On Wallis' continued fractions *

Leonhard Euler

§1 After Brouncker had found his memorable continued fraction for the quadrature of the circle and communicated it with Wallis without a proof, the latter mostly spent his eagerness on that that he detects the source from which Brouncker derived this extraordinary formula. But he (Wallis) believed that he (Brouncker) had used that astounding formulas which he (Wallis) found in his work *Arithmetica infinitorum*. Yes, hence by rather non straightforward calculations he not only found the Brounckerian continued fraction but additionally rescued many other similar ones, which, as the Brounckerian expression, are to considered worth one's attention, from oblivion.

§2 But what from Wallis' *Arithmetica infinitorum*, written long before the invention of the Analysis of the Infinite, extends to this, can now in customary manner be represented in such a way that, having extended the integral formulas from the limit $x = 0$ to $x = 1$, the following quadratures are exhibited:

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$$\int \frac{x\partial x}{\sqrt{1-xx}} = 1 = 1,$$

$$\int \frac{x^3\partial x}{\sqrt{1-xx}} = \frac{2}{3} = \frac{2\cdot 2}{2\cdot 3},$$

$$\int \frac{x^5\partial x}{\sqrt{1-xx}} = \frac{2\cdot 4}{3\cdot 5} = \frac{2\cdot 2\cdot 4\cdot 4}{2\cdot 3\cdot 4\cdot 5},$$

$$\int \frac{x^7\partial x}{\sqrt{1-xx}} = \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7},$$

$$\int \frac{x^9\partial x}{\sqrt{1-xx}} = \frac{2\cdot 4\cdot 6\cdot 8}{3\cdot 5\cdot 7\cdot 9} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}$$

etc.

§**3** I arranged those formulas in the third column in such a way that the denominators obviously admit an interpolation; and so it just remains that also the numerators are transformed in such a way that they likewise allow an interpolation, what will happen, if such a series progressing according to an uniform law i.e. A, B, C, D, E, F etc. is investigated that

$$AB = 1\cdot 1, \quad BC = 2\cdot 2, \quad CD = 3\cdot 3, \quad DE = 4\cdot 4 \quad \text{etc.},$$

which is that in which Wallis revealed the highest ingenuity of his mind, but which investigation I will expedite a lot more generally and by a much easier calculation in the following.

§**4** But having found this series of the letters A, B, C, D etc. the whole task will be completely done. For, since, as the following table shows:

$$\int \frac{x\partial x}{\sqrt{1-xx}} = 1 = \frac{1}{A}\cdot\frac{A}{1},$$

$$\int \frac{x^3\partial x}{\sqrt{1-xx}} = \frac{BC}{2\cdot 3} = \frac{1}{A}\cdot\frac{ABC}{1\cdot 2\cdot 3},$$

$$\int \frac{x^5\partial x}{\sqrt{1-xx}} = \frac{BCDE}{2\cdot 3\cdot 4\cdot 5} = \frac{1}{A}\cdot\frac{ABCDE}{1\cdot 2\cdot 3\cdot 4\cdot 5},$$

$$\int \frac{x^7\partial x}{\sqrt{1-xx}} = \frac{BCDEFG}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} = \frac{1}{A}\cdot\frac{ABCDEFG}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}$$

etc.,

the interpolation gives us the following quadratures:

$$\int \frac{\partial x}{\sqrt{1-xx}} = \frac{1}{A}\cdot 1,$$

$$\int \frac{xx\partial x}{\sqrt{1-xx}} = \frac{1}{A}\cdot\frac{AB}{1\cdot 2},$$

$$\int \frac{x^4\partial x}{\sqrt{1-xx}} = \frac{1}{A}\cdot\frac{ABCD}{1\cdot 2\cdot 3\cdot 4},$$

$$\int \frac{x^6\partial x}{\sqrt{1-xx}} = \frac{1}{A}\cdot\frac{ABCDEF}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}$$

etc.

§5 Now, since

$$\int \frac{\partial x}{\sqrt{1-xx}} = \frac{\pi}{2}$$

while π denotes the circumference of the circle with diameter $= 1$, for the sake of brevity, let us write $q = \frac{\pi}{2}$, then all values of the letters A, B, C, D etc. will be expressed in terms if the quadrature of the circle in the following way:

		Differences
$A = \dfrac{1}{q}$	$= 0,636620$	
		$0,934176$
$B = q$	$= 1,570796$	
		$0,975683$
$C = \dfrac{4}{q}$	$= 2,546479$	
		$0,987813$
$D = \dfrac{9q}{4}$	$= 3,534292$	
		$0,992782$
$E = \dfrac{4 \cdot 16}{9q}$	$= 4,527074$	
		$0,995257$
$F = \dfrac{9 \cdot 25}{4 \cdot 16}q$	$= 5,522331$	
	etc.	

