On the infinite Series $1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}xx + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\cdot\gamma(\gamma+1)(\gamma+2)}x^3 + \text{etc.}$ - Part Two*

Carl Friedrich Gauss

38.

For the sake of brevity setting $F(\alpha, \beta, \gamma, x) = P$ by art. 4 we have

$$\frac{dP}{dx} = \frac{\alpha\beta}{\gamma}F(\alpha+1, \beta+1, \gamma+1, x)$$

and hence differentiating again

$$\frac{ddP}{dx^2} = \frac{\alpha\beta(\alpha+1)(\beta+1)}{\gamma(\gamma+1)}F(\alpha+2, \beta+2, \gamma+2, x)$$

Hence equation IX of art. 10 yields

$$[80] \quad \alpha\beta P - (\gamma - (\alpha+\beta+1)x)\frac{dP}{dx} - (x - xx)\frac{ddP}{dx}$$

This differential equation of second order can be considered as more exact definition of our function; but since $P = F(\alpha, \beta, \gamma, x)$ is not the complete integral but only a particular solution (since it does not contain any new

*Original Title: „Circa seriem infinitam $1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}xx + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\cdot3\gamma(\gamma+1)(\gamma+2)}x^3 +$ etc. - Pars secunda", actually taken from Gauß's "Nachlass"the work has no official title but is clearly intended as a sequel to Gauß's first paper on the hypergeometric series; reprinted in „*Carl Friedrich Gauß Werke*: Volume 3 pp. 207 - 229", translated by: Alexander Aycock for the project „Euler-Kreis Mainz"

constants), one has to add the condition that P starts from the value 1 for $x = 0$ and for the same value of x $\frac{dP}{dx} = \frac{\alpha\beta}{\gamma}$ and $\frac{ddP}{dx^2} = \frac{\alpha\beta(\alpha+1)(\beta+1)}{\gamma(\gamma+1)}$.
So each value of x, to which you get step by step from the value $x = 0$, nevertheless in such a way that you to not reach the value $x = 1$, for which $x - xx = 0$, P will be determined completely; but obviously *this way* you can only get to a real positive values of x larger than 1 by going through imaginary values, since which can be done in infinitely many different ways and without prejudice of continuity, it will hence not be plain, whether to the same value x several, or even infinitely many, discrete values of P correspond, as it is known to happen in many more common transcendental functions. But let us keep the discussion about this for later, since here mainly the case, where x is taken below or at least not beyond 1, is considered and P is considered equal to the series $F(\alpha, \beta, \gamma, x)$.

39.

Writing $1 - y$ for x in equation 80, it goes over into this one

$$0 = \alpha\beta P - (\alpha + \beta + 1 - \gamma) - (\alpha + \beta + 1)y)\frac{dP}{dy} - (y - yy)\frac{ddP}{dy^2}$$

which has a similar form. Hence immediately another particular integral follows

$$P = F(\alpha, \beta, \alpha + \beta + 1 - \gamma, y) = F(\alpha, \beta, \alpha + \beta + 1 - \gamma, 1 - x)$$

whence by known principles the complete integral of equation 80 follows,

$$[81] \qquad P = MF(\alpha, \beta, \gamma, x) + NF(\alpha, \beta, \alpha + \beta + 1 - \gamma, 1 - x)$$

while M and N denote arbitrary constants.
Furthermore, we observe here that the following more general equation can easily be reduced to the form of equation 80

$$0 = AP + (B + Cy)\frac{dP}{dy} + (D + Ey + Fyy)\frac{ddP}{dy^2}$$

For, having taken the roots of the equation $0 = D + Ey + Fyy$, they are $y = a$, $y = b$ or $D + Ey + Fyy$ is equal to the product $F(y - a)(y - b)$ and, setting $\frac{y-a}{b-a} = x$ and determining α, β, γ so that

$$\alpha\beta = \frac{A}{F}, \quad \alpha+\beta+1 = \frac{C}{F}, \quad \gamma = -\frac{B+aC}{F(b-a)}$$

it is plain that the initial equation goes over into equation 80.

40.

