = \frac{1 - 4D + 10DE - 20DEF + 35DEFG - \text{etc.}}{1 - 4E + 10EF - 20EFG + 35EFGH - \text{etc.}}.$$

§6 Hence doing the calculation as above, since

$$E - D = b, \quad F - D = 2b, \quad G - D = 3b \quad \text{etc.},$$

it will be

$$V = 1 + \frac{4b - 2 \cdot 10bE + 3 \cdot 20bEF - 4 \cdot 35bEFG + \text{etc.}}{1 - 4E + 10EF - 20EFG + 35EFGH - \text{etc.}}.$$

Let $V = 1 + \frac{4b}{X}$ that

$$X = \frac{1 - 4E + 10EF - 20EFG + 35EFGH - \text{etc.}}{1 - 5E + 15EF - 35EFG + 70EFGH - \text{etc.}};$$

that expression is reduced to this one

$$X = 1 + \frac{E - 5EF + 15EFG - 35EFGH + \text{etc.}}{1 - 5E + 15EF - 35EFG + \text{etc.}}.$$

Let $X = 1 + \frac{E}{Y}$ and it will be

$$Y = \frac{1 - 5E + 15EF - 35EFG + 70EFGH - \text{etc.}}{1 - 5F + 15FG - 35FGH + 70FGHI - \text{etc.}}.$$

§7 Therefore, since

$$F - E = b, \quad G - E = 2b, \quad H - E = 3b \quad \text{etc.},$$

it will be

$$Y = 1 + \frac{5b - 2 \cdot 15bF + 3 \cdot 35bFG - 4 \cdot 70FGH + \text{etc.}}{1 - 5F + 15FG - 35FGH + 70FGHI - \text{etc.}}.$$

Now let $Y = 1 + \frac{5b}{Z}$ that

$$Z = \frac{1 - 5F + 15FG - 35FGH + 70FGHI - \text{etc.}}{1 - 6F + 21FG - 56FGH + 126FGHI - \text{etc.}}.$$

Therefore, since we initially set $\frac{1}{S} = 1 + \frac{A}{P}$, the sum in question will be

$$S = \cfrac{1}{1 + \cfrac{A}{P}};$$

but then the following substitutions were made

$$P = 1 + \frac{b}{Q}, \qquad S = 1 + \frac{C}{T}, \qquad V = 1 + \frac{4b}{X},$$

$$Q = 1 + \frac{B}{R}, \qquad T = 1 + \frac{3b}{U}, \qquad X = 1 + \frac{E}{Y},$$

$$R = 1 + \frac{2b}{S}, \qquad U = 1 + \frac{D}{V}, \qquad Y = 1 + \frac{5b}{Z}$$

etc.;

having substituted these values in order this continued fraction will result:

$$S = \cfrac{1}{1 + \cfrac{A}{1 + \cfrac{b}{1 + \cfrac{B}{1 + \cfrac{2b}{1 + \cfrac{C}{1 + \cfrac{3b}{1 + \cfrac{D}{1 + \cfrac{4b}{1 + \text{etc.}}}}}}}}}}$$

Therefore, if we substitute the assumed values for the letters A, B, C, D etc. again, that we have this divergent series

$$1 - a + a(a+b) - a(a+b)(a+2b) + a(a+b)(a+2b)(a+3b) - \text{etc.},$$

its sum will be expressed by the following continued fraction

$$S = \cfrac{1}{1+\cfrac{a}{1+\cfrac{b}{1+\cfrac{a+b}{1+\cfrac{2b}{1+\cfrac{a+2b}{1+\cfrac{3b}{1+\cfrac{a+3b}{1+\cfrac{4b}{1+\text{etc.}}}}}}}}}}$$

which is the same form I had once given.

§**8** This transformation is even more remarkable since it opens a very safe and maybe unique way to determine the value of the series approximately. For, if in usual manner the continued fraction is resolved into simple fractions

$$1, \quad \frac{1}{1+a}, \quad \frac{1+b}{1+a+b} \quad \text{etc.},$$

they alternately are greater and smaller than the value of the divergent series and come continuously closer to that value. But then I even explained extraordinary artifices which lead to the true value a lot faster.

§**9** Furthermore, it will indeed be helpful to have noted that such a continued fraction

$$1+\cfrac{\alpha}{1+\cfrac{\beta}{1+\cfrac{\gamma}{1+\cfrac{\delta}{1+\text{etc.}}}}}$$

can in general be reduced to half the number of parts in convenient manner. For, having put its value $= S$, it will possible to represent it that way

$$S = 1 + \cfrac{\alpha}{1 + \cfrac{\beta}{P}}, \quad P = 1 + \cfrac{\gamma}{1 + \cfrac{\delta}{Q}}, \quad Q = 1 + \cfrac{\varepsilon}{1 + \cfrac{\zeta}{R}} \quad \text{etc.}$$

In the first of these formulas it will be

$$S = 1 + \frac{\alpha P}{P + \beta} = 1 + \alpha - \frac{\alpha\beta}{\beta + P},$$

further, the second formula gives

$$P = 1 + \frac{\gamma Q}{Q + \delta} = 1 + \gamma - \frac{\gamma\delta}{\delta + Q},$$

the same way the third yields

$$Q = 1 + \frac{\varepsilon R}{R + \zeta} = 1 + \varepsilon - \frac{\varepsilon\zeta}{\zeta + R}$$

etc.