§6 Here I added the third column, which exhibits the numerical values of these letters that it becomes more clear how those numbers increase according to an uniform law, what would not have happened, if I had taken a wrong value for q. Having explained these things, I will give a much simpler method by which for each of these letters continued fractions can be found, and in the same step I will make this operation a lot more general, while I resolve the following problem.

Problem

To find a series of letters A, B, C, D etc. progressing according to an uniform law such that

$$AB = ff, \quad BC = (f+a)^2, \quad CD = (f+2a)^2 \quad etc.$$

SOLUTION

§7 Here it is plain immediately what kind of function A was of f, that such a function B must be of $f+a$, but then C of $f+2a$, D of $f+3a$ and so forth. If, having observed this law, we set

$$A = f - \frac{1}{2}a + \frac{\frac{1}{2}s}{A'},$$

one will have to put

$$B = f + \frac{1}{2}a + \frac{\frac{1}{2}s}{B'},$$

where the letters A' and B' have the same ratio to each other such that B' originates from A', if one writes $f+a$ instead of f. Therefore, since after having removed the fractions

$$2A = 2f - a + \frac{s}{A'} \quad \text{and} \quad 2B = 2f + a + \frac{s}{B'},$$

the product of these formula is to be put equal to $4ff$, whence this equation, already freed from fractions, results:

$$aaA'B' - A's(2f-a) - B's(2f+a) - ss = 0.$$

Therefore, let us take $s = aa$ that the equation, divided by aa, is

$$A'B' - A'(2f-a) - B'(2f+a) = aa,$$

which can be expressed conveniently in terms of factors this way:

$$(A' - 2f - a)(B' + 2f + a) = 4ff.$$

§8 Now since, if both letters A' and B' would be equal, from the left-hand side it would be $A' = B' = 4f$, following the law mentioned above let us set:

$$A' = 4f - 2a + \frac{s'}{A''}$$

and

$$B' = 4f + 2a + \frac{s'}{B''},$$

having substituted which the last equation will take this form:

$$\left(2f - 3a + \frac{s'}{A''}\right)\left(2f + 3a + \frac{s'}{B''}\right) = 4ff.$$

Therefore, after the expansion and having got rid of the fractions the following equation will result:

$$9aaA''B'' - A''s'(2f - 3a) - B''s'(2f + 3a) - s's' = 0.$$

Therefore, take $s' = 9aa$ here that one has this equation:

$$A''B'' - A''(2f - 3a) - B''(2f + 3a) = 9aa,$$

which again can be represented in terms of factors this way:

$$(A'' - 2f - 3a)(B'' - 2f + 3a) = 4ff.$$

§**9** Since now again the intermediate value between A'' and B'' is $4f$, let us further set

$$A'' = 4f - 2a + \frac{s''}{A'''} \quad \text{and} \quad B'' = 4f + 2a + \frac{s''}{B'''},$$

and after the substitution this equation will emerge:

$$\left(2f - 5a + \frac{s''}{A'''}\right)\left(2f + 5a + \frac{s''}{B'''}\right) = 4ff.$$

Therefore, having done the expansion and having got rid of the fractions it will be

$$25aaA'''B''' - A'''s''(2f - 5a) - B'''s''(2f + 5a) - s''s'' = 0.$$

Set $s'' = 25aa$, and that equation will take on this form:

$$A'''B''' - A'''(2f - 5a) - B'''(2f + 5a) = 25aa,$$

which can be represented in terms of factors this way:

$$(A''' - 2f - 5a)(B''' - 2f + 5a) = 4ff.$$

§10 Again, as before, set

$$A''' = 4f - 2a + \frac{s'''}{A^{\mathrm{IV}}} \quad \text{and} \quad B''' = 4f + 2a + \frac{s'''}{B^{\mathrm{IV}}}$$

and after the substitution it will be

$$\left(2f - 7a + \frac{s'''}{A^{\mathrm{IV}}}\right)\left(2f + 7a + \frac{s'''}{A^{\mathrm{IV}}}\right) = 4ff,$$

having expanded and ordered which equation one obtains

$$A^{\mathrm{IV}}B^{\mathrm{IV}} - A^{\mathrm{IV}}(2f - 7a) - B^{\mathrm{IV}}(2f + 7a) = 49aa,$$

where we put $s''' = 49aa$; but then in terms of factors it will be

$$(A^{\mathrm{IV}} - 2f - 7a)(B^{\mathrm{IV}} - 2f + 7a) = 4ff.$$

Hence it is perspicuous how these operations are to be continued.