By means of the differential equation 80 it is possible to find many most memorable theorems on our series, first general ones, then more special ones, and there is no doubt that still many more and more important ones are to be discovered, but they require more advances techniques. We will now present, what we were able to discover.
Let us set $P = (1-x)^{\mu} P'$ and it will be

$$\begin{aligned}\frac{dP}{dx} &= -\mu(1-x)^{\mu-1}P' + (1-x)^{\mu}\frac{dP'}{dx} \\ \frac{ddP}{dx^2} &= \mu(\mu-1)(1-x)^{\mu-2}P' - 2\mu(1-x)^{\mu-1}\frac{dP'}{dx} + (1-x)^{\mu}\frac{ddP'}{dx^2}\end{aligned}$$

Having substituted these values in equation 80, dividing by $(1-x)^{\mu}$, it results

$$\begin{aligned}0 = P'\left\{\alpha\beta(1-x) + (\gamma-(\alpha+\beta+1)x)\mu - x(\mu\mu-\mu)\right\} \\ -\frac{dP'}{dx}\left\{(g-(\alpha+\beta+1)x) - 2\mu x\right\}(1-x) - \frac{ddP'}{dx^2}\left\{x-xx\right\}(1-x)\end{aligned}$$

Let us determine μ in such a way that the multiplicator of P' is divisible by $1-x$, what will happen either by setting $\mu = 0$ or $\mu = \gamma - \alpha - \beta$. The first assumption would lead to nothing new, but the second value gives

$$0 = P'\left\{\alpha\beta - \alpha\gamma - \beta\gamma + \gamma\gamma\right\} - \frac{dP'}{dx}\left\{\gamma - ((\gamma-\alpha)+(\gamma-\beta)+1)x\right\} - \frac{ddP'}{dx^2}(x-xx)$$

which has completely the same form as equation 80. Therefore, since for $x = 0$ obviously $P' = 1$ and $\frac{dP'}{dx} = \frac{\alpha\beta}{\gamma} - \mu = \frac{(\gamma-\alpha)(\gamma-\beta)}{\gamma}$, it is plain that its integral is $P' = F(\gamma-\alpha, \gamma-\beta, \gamma, x)$ so that one has in general

$$[82] \qquad F(\gamma-\alpha, \gamma-\beta, \gamma, x) = (1-x)^{\alpha+\beta-\gamma}F(\alpha, \beta, \gamma, x)$$

Hence one has to derive the transformation of the series

$$1+\frac{2\cdot 8}{9}x+\frac{3\cdot 8\cdot 10}{9\cdot 11}xx+\frac{4\cdot 8\cdot 10\cdot 12}{9\cdot 11\cdot 13}x^3+\text{etc.}=F\left(2,4,\frac{9}{2},x\right)$$

into

$$(1-x)^{-\frac{3}{2}}\left(1+\frac{1\cdot 5}{2\cdot 9}x+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 9\cdot 11}xx+\text{etc.}\right)=(1-x)^{-\frac{8}{2}}F\left(\frac{5}{2},\frac{1}{2},\frac{9}{2},x\right)$$

which we indicated in the Ephemeridibus Astronomicis Berolinensibus 1814 p. 257 [Addendum to Art. 90 and 100 of the Theoria motus] without a proof.

41.

Further, let us set $P=x^{\mu}P'$ so that

$$\begin{aligned}\frac{dP}{dx}&=\mu x^{\mu-1}P'+x^{\mu}\frac{dP'}{dx}\\ \frac{ddP}{dx^2}&=(\mu\mu-\mu)x^{\mu-2}P'+2\mu x^{\mu-1}\frac{dP'}{dx}+x^{\mu}\frac{ddP'}{dx^2}\end{aligned}$$

having substituted which values in 80, dividing by $x^{\mu-1}$ we find

$$\begin{aligned}0=&P'\{\alpha\beta x-(\gamma-(\alpha+\beta+1)x)\mu-(1-x)(\mu\mu-\mu)\}\\ &-\frac{dP'}{dx}\{\gamma-(\alpha+\beta+1)x+2\mu(1-x)\}x\\ &-\frac{ddP'}{dx^2}(xx-x^3)\end{aligned}$$