Therefore, these values substituted successively will produce this new continued fraction

$$S = 1 + \alpha - \cfrac{\alpha\beta}{1 + \beta + \gamma - \cfrac{\gamma\delta}{1 + \delta + \varepsilon - \cfrac{\varepsilon\zeta}{1 + \zeta + \eta - \cfrac{\eta\theta}{1 + \theta + \iota - \text{etc.}}}}}$$

§10 Therefore, since in our case the divergent series

$$S = 1 - a + a(a+b) - a(a+b)(a+2b) + a(a+b)(a+2b)(a+3b) - \text{etc.}$$

is reduced to this continued fraction

$$S=\cfrac{1}{1+\cfrac{a}{1+\cfrac{b}{1+\cfrac{a+b}{1+\cfrac{2b}{1+\cfrac{a+2b}{1+\cfrac{3b}{1+\cfrac{a+3b}{1+\text{etc.}}}}}}}}}$$

let us take the following here

$$\alpha=a,\quad \beta=b,\quad \gamma=a+b,\quad \delta=2b,\quad \varepsilon=a+2b\quad \text{etc.}$$

and it will be

$$S=1+a-\cfrac{ab}{1+a+2b-\cfrac{2b(a+b)}{1+a+4b-\cfrac{3b(a+2b)}{1+a+6b-\cfrac{4b(a+3b)}{1+a+\text{etc.}}}}}$$

APPENDIX

ON THE BROUNCKERIAN CONTINUED FRACTION

§11 After I had one been occupied a lot by reconstructing the analysis which led Brouncker to this extraordinary continued fraction, since it seemed improbable to me that he was led there in such a unintutive way as Wallis described it, I think to have shown sufficiently convincingly eventually that Brouncker deduced this form from the Leibnizian series

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\text{etc.},$$

which had already been found by Gregory before, rather than from the interpolation of the series

$$1, \quad \frac{1}{2}, \quad \frac{1\cdot 3}{2\cdot 4}, \quad \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}, \quad \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8} \quad \text{etc.},$$

as Wallis did it, since the consideration of that Leibnizian series leads to Brounckerian form by sufficiently plain reasoning.

§**12** But this observation seems to be worth of even greater attention, after Daniel Bernoulli had not missed to renew the memory of the Brounckerian form. Therefore, since not that long ago I explained a method to derive that form from the series

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\text{etc.},$$

I think it will not be inappropiate to the Geometers, if I present a method by means of which the Brounckerian formula can vice versa be reduced to the Leibnizian series.

§**13** Therefore, I will consider that continued fraction as if its value is non known yet, setting

$$S=\cfrac{1}{1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\cfrac{81}{2+\text{etc.}}}}}}}$$

which I represent part by part in the following way:

$$S=\cfrac{1}{1+\cfrac{1}{-1+P}}, \quad P=3+\frac{9}{-3+Q}, \quad Q=5+\frac{25}{-5+R}, \quad R=7+\frac{49}{-7+S} \quad \text{etc.}$$

§**14** Therefore, let us expand each of these parts separately; and the first, reduced to a simple fraction, yields

$$S = \frac{P-1}{P}$$

and hence $S = 1 - \dfrac{1}{P}$, the second on the other hand will be $\frac{3Q}{Q-3}$, whence

$$\frac{1}{P} = \frac{Q-3}{3Q} \quad \text{or} \quad \frac{1}{P} = \frac{1}{3} - \frac{1}{Q};$$

in like manner, the third part gives

$$Q = \frac{5R}{R-5}$$

and hence $\frac{1}{Q} = \frac{1}{5} - \frac{1}{R}$; in the same way from the following parts we will obtain

$$\frac{1}{R} = \frac{1}{7} - \frac{1}{S}, \quad \frac{1}{S} = \frac{1}{9} - \frac{1}{T} \quad \text{etc.}$$

Hence if those value are substituted successively, we will obtain this expression

$$S = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \text{etc.},$$

such that we are certain that $S = \frac{\pi}{4}$.

§**15** In like manner, it will even be possible to find the value of other continued fractions of this kind. As if, e.g., this form was propounded

$$S = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{4}{1 + \cfrac{9}{1 + \cfrac{16}{1 + \text{etc.}}}}}}$$

distribute it into parts in the following way

$$S = \cfrac{1}{1 + \cfrac{1}{-1+P}}, \quad P = 2 + \frac{4}{-2+Q}, \quad Q = 3 + \frac{9}{-3+R}, \quad R = 4 + \frac{16}{-4+S} \quad \text{etc.};$$

For, having expanded each of these parts one will find

$$S = 1 - \frac{1}{P}, \quad \frac{1}{P} = \frac{1}{2} - \frac{1}{Q}, \quad \frac{1}{Q} = \frac{1}{3} - \frac{1}{R}, \quad \frac{1}{R} = \frac{1}{4} - \frac{1}{S} \quad \text{etc.},$$

whence it follows that it will be

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \text{etc.} = \log 2.$$

Therefore, this method seems to promise a lot of nice applications in the future.