§11 Therefore, collecting them, because of

$$s = aa, \quad s' = 9aa, \quad s'' = 25aa, \quad s''' = 49aa \quad \text{etc.}$$

for $2A$ we will obtain the following continued fraction:

$$2A = 2f - a + \cfrac{aa}{4f - 2a + \cfrac{9aa}{4f - 2a + \cfrac{25aa}{4f - 2a + \cfrac{49aa}{4f - 2a + \text{etc.}}}}}$$

where, if instead of f in order we write $f + a$, $f + 2a$, $f + 3a$ etc., similar continued fractions for $2B$, $2C$, $2D$ etc. will result, which will look as follows:

$$2B = 2f + a + \cfrac{aa}{4f + 2a + \cfrac{9aa}{4f + 2a + \cfrac{25aa}{4f + 2a + \cfrac{49aa}{4f + 2a + \text{etc.}}}}}$$

$$2C = 2f + 3a + \cfrac{aa}{4f + 6a + \cfrac{9aa}{4f + 6a + \cfrac{25aa}{4f + 6a + \cfrac{49aa}{4f + 6a + \text{etc.}}}}}$$

$$2D = 2f + 5a + \cfrac{aa}{4f + 10a + \cfrac{9aa}{4f + 10a + \cfrac{25aa}{4 + 10a + \cfrac{49aa}{4f + 10a + \text{etc.}}}}}$$

etc.

§12 If we now put $f = 1$ and $a = 1$ here, the cases treated by Wallis will result, whence the continued fractions found by Wallis together with its values expressed in terms of the quadrature of the circle will be the following:

$$2A = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \text{etc.}}}}} = \frac{2}{q} = \frac{4}{\pi},$$

$$2B = 3 + \cfrac{1}{6 + \cfrac{9}{6 + \cfrac{25}{6 + \cfrac{49}{6 + \text{etc.}}}}} = 2q = \pi,$$

$$2C = 5 + \cfrac{1}{10 + \cfrac{9}{10 + \cfrac{25}{10 + \cfrac{49}{10 + \text{etc.}}}}} = \frac{8}{q} = \frac{16}{\pi},$$

$$2D = 7 + \cfrac{1}{14 + \cfrac{9}{14 + \cfrac{25}{14 + \cfrac{49}{14 + \text{etc.}}}}} = \frac{9q}{2} = \frac{9\pi}{4},$$

$$2E = 9 + \cfrac{1}{18 + \cfrac{9}{18 + \cfrac{25}{18 + \cfrac{49}{18 + \text{etc.}}}}} = \frac{128}{9q} = \frac{256}{9\pi},$$

the first of which is the continued fraction found by Brouncker.

§**13** But it is not probable at all that Brouncker got to its formula in such an non straight-forward way; I believe that he rather had derived it from the consideration of this very well-known series

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\text{etc.}=\frac{\pi}{4},$$

- which is usually attributed to Leibniz, but had been found much earlier by Jacob Gregory, from whom Brouncker could have known it -, which was possible by sufficiently easy and obvious operations in the following way:

Having put	it will be
$\frac{\pi}{4}=1-\alpha$	$\frac{4}{\pi}=\frac{1}{1-\alpha}=1+\frac{\alpha}{1-\alpha}=1+\frac{1}{-1+\frac{1}{\alpha}}$
$\alpha=\frac{1}{3}-\beta$	$\frac{1}{\alpha}=\frac{3}{1-3\beta}=3+\frac{9\beta}{1-3\beta}=3+\frac{9}{-3+\frac{1}{\beta}}$
$\beta=\frac{1}{5}-\gamma$	$\frac{1}{\beta}=\frac{5}{1-5\gamma}=5+\frac{25\gamma}{1-5\gamma}=5+\frac{25}{-7+\frac{1}{\gamma}}$
$\gamma=\frac{1}{7}-\delta$	$\frac{1}{\gamma}=\frac{7}{1-7\delta}=7+\frac{49\delta}{1-7\delta}=7+\frac{49}{-7+\frac{1}{\delta}}$
etc.	etc.

If now here for $\frac{1}{\alpha}$, $\frac{1}{\beta}$, $\frac{1}{\gamma}$ etc. the values just found are substituted, eventually Brouncker's continued fraction shows itself, since hence follows that it will be

$$\frac{4}{\pi}=1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\text{etc.}}}}}$$

§**14** But concerning our general solution of the problem, it even allows to express the value of each continued fraction by certain quadratures, what we will show in the following problem.

PROBLEM

Having propounded the series A, B, C, D etc. progressing according to an uniform law such that

$$AB = ff, \quad BC = (f+a)^2, \quad CD = (f+2a)^2 \quad etc.,$$

to investigate the value of each of these letters first expressed as infinite products, but then in terms of integral formulas.

SOLUTION

§15 Therefore, since

$$A = \frac{ff}{B}, \quad B = \frac{(f+a)^2}{C}, \quad C = \frac{(f+2a)^2}{D} \quad \text{etc.},$$

having substituted theses values continuously one will find

$$A = \frac{ff(f+2a)^2(f+4a)^2(f+6a)^2 \cdot \text{etc.}}{(f+a)^2(f+3a)^2(f+5a)^2 \cdot \text{etc.}}$$

to infinity. But since this way no definite value results, since, wherever it is terminated, either in the numerators or the denominators one factor is redundant, this inconvenience will be avoided, if we arrange the simple factors in the following way:

$$A = f \cdot \frac{f(f+2a)}{(f+a)(f+a)} \cdot \frac{(f+2a)(f+4a)}{(f+3a)(f+3a)} \cdot \frac{(f+4a)(f+6a)}{(f+5a)(f+5a)} \cdot \text{etc.}$$

For, this way the factors will get continuously closer to 1 and at infinity will become equal to it, and so that expression will certainly have a definite value.