The multiplicator of P' in this formula becomes divisible by x setting $\mu=0$ or $\mu=1-\gamma$; the second value gives

$$\begin{aligned}0=&P'(\alpha\beta+\alpha+\beta+1-2\gamma-\alpha\gamma-\beta\gamma+\gamma\gamma)\\ &-\frac{dP'}{dx}(2-\gamma-(\alpha+\beta+3-2\gamma)x)\\ &-\frac{ddP'}{dx^2}(x-xx)\end{aligned}$$

Comparing this equation to 80, whose form is completely the same, it is plain that what was P, α, β, γ there is P', $\alpha+1-\gamma$, $\beta+1-\gamma$, $2-\gamma$ here: Therefore, since we assigned the complete integral of that equation, it is obvious that P' will be contained in the formula

$$\begin{aligned} P' &= MF(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma, x) \\ &\quad + NF(\alpha+1-\gamma, \beta+1-\gamma, \alpha+\beta+1-\gamma, 1-x) \end{aligned}$$

while M, N denote constant quantities, or

$$[83] \qquad \begin{aligned} F(\alpha,\beta,\gamma,x) &= Mx^{1-\gamma}F(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma, x) \\ &\quad + Nx^{1-\gamma}F(\alpha+1-\gamma, \beta+1-\gamma, \alpha+\beta+1-\gamma, 1-x) \end{aligned}$$

where the constants M, N will depend on the elements α, β, γ.

42.

From equation 82 it follows

$$\begin{aligned} &F(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma, x) = (1-x)^{\gamma-\alpha-\beta}F(1-\alpha, 1-\beta, 2-\gamma, x) \\ &F(\alpha+1-\gamma, \beta+1-\gamma, \alpha+\beta+1-\gamma, 1-x) = x^{\gamma-1}F(\alpha, \beta, \alpha+\beta+1-\gamma, 1-x) \end{aligned}$$

whence setting

$$\frac{1}{N} = f(\alpha,\beta,\gamma), \quad \frac{M}{N} = g(\alpha,\beta,\gamma)$$

equation 83 becomes

$$\begin{gathered} F(\alpha,\beta,\alpha+\beta+1-\gamma,1-x) \\ = f(\alpha,\beta,\gamma)F(\alpha,\beta,\gamma,x) \\ +g(\alpha,\beta,\gamma)(1-x)^{\gamma-\alpha-\beta}x^{1-\gamma}F(1-\alpha,1-\beta,2-\gamma,x) \end{gathered}$$

By means of formula 82 the same equation can also cast into this form

$$\begin{aligned} &x^{1-\gamma}F(\alpha+1-\gamma, \beta+1-\gamma, \alpha+\beta+1-\gamma, 1-x) = \\ &f(\alpha,\beta,\gamma)(1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha, \gamma-\beta, \gamma, x) + g(\alpha,\beta,\gamma)x^{1-\gamma}F(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma, x) \end{aligned}$$

or dividing by $x^{1-\gamma}$ and changing α, β, γ into $\alpha+1-\gamma$, $\beta+1-\gamma$, $2-\gamma$ respectively

$$\begin{gathered}F(\alpha,\beta,\alpha+\beta+1-\gamma,1-x)\\ = g(\alpha+1-\gamma,\beta+1-\gamma,2-\gamma)F(\alpha,\beta,\gamma,x)\\ +f(\alpha+1-\gamma,\beta+1-\gamma,2-\gamma)(1-x)^{\gamma-\alpha-\beta}x^{1-\gamma}F(1-\alpha,1-\beta,2-\gamma,x)\end{gathered}$$

43.