§16 But to show how its value has to be reduced to integral formulas, let us recall this lemma:

Having extended the integrals from $x = 0$ to $x = 1$ it will be

$$\int \frac{x^{m-1}\partial x}{\sqrt[n]{(1-x^n)^{n-k}}} = \frac{m+k}{m} \cdot \frac{m+k+n}{m+n} \cdot \frac{m+k+2n}{m+2n} \cdot \frac{m+k+3n}{m+3n}$$

$$\cdot \frac{m+k+4n}{m+4n} \cdots\cdots \int \frac{x^\infty \partial x}{\sqrt[n]{(1-x^n)^{n-k}}}.$$

To accommodate this lemma to our cases, since in our terms each factor increases by $2a$, one has to set $n = 2a$; but then having taken $m = f$ and $k = a$ we will have

$$\int \frac{x^{f-1}\partial x}{\sqrt{1-x^{2a}}} = \frac{f+a}{f} \cdot \frac{f+3a}{f+2a} \cdot \frac{f+5a}{f+4a} \cdots\cdots \int \frac{x^\infty \partial x}{\sqrt{1-x^{2a}}},$$

which expression, inverted, yields the first factors of each term. For the others let us take $m = f + a$ while still $k = a$, and having done so it will be

$$\int \frac{x^{f+a-1}\partial x}{\sqrt{1-x^{2a}}} = \frac{f+2a}{f+a} \cdot \frac{f+4a}{f+3a} \cdot \frac{f+6a}{f+5a} \cdots\cdots \int \frac{x^\infty \partial x}{\sqrt{1-x^{2a}}}.$$

§**17** Now it is evident that the second formula divided by the first exhibits our infinite product, and this way the infinitesimal integrals cancel each other, as a logical consequence, we have

$$A = \int \frac{x^{f+a-1}\partial x}{\sqrt{1-x^{2a}}} : \int \frac{x^{f-1}\partial x}{\sqrt{1-x^{2a}}}.$$

In like manner, immediately

$$A = \int \frac{x^{f+2a-1}\partial x}{\sqrt{1-x^{2a}}} : \int \frac{x^{f+a-1}\partial x}{\sqrt{1-x^{2a}}}$$

$$B = \int \frac{x^{f+3a-1}\partial x}{\sqrt{1-x^{2a}}} : \int \frac{x^{f+2a-1}\partial x}{\sqrt{1-x^{2a}}}$$

etc.

But this investigation can indeed be generalised even more, as the following problem will teach.

More general Problem

To find a series A, B, C, D etc. progressing according to an uniform law such that

$$AB = ff + c, \quad BC = (f+a)^2 + c, \quad CD = (f+2a)^2 + c,$$

$$DE = (f + 3a)^2 + c \quad etc.,$$

where in each of the products the letter f is increased by the quantity a.

First Solution by continued Fractions

§**18** Here it is evident again, what a function A was of f, that such a function B must be of $f + a$, C of $f + 2a$, D of $f + 3a$ and so forth. Therefore, since $AB = ff + c$, if A and B would be equal, having omitted c it would be $A = B = f$. Therefore, by what amount A is taken smaller than f, by that amount B has to greater; hence having put $A = f - x$ it will be $B = f + x$. But since B results from A, if one writes $f + a$ instead of f, it must also be $B = f + a - x$, whence we conclude that $x = \frac{1}{2}a$; and so the principal parts for A and B will be

$$A = f - \frac{1}{2}a \quad \text{and} \quad B = f + \frac{1}{2}a$$

or

$$2A = 2f - a \quad \text{and} \quad 2B = 2f + a$$

and hence for the following

$$2C = 2f + 3a, \quad 2D = 2f + 5a, \quad 2E = 2f + 7a \quad \text{etc.}$$

§**19** Having found the principal values let us put that actually

$$2A = 2f - a + \frac{s}{A'} \quad \text{and} \quad 2B = 2f + a + \frac{s}{B'}.$$

But for s an appropriate value will emerge soon. Therefore, hence it will be

$$4AB = 4ff - aa + \frac{s}{A'}(2f + a) + \frac{s}{B'}(2f - a) + \frac{ss}{A'B'} = 4ff + 4c,$$

which equation, having got rid of the fractions, takes this form:

$$A'B'(aa + 4c) - A's(2f - a) - B's(2f + a) - ss = 0.$$

Now let us take $s = aa + 4c$, and after a division it will be

$$A'B' - A'(2f - a) - B'(2f + a) = aa + 4c,$$

which equation we want to represent this way in terms of factors:

$$(A' - 2f - a)(B' - 2f + a) = 4ff + 4c.$$

§20 Now reasoning the same way as before it is understood, if A' and B' were equal, that then the left-hand side will be

$$A'A' - 4fA' = 0 \quad \text{and hence} \quad A' = B' = 4f.$$

But since B' must result from A', if one writes $f + a$ instead of f, it is evident that the principal parts will be

$$A' = 4f - 2a \quad \text{and} \quad B' = 4f - 2a.$$

Therefore, let us put that actually

$$A' = 4f - 2a + \frac{s'}{A''} \quad \text{and} \quad B' = 4f + 2a + \frac{s'}{B''},$$

whence, if these values are substituted, the preceding equation exhibited in terms of factors will take this form:

$$\left(2f - 3a + \frac{s'}{A''}\right)\left(2f + 3a + \frac{s'}{B''}\right) = 4ff + 4c,$$

which after the expansion leads to this equation:

$$(4ff - 9aa) + \frac{s'}{A''}(2f + 3a) + \frac{s'}{B''}(2f - 3a) + \frac{s's'}{A''B''} = 4ff + 4c,$$

and having carried away the fractions it goes over into this one:

$$A''B''(9aa + 4c) - A''s'(2f - 3a) - B''s'(2f + 3a) - s's' = 0.$$

Therefore, having taken $s' = 9aa + 4c$ and after a division this equation results:

$$A''B'' - A''(2f - 3a) - B''(2f + 3a) = 9aa + 4c,$$

which can be represented via factors this way:

$$(A'' - 2f - 3a)(B'' - 2f + 3a) = 4ff + 4c.$$

§**21** Since this equation is similar to the preceding and for the case $A'' = B''$ again $A'' = B'' = 4f$ would result, further set

$$A'' = 4f - 2a + \frac{s''}{A'''} \quad \text{and} \quad B'' = 4f + 2a + \frac{s''}{B'''},$$

whence the last equation via factors would be

$$\left(2f - 5a + \frac{s''}{A'''}\right)\left(2f + 5a + \frac{s''}{B'''}\right) = 4ff + 4c.$$

But having done the expansion and carried away the fractions it results:

$$A'''B'''(25aa + 4c) - A'''s''(2f - 5a) - B'''s''(2f + 5a) - s''s'' = 0.$$

Therefore, taking $s''' = 25aa + 4c$ and dividing by s'' it will be

$$A'''B''' - A'''(2f - 5a) - B'''(2f + 5a) = 25aa + 4c$$

or as a product

$$(A''' - 2f - 5a)(B''' - 2f + 5a) = 4ff + 4c.$$

§**22** Further, set

$$A''' = 4f - 2a + \frac{s'''}{A^{\mathrm{IV}}} \quad \text{and} \quad B''' = 4f + 2a + \frac{s'''}{B^{\mathrm{IV}}},$$

and the above equation in terms of products having substituted these values will be

$$\left(2f - 7a + \frac{s'''}{A^{\mathrm{IV}}}\right)\left(2f + 7a + \frac{s'''}{B^{\mathrm{IV}}}\right) = 4ff + 4c,$$

which having repeated the same operations and taken $s''' = 49aa + 4c$ is reduced to the following form:

$$A^{\mathrm{IV}}B^{\mathrm{IV}} - A^{\mathrm{IV}}(2f - 7a) - B^{\mathrm{IV}}(2f + 7a) = 49aa + 4c,$$

or in terms of factors it will be

$$(A^{\mathrm{IV}} - 2f - 7a)(B^{\mathrm{IV}} - 2f + 7a) = 4ff + 4c.$$

From these it is abundantly clear how the calculation is to be continued.

§**23** Therefore, having successively substituted these values, because of

$$s = aa + 4c, \quad s' = 9aa + 4c, \quad s'' = 25aa + 4c, \quad s''' = 49aa + 4c \quad \text{etc.},$$

for A we will obtain the following continued fraction:

$$2A = 2f - a + \cfrac{aa + 4c}{4f - 2a + \cfrac{9aa + 4c}{4f - 2a + \cfrac{25aa + 4c}{4f - 2a + \cfrac{49aa + 4c}{4f - 2a + \text{etc.}}}}}$$

In like manner, it will hence be

$$2B = 2f + a + \cfrac{aa + 4c}{4f + 2a + \cfrac{9aa + 4c}{4f + 2a + \cfrac{25aa + 4c}{4f + 2a + \cfrac{49aa + 4c}{4f + 2a + \text{etc.}}}}}$$

$$2C = 2f + 3a + \cfrac{aa + 4c}{4f + 6a + \cfrac{9aa + 4c}{4f + 6a + \cfrac{25aa + 4c}{4f + 6a + \cfrac{49aa + 4c}{4f + 6a + \text{etc.}}}}}$$

$$2D = 2f + 5a + \cfrac{aa + 4c}{4f + 10a + \cfrac{9aa + 4c}{4f + 10a + \cfrac{25aa + 4c}{4f + 10a + \cfrac{49aa + 4c}{4f + 10a + \text{etc.}}}}}$$

etc.