Now to find the nature of the function $f(\alpha,\beta,\gamma)$, let us set $x=0$. Then it is clear that $F(\alpha,\beta,\gamma,x)=1$, $x^{1-\gamma}=0$, if $1-\gamma$ was a positive quantity, of course. But by equation 48 we have

$$F(\alpha,\beta,\alpha+\beta+1-\gamma,1)=\frac{\Pi(\alpha+\beta-\gamma)\Pi(-\gamma)}{\Pi(\alpha-\gamma)\Pi(\beta-\gamma)}$$

Therefore, under the same restriction it is demonstrated that

$$[85]\qquad f(\alpha,\beta,\gamma)=\frac{\Pi(\alpha+\beta-\gamma)\Pi(-\gamma)}{\Pi(\alpha-\gamma)\Pi(\beta-\gamma)}$$

That this formula is indeed general is proven as follows. Differentiating equation 84 it results

$$\begin{aligned}&-\frac{\alpha\beta}{\alpha+\beta+1-\gamma}F(\alpha+1,\beta+1,\alpha+\beta+2-\gamma,1-x)\\ &=\frac{\alpha\beta}{\gamma}f(\alpha,\beta,\gamma)F(\alpha+1,\beta+1,\gamma+1,x)\\ &+f(\alpha+1-\gamma,\beta+1-\gamma,2-\gamma)(1-x)^{\gamma-\alpha-\beta-1}x^{-\gamma}\\ &\times\Big\{((1-\gamma)(1-x)-(\gamma-\alpha-\beta)x)F(1-\alpha,1-\beta,2-\gamma,x)\\ &+\frac{(1-\alpha)(1-\beta)}{2-\gamma}(x-xx)F(2-\alpha,2-\beta,3-\gamma,x)\Big\}\end{aligned}$$

But by formula IX of art. 10 changing α, β, γ into $-\alpha$, $-\beta$, $1-\gamma$

$$(1-\gamma)(2-\gamma)F(-\alpha,-\beta,1-\gamma,x) = (2-\gamma)(1-\gamma+(\alpha+\beta-1)x)F(1-\alpha,1-\beta,2-\gamma,x) + (1-\alpha)(1-\beta)(x-xx)F(2-\alpha,2-\beta,3-\gamma,x)$$

whence the preceding equation goes over into this one

$$F(\alpha+1,\beta+1,\alpha+\beta+2-\gamma,1-x)$$

$$= -\frac{\alpha+\beta+1-\gamma}{\gamma} f(\alpha,\beta,\gamma)F(\alpha+1,\beta+1,\gamma+1,x)$$

$$-\frac{(\alpha+\beta+1-\gamma)(1-\gamma)}{\alpha\beta} f(\alpha+1-\gamma,\beta+1-\gamma,2-\gamma)(1-x)^{\gamma-\alpha-\beta-1}x^{-\gamma}F(-\alpha,-\beta,1-\gamma,x)$$

But changing α, β, γ into $\alpha+1$, $\beta+1$, $\gamma+1$ in equation 84

$$\begin{aligned} &F(\alpha+1,\beta+1,\alpha+\beta+2-\gamma,1-x) \\ &= f(\alpha+1,\beta+1,\gamma+1)F(\alpha+1,\beta+1,\gamma+1,x) \\ &+f(\alpha+1-\gamma,\beta+1-\gamma,1-\gamma)(1-x)^{\gamma-\alpha-\beta-1}x^{-\gamma}F(-\alpha,-\beta,1-\gamma,x) \end{aligned}$$

Therefore, since it is easily seen that these two equations must be identical, in general

$$f(\alpha+1,\beta+1,\gamma+1) = \frac{\alpha+\beta+1-\gamma}{-\gamma} f(\alpha,\beta,\gamma)$$

or changing α, β, γ into $\alpha-1$, $\beta-1$, $\gamma-1$

$$\begin{aligned} f(\alpha,\beta,\gamma) &= \frac{\alpha+\beta-\gamma}{1-\gamma} f(\alpha-1,\beta-1,\gamma-1) \\ &= \frac{(\alpha+\beta-\gamma)\cdot(\alpha+\beta-\gamma-1)}{(1-\gamma)(2-\gamma)} f(\alpha-2,\beta,\gamma-2) \end{aligned}$$

etc. whence it is easily concluded that generally for each integer value of k

$$f(\alpha,\beta,\gamma) = \frac{\Pi(\alpha+\beta-\gamma)\Pi(-\gamma)}{\Pi(\alpha+\beta-\gamma-k)\Pi(k-\gamma)} \cdot f(\alpha-k,\beta-k,\gamma-k)$$