Second Solution by infinite Products

§**24** Since

$$AB = ff + c, \quad BC = (f+a)^2 + c, \quad CD = (f+2a)^2 + c, \quad DE = (f+3a)^2 + c \quad \text{etc.},$$

it will be

$$A = \frac{(ff+c)((f+2a)^2+c)((f+4a)^2+c)((f+6a)^2+c)\text{etc.}}{((f+a)^2+c)((f+3a)^2+c)((f+5a)^2+c)\text{etc.}}$$

But indeed in this expression, wherever one is positioned, either in the numerator or in the denominator there will be one redundant factor. For this to become more clear, let us first stop at the letter F, and it will be

$$A = \frac{ff+c}{(f+a)^2+c} \cdot \frac{(f+2a)^2+c}{(f+3a)^2+c} \cdot ((f+4a)^2+c) \cdot \frac{1}{F}.$$

But whenever we stop at the following letter, G, it will be

$$A = \frac{ff+c}{(f+a)^2+c} \cdot \frac{(f+2a)^2+c}{(f+3a)^2} \cdot \frac{(f+4a)^2+c}{(f+5a)^2+c} \cdot G.$$

§25 Therefore, if these two expressions are continued to infinity and are multiplied by each other, the last literal factor, which is $\frac{G}{F}$ here, obviously becomes equal to 1. But since in this case the number of factors in the number is greater by one, let us write its first factor separately in front of it, and the product will be expressed in the following way:

$$A^2 = (ff+c) \cdot \frac{(ff+c)((f+2a)^2+c)}{((f+a)^2+c)((f+a)^2+c)} \cdot \frac{((f+2a)^2+c)((f+4a)^2+c)}{((f+3a)^2+c)((f+3a)^2+c)} \cdot \text{etc.},$$

where the infinitesimal factors will become equal to 1 and so that expression proceeds in an uniform way.

But here it will be convenient to distinguish two cases, depending on whether c was a negative or positive number.

Case 1 in which $c = -bb$

§26 In the first case each factor will admit to be resolved into two others. Therefore, let us first set $c = -bb$ in which case the continued fraction can be exhibited in the following way:

$$2A = 2f - a + \cfrac{(a+2b)(a-2b)}{4f - 2a + \cfrac{(3a+2b)(3a-2b)}{4f - 2a + \cfrac{(5a+2b)(5a-2b)}{4f - 2a + \cfrac{(7a+2b)(7a-2b)}{4f - 2a + \text{etc.}}}}}$$

and instead of the expression as infinite product we will now have the following for the simple letter A

$$A = (f-b) \cdot \frac{(f+b)(f+2a-b)}{(f+a+b)(f+a-b)} \cdot \frac{(f+2a+b)(f+4a-b)}{(f+3a+b)(f+3a-b)} \cdot \text{etc.},$$

in each term of which expression the sum of the factors of the numerator become equal to sum of the factors of the denominator; because of this property these factors can be expressed via an integral formula.

§**27** For, it is known, if this integral formula:

$$\int \frac{x^{m-1}\partial x}{\sqrt[n]{(1-x^n)^{n-k}}}$$

is extended from $x = 0$ to $x = 1$, that the value is reduced to the following infinite product:

$$\frac{m+k}{m} \cdot \frac{m+k+n}{m+n} \cdot \frac{m+k+2n}{m+2n} \cdot \cdots \cdot \int \frac{x^{\infty}\partial x}{\sqrt[n]{(1-x^n)^{n-k}}}.$$

Therefore, to accommodate this form to our expression, since each factor is increased by the quantity $2a$ in the following term, one has to take $n = 2a$; but then having put $m = f + b$ and $k = a$ one will find that it will be

$$\frac{f+a+b}{f+b} \cdot \frac{f+3a+b}{f+2a+b} \cdot \frac{f+5a+b}{f+4a+b} \cdot \cdots \cdot \int \frac{x^{\infty}\partial x}{\sqrt{1-x^{2a}}} = \int \frac{x^{f+b-1}\partial x}{\sqrt{1-x^{2a}}},$$

which expression, inverted, contains the first factors of each term. But for the second, while $n = 2a$, take $m = f + a - b$ and $k = a$, having done which this equation will results:

$$\frac{f+2a-b}{f+a-b}\cdot\frac{f+4a-b}{f+3a-b}\cdot\frac{f+6a-b}{f+5a-b}\cdot\dots\cdot\int\frac{x^{\infty}\partial x}{\sqrt{1-x^{2a}}}=\int\frac{x^{f+a-b-1}\partial x}{\sqrt{1-x^{2a}}}.$$