But if $1-(\gamma-k)$ or $k+1-\gamma$ is a positive quantity, we demonstrated that (formula 85)

$$f(\alpha-k,\beta-k,\gamma-k)=\frac{\Pi(\alpha+\beta-\gamma-k)\Pi(k-\gamma)}{\Pi(\alpha-\gamma)\Pi(\beta-\gamma)}$$

Hence, since k, whatever γ is, can always assumed so large that $k+1-\gamma$ becomes a positive quantity, it will be *in general*

$$f(\alpha,\beta,\gamma)=\frac{\Pi(\alpha+\beta-\gamma)\Pi(-\gamma)}{\Pi(\alpha-\gamma)\Pi(\beta-\gamma)}$$

and therefore

$$f(\alpha+1-\gamma,\beta+1-\gamma,2-\gamma)=\frac{\Pi(\alpha+\beta-\gamma)\Pi(\gamma-2)}{\Pi(\alpha-1)\Pi(\beta-1)}$$

so that our formula becomes

$$[86]\quad F(\alpha,\beta,\alpha+\beta+1-\gamma,1-x)$$

$$=\frac{\Pi(\alpha+\beta-\gamma)\Pi(-\gamma)}{\Pi(\alpha-\gamma)(\Pi(\beta-\gamma)}F(\alpha,\beta,\gamma,x)$$

$$+\frac{\Pi(\alpha+\beta-\gamma)\Pi(\gamma-2)}{\Pi(\alpha-1)\Pi(\beta-1)}x^{1-\gamma}(1-x)^{\gamma-\alpha-\beta}F(1-\alpha,1-\beta,2-\gamma,x)$$

or having changed γ into $\alpha+\beta+1-\gamma$

$$[87]\quad F(\alpha,\beta,\gamma,1-x)$$

$$=\frac{\Pi(\gamma.1)\Pi(\gamma-\alpha-\beta-1)}{\Pi(\gamma-\alpha-1)\Pi(\gamma-\beta-1)}F(\alpha,\beta,\alpha+\beta+1-\gamma,x)$$

$$+\frac{\Pi(\gamma-1)\Pi(\alpha+\beta-\gamma-1)}{\Pi(\alpha-1)\Pi(\beta-1)}x^{\gamma-\alpha-\beta}(1-x)^{\gamma-\alpha-\beta}F(1-\alpha,1-\beta,\gamma+1-\alpha-\beta,x)$$

If you like a more, it is possible in formula 86 to write

$$F(\alpha+1-\gamma,\beta+1-\gamma,2-\gamma,x)\quad\text{for}\quad(1-x)^{\gamma-\alpha-\beta}F(1-\alpha,1-\beta,2-\gamma,x)$$

in formula 87

$$F(\gamma-\alpha,\gamma-\beta,\gamma+1-\alpha-\beta,x) \quad \text{for} \quad (1-x)^{1-\gamma}F(1-\alpha,1-\beta,\gamma+1-\alpha-\beta,x)$$

44.

Therefore, if in a certain series contained in our formula a value between 0.5 and 1 is attributed to the fourth element, the slow convergence can be avoided by the preceding formulas, which split that series into two other similar ones converging the faster the slower the initial one converged. But one has to exclude special cases, where this transformation, if in the series to be transformed the difference of the third element and the sum of the first two elements is an integer number. For, if in formula 86 $\gamma = 0$ or equal to a negative integer number, $F(\alpha,\beta,\gamma,x)$ obviously becomes an inept series (art. 2) and the factor $\Pi(\gamma-2)$ is infinite; but if γ is an integer larger than 1, $F(1-\alpha,1-\beta,2-\gamma,x)$ and $F(\alpha+1-\gamma,\beta+1-\gamma,2-\gamma,x)$ become inept series and $\Pi(-\gamma)$ becomes infinite; finally, if $\gamma = 1$, the two transformed series $F(\alpha,\beta,\gamma,x)$ and $F(1-\alpha,1-\beta,2-\gamma,x)$ or $F(\alpha+1-\gamma,\beta+1-\gamma,2-\gamma,x)$, which becomes identical to $F(\alpha,\beta,\gamma,x)$, do not have this problem, but nevertheless the transformation is of no use, since each of both transformed series is multiplied by the infinite coefficient $\Pi(-1)$. Therefore, it will worth one's while to show, how even in these cases the convergence can be accelerated.

45.

Let k be a positive integer number (or even $= 0$) and let us denote the $k+1$ first terms of the series $F(\alpha,\beta,\gamma,x)$ by X. The following term will be

$$= \frac{\alpha\cdot(\alpha+1)\cdot(\alpha+2)\cdots(\alpha+k)\cdot\beta\cdot(\beta+1)\cdot(\beta+2)\cdots(\beta+k)}{1\cdot 2\cdot 3\cdots(k+1)\cdot\gamma\cdot(\gamma+1)\cdot(\gamma+2)\cdots(\gamma+k)}x^{k+1}$$

and in like manner the following terms. Hence it is concluded that

I. $$\frac{\Pi(\alpha+\beta-\gamma)\Pi(-\gamma)}{\Pi(\alpha-\gamma)\Pi(\beta-\gamma)}\cdot F(\alpha,\beta,\gamma,x) \quad \text{can be expressed by}$$

$$\frac{\Pi(\alpha+\beta-\gamma)\Pi(-\gamma)}{\Pi(\alpha-\gamma)\Pi(\beta-\gamma)}\cdot X+\frac{\Pi(\alpha+\beta-\gamma)\Pi(-\gamma)\Pi(\gamma-1)}{\Pi(\alpha-1)\Pi(\beta-1)\Pi(\alpha-\gamma)\Pi(\beta-\gamma)}\Sigma\left\{\frac{\Pi(a+k+t)\Pi(\beta+k+t)}{\pi(k+t+1)\Pi(\gamma+k+t)}x^{k+1+t}\right\}$$

if for t all values 0, 1, 2, 3 etc. to infinity are understood to be substituted. In like manner $F(\alpha+1-\gamma,\beta+1-\gamma,2-\gamma,x)$ can be expressed by

$$\frac{\Pi(1-\gamma)}{\Pi(\alpha-\gamma)\Pi(\beta-\gamma)}\sum\left(\frac{\Pi(\alpha-\gamma+t)\Pi(\beta-\gamma+t)}{\Pi l\Pi(1-\gamma+t}x^t\right)$$

having determined t exactly as before, and hence, since $\Pi(1-\gamma)=(1-\gamma)\Pi(-\gamma)$ and $\Pi(\gamma-1)=-(1-\gamma)\Pi(\gamma-2)$, it is plain that

$$\text{II.}\qquad \frac{\Pi(\alpha+\beta-\gamma)\pi(\gamma-2)}{\Pi(\alpha-1)\Pi(\beta-1)}x^{1-\gamma}F(\alpha+1-\gamma,\beta+1-\gamma,2-\gamma,x)$$
$$=-\frac{\Pi(\alpha+\beta-\gamma)\Pi(-\gamma)\Pi(\gamma-t)}{\Pi(\alpha-1)\Pi(\beta-1)\Pi(\alpha-\gamma)\Pi(\beta-\gamma)}\Sigma\left\{\frac{\Pi(\alpha-\gamma+t)\Pi(\beta-\gamma+t)}{\Pi t\pi(1-\gamma+t)x^{1+t-\gamma}}\right\}$$

Hence formula 86 can also be exhibited this way:

$$F(\alpha,\beta,\alpha+\beta+1-\gamma,1-x)$$
$$=\frac{\Pi(\alpha+\beta-\gamma)\Pi(-\gamma)}{\Pi(\alpha-\gamma)\Pi(\beta-\gamma)}X+\frac{\Pi(\alpha+\beta-\gamma)\Pi(-\gamma)\Pi(\gamma-1)}{\Pi(\alpha-1)\Pi(\beta-1)\Pi(\alpha-\gamma)\Pi(\beta-\gamma)}$$
$$\times\Sigma\left\{\frac{\Pi(\alpha+k+l)\Pi(\beta+k+z)}{\Pi(k+t+1)\Pi(\gamma+k+t)}x^{k+1+t}-\frac{\Pi(\alpha-\gamma+t)\Pi(\beta-\gamma+t)}{\Pi(t-\gamma+t)\Pi t}x^{1+t-\gamma}\right\}$$