Therefore, if this equation is divided by the preceding, the last integral factors will cancel each other and an infinite value converging to the value A and expressed via integral formulas will result such that

$$A=(f-b)\cdot\int\frac{x^{f+a-b-1}\partial x}{\sqrt{1-x^{2a}}}:\int\frac{x^{f+b-1}\partial x}{\sqrt{1-x^{2a}}}.$$

§28 To illustrate these things in an example, let us take $f=2$, $a=1$, $b=1$ that we have these values:

$$AB=3,\quad BC=8,\quad CD=15,\quad DE=24\quad\text{etc.},$$

and in this case the continued fraction becomes

$$2A=3-\cfrac{3}{6+\cfrac{5}{6+\cfrac{21}{6+\cfrac{45}{6+\cfrac{77}{6+\text{etc.}}}}}}$$

But by an infinite product it will be

$$A=\frac{3\cdot 3}{2\cdot 4}\cdot\frac{5\cdot 5}{4\cdot 6}\cdot\frac{7\cdot 7}{6\cdot 8}\cdot\frac{9\cdot 9}{8\cdot 10}\cdot\text{etc.}$$

But then via integral formulas one will have

$$A=\int\frac{x\partial x}{\sqrt{1-xx}}:\int\frac{xx\partial x}{\sqrt{1-xx}}.$$

But it is known that for our limits of integration, from $x=0$ to $x=1$,

$$\int\frac{x\partial x}{\sqrt{1-xx}}=1\quad\text{and}\quad\int\frac{xx\partial x}{\sqrt{1-xx}}=\frac{\pi}{4},$$

whence one concludes $A=\frac{4}{\pi}$, what agrees with the Wallisian product, by which

$$\frac{\pi}{2} = \frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdot \frac{6 \cdot 6}{5 \cdot 5} \cdot \text{etc.},$$

extraordinarily.

CASE 2 IN WHICH $c = +bb$

§**29** Now let us also expand the other case $c = +bb$, for which the continued fraction takes this form:

$$2A = 2f - a + \cfrac{aa + 4bb}{4f - 2a + \cfrac{9aa + 4bb}{4f - 2a + \cfrac{25aa + 4bb}{4f - 2a + \cfrac{49aa + 4bb}{4f - 2a + \text{etc.}}}}}$$

But the infinite product on the other hand results from the preceding form writing $b\sqrt{-1}$ instead of b expressed via imaginary quantities this way:

$$A = (f - b\sqrt{-1}) \cdot \frac{(f + b\sqrt{-1})(f + 2a - b\sqrt{-1})}{(f + a + b\sqrt{-1})(f + a - b\sqrt{-1})} \cdot \frac{(f + 2a + b\sqrt{-1})(f + 4a - b\sqrt{-1})}{(f + 3a + b\sqrt{-1})(f + 3a - b\sqrt{-1})} \cdot \text{etc.}$$

But it is evident that in the same expression mentioned in § 26 one could also have written $-b\sqrt{-1}$ instead of b, whence it would have resulted:

$$A = (f + b\sqrt{-1}) \cdot \frac{(f - b\sqrt{-1})(f + 2a + b\sqrt{-1})}{(f + a - b\sqrt{-1})(f + a + b\sqrt{-1})} \cdot \frac{(f + 2a - b\sqrt{-1})(f + 4a + b\sqrt{-1})}{(f + 3a - b\sqrt{-1})(f + 3a + b\sqrt{-1})} \cdot \text{etc.}$$

Therefore, the product of these two expressions becomes real; for, it will be

$$A^2 = (ff + bb)\frac{(ff + bb)((f + 2a)^2 + bb)}{((f + a)^2 + bb)((f + a)^2 + bb)} \cdot \frac{((f + 2a)^2 + bb)((f + 4a)^2 + bb)}{((f + 3a)^2 + bb)((f + 3a)^2 + bb)} \cdot \text{etc.},$$

which expression agrees with the one above in § 25.

§30 But on the other hand the expression in terms of integral formulas becomes imaginary. For, if in the formulas of § 27 one writes $b\sqrt{-1}$ instead of b, the following expression will result:

$$A = (f - b\sqrt{-1}) \int \frac{x^{f+a-1+b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}} : \int \frac{x^{f-1+b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}}.$$

But having changed the signs of the imaginary signs it will be

$$A = (f + b\sqrt{-1}) \int \frac{x^{f+a-1-b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}} : \int \frac{x^{f-1-b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}},$$

where there is no doubt that in each of both expression the imaginary quantities cancel each other, even though there is no method to actually expand this mutual cancellation of the imaginary quantities.