This expression shows clearly, that the single differences, which are contained in the sum $\sum$, become $=0$, if one takes $\gamma=-k$, but since here at the same time $\Pi(\gamma-1)$ is an infinitely large quantity, the product is understood to be able to become finite. To express its value by finite quantities, first let us set $\gamma+k=\omega$, whence

$$\Pi(\gamma-1)\cdot\gamma\cdot(\gamma+1)(\gamma+2)\cdots(\gamma+k-1)\omega=\Pi\omega$$

or

$$\Pi(\gamma-1)=\frac{\Pi\omega}{\omega(\omega-1)(\omega-2)\cdots(\omega-k)}$$

Therefore, the sum in discussion is changed in such a way that we see

$$\frac{1}{\omega}\left\{\frac{\Pi(\alpha-\gamma+t+\omega)(\Pi(\beta-\gamma+t+\omega)}{\Pi(t-\gamma+1+\omega)\Pi(t+\omega)}x^{1+t-\gamma+\omega}-\frac{\Pi(\alpha-\gamma+t)\Pi(\beta-\gamma+t)}{\Pi(t-\gamma+1)\Pi t}x^{1+t-\gamma}\right\}$$

if ω decreases to zero. But by known principles hence it results

$$-\frac{dU}{d\gamma}$$

if, for the sake of brevity, we set

$$\frac{\Pi(\alpha-\gamma+t)\Pi(\beta-\gamma+t)}{\Pi(t-\gamma+1)\Pi(t-k-\gamma)}x^{1+t-\gamma} = U$$

and consider only γ as a variable. But hence

$$\frac{dU}{Ud\gamma} = -\Psi(\alpha-\gamma+t) - \Psi(\beta-\gamma+t) + \Psi(t-\gamma+1) + \Psi(t-k-\gamma) - \log x$$

Hence for $\gamma = -k$ one concludes

$$\begin{aligned}
[88] \quad & F(\alpha,\beta,\alpha+\beta+1+k,1-x) \\
= \; & \frac{\Pi(\alpha+\beta+k)\Pi k}{\Pi(\alpha+k)\Pi(\beta+k)}X \\
& + \frac{\Pi(\alpha+\beta+k)\Pi k}{\Pi(\alpha-1)\Pi(\beta-1)\Pi(\alpha+k)\Pi(\beta+k)(-1)(-2)\cdots(-k)}\Big\{(\log x \\
& + \Psi(\alpha+t+k) + \psi(\beta+t+k) - \Psi(t+k+1) - \Psi t)\frac{\Pi(\alpha+t+k)\Pi(\beta+t+k)}{\Pi(t+k+1)\Pi t}x^{1+t+k}\Big\} \\
= \; & \frac{\Pi(\alpha+\beta+k)\Pi k}{\Pi(\alpha+k)\Pi(\beta+k)}X \pm \frac{\Pi(\alpha+\beta+k)x^{1+k}}{\Pi(\alpha-1)\Pi(\beta-1)\Pi(k+1)}Y
\end{aligned}$$

where

$$Y = \{\log x + \Psi(\alpha+k) + \psi(\beta+k) - \Psi(k+1) - \Psi(0)\}\, F(\alpha+k+1, \beta+k+1, k+2, x)$$

$$+A\frac{(\alpha+k+1)\cdot(\beta+k+1)}{1\cdot(k+2)}x$$

$$+(A+B)\frac{(\alpha+k+1)\cdot(\alpha+k+2)\cdot(\beta+k+1)\cdot(\beta+k+2)}{1\cdot 2\cdot 3\cdot(k+2)\cdot k+3}xx$$

$$+(A+B+C)\frac{(\alpha+k+1)(\alpha+k+2)(\alpha+k+3)(\beta+k+1)(\beta+k+2)(\beta+k+3)}{1\cdot 2\cdot 3\cdot(k+2)\cdot(k+3)\cdot(k+4)}x^3$$

$$+\text{etc.}$$

and

$$A = \frac{1}{\alpha+k+1} + \frac{1}{\beta+k+1} - \frac{1}{k+2} - 1$$

$$B = \frac{1}{\alpha+k+2} + \frac{1}{\beta+k+2} - \frac{1}{k+3} - \frac{1}{2}$$

$$C = \frac{1}{\alpha+k+3} + \frac{1}{\beta+k+3} - \frac{1}{k+4} - \frac{1}{3},$$

the upper or lower sign is to be taken, depending on whether k is an even or odd number.