§31 But if both of these expression are multiplied by each other, then this cancellation can easily be shown. For, since the product is

$$A^2 = (ff + bb) \frac{\int \frac{x^{f+a-1-b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}} \cdot \int \frac{x^{f+a-1+b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}}}{\int \frac{x^{f-1-b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}} \cdot \int \frac{x^{f-1+b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}}},$$

it can be shown that so in the numerator as in the denominator the imaginary quantities cancels, to have shown which for the denominator will certainly suffice, since the numerator arises from it writing $f + a$ instead of f.

§32 To shorten the proof, for the sake of brevity, let us put

$$\frac{x^{f-1}\partial x}{\sqrt{1-x^{2a}}} = \partial V,$$

having done which the denominator of our expression affected by imaginary quantities will be

$$\int x^{+b\sqrt{-1}}\partial V \cdot \int x^{-b\sqrt{-1}}\partial V.$$

Now set

$$\text{the sum of the products} = \int (x^{b\sqrt{-1}} + x^{-b\sqrt{-1}})\partial V = p,$$

$$\text{the difference of the products} = \int (x^{b\sqrt{-1}} - x^{-b\sqrt{-1}})\partial V = q,$$

and it is known that the propounded product will be

$$\int x^{b\sqrt{-1}}\partial V \cdot \int x^{-b\sqrt{-1}}\partial V = \frac{pp-qq}{4}.$$

Therefore, I will show that so pp as qq can be reduced to real quantities.

§**33** To this end, let us write $e^{\log x}$ instead of x in the imaginary powers that

$$p = \int \left(e^{b\log x\sqrt{-1}} + e^{-b\log x\sqrt{-1}}\right)\partial V,$$

$$q = \int \left(e^{b\log x\sqrt{-1}} - e^{-b\log x\sqrt{-1}}\right)\partial V.$$

Therefore, since we now that

$$e^{\varphi\sqrt{-1}} + e^{-\varphi\sqrt{-1}} = 2\cos\varphi$$

and

$$e^{\varphi\sqrt{-1}} - e^{-\varphi\sqrt{-1}} = 2\sqrt{-1}\sin\varphi,$$

for the sake of brevity having set $b\log x = \varphi$ it will be

$$p = 2\int \partial V\cos\varphi \quad \text{and} \quad q = 2\sqrt{-1}\int \partial V\sin\varphi,$$

whence without any effort the denominator follows

$$\frac{pp-qq}{4} = \left(\int \partial V\cos\varphi\right)^2 + \left(\int \partial V\sin\varphi\right)^2,$$

an expression which is obviously real.

§34 Hence the value of the numerator is easily concluded, which will, of course, be

$$\left(\int x^a \partial V \cos\varphi\right)^2 + \left(\int x^a \partial V \sin\varphi\right)^2,$$

such that our expression disturbed by imaginary quantities is expressed in just real quantities in the following way:

$$A^2 = (ff + bb)\frac{\left(\int x^a \partial V \cos\varphi\right)^2 + \left(\int x^a \partial V \sin\varphi\right)^2}{\left(\int \partial V \cos\varphi\right)^2 + \left(\int \partial V \sin\varphi\right)^2}$$

while

$$\partial V = \frac{x^{f-1}\partial x}{\sqrt{1-x^{2a}}} \quad \text{and} \quad \varphi = b\log x.$$

§35 But in analysis one still desires a direct method to treat formulas of the following kind by integration:

$$\int \frac{x^{f-1}\partial x \cos b\log x}{\sqrt{1-x^{2a}}} \quad \text{and} \quad \int \frac{x^{f-1}\partial x \sin b\log x}{\sqrt{1-x^{2a}}}.$$

Nevertheless, if it was not for the denominator, each of both formulas could indeed be integrated, which will be worth one's while to have shown in the following way.

§36 For, this can be achieved by means of the very well known reduction

$$\int P\partial Q = PQ - \int Q\partial P.$$

If for the first formula one takes

$$P = \cos b\log x \quad \text{and} \quad \partial Q = x^{f-1}\partial x,$$

it will be

$$\int x^{f-1}\partial x \cos b\log x = \frac{x^f}{f}\cos b\log x + \frac{b}{f}\int x^{f-1}\partial x \sin b\log x.$$

But for the other, having taken

$$P = \sin b \log x \quad \text{and} \quad \partial Q = x^{f-1} \partial x,$$

it will be

$$\int x^{f-1} \partial x \sin b \log x = \frac{x^f}{f} \sin b \log x - \frac{b}{f} \int x^{f-1} \partial x \cos b \log x.$$

Hence by substitution one further concludes

$$\begin{aligned} \int x^{f-1} \partial x \cos b \log x &= \frac{x^f}{ff + bb} (f \cos b \log x + b \sin b \log x), \\ \int x^{f-1} \partial x \sin b \log x &= \frac{x^f}{ff + bb} (f \sin b \log x - b \cos b \log x). \end{aligned}$$

But on the other hand, if the denominator is not absent, nothing more is understood than that the integral is reduced to a still unknown class of most transcendental quantities.