46.

Therefore, this way $F(\alpha, \beta, \alpha+\beta+1-\gamma, 1-x)$ is transformed, if γ is 0 or a negative integer. We can treat the case $\gamma = +1$ in completely the same way, or, for a shorter calculation, we can set $k = -1$ in the preceding arguments, whence X vanishes completely and we obtain:

[89] $F(\alpha, \beta, \alpha+\beta, 1-x)$

$$= -\frac{\Pi(\alpha+\beta-1)}{\Pi(\alpha-1)\Pi(\beta-1)}\left\{\log x + \Psi(\alpha-1) + \Psi(\beta-1) - 2\Psi(0)\right\} F(\alpha, \beta, 1, x)$$

$$-\frac{\Pi(\alpha+\beta-1)}{\Pi(\alpha-1)\Pi(\beta-1)}\left\{A\frac{\alpha\cdot\beta}{1}x\right.$$

$$+(A+B)\frac{\alpha(\alpha+1)\cdot\beta\cdot(\beta+1)}{1\cdot 2\cdot 1\cdot 2}xx$$

$$+(A+B+C)\frac{\alpha\cdot(\alpha+1)\cdot(\alpha+2)\cdot\beta\cdot(\beta+1)\cdot(\beta+2)}{1\cdots 2\cdot 3\cdot 1\cdot 2\cdot 3}x^3$$

$$\left.+\text{etc.}\right\}$$

where

$$A = \frac{1}{\alpha} + \frac{1}{\beta} - 2, \quad B = \frac{1}{\alpha+1} + \frac{1}{\beta+2} - \frac{2}{2}, \quad C = \frac{1}{\alpha+2} + \frac{1}{\beta+2} - \frac{2}{3} \quad \text{etc.}$$

So, e. g., for $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$ we obtain (confer for, 52, 71)

[90] $F\left(\frac{1}{2}, \frac{1}{2}, 1, 1-x\right)$

$$= -\frac{1}{\pi}\log\frac{1}{16}x\cdot F\left(\frac{1}{2}, \frac{1}{2}, 1, x\right)$$

$$-\frac{1}{\pi}\left\{2\cdot\frac{1\cdot 1}{2\cdot 2}x + \left(2+\frac{1}{3}\right)\frac{1\cdot 1\cdot 3\cdot 3}{2\cdot 2\cdot 4\cdot 4}xx + \left(2+\frac{1}{3}+\frac{2}{15}\right)\frac{1\cdot 1\cdot 3\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 6}\right.$$

$$\left.+\left(2+\frac{4}{3\cdot 4}+\frac{4}{5\cdot 6}+\frac{4}{7\cdot 8}\right)\frac{1\cdot 1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8}x^4 + \text{etc.}\right\}$$

$$= -\left\{\log\frac{1}{16}xF\left(\frac{1}{2}, \frac{1}{2}, 1, x\right) + \frac{1}{2}x + \frac{21}{64}xx + \frac{185}{768}x^3 + \frac{18655}{98394}x^4\right.$$

$$\left.+\frac{102501}{655360}x^5 + \frac{1394239}{10485760}x^6 + \text{etc.}\right\}$$

Finally, it is not necessary to treat the third case, where γ is a positive integer larger than 1, separately, since

$$F(\alpha, \beta, \alpha+\beta+1-\gamma, 1-x) = x^{\gamma-1}F(\alpha+1-\gamma, \beta+1-\gamma, \alpha+\beta+1-\gamma, 1-x)$$

and the transformation of the series $F(\alpha+1-\gamma, \beta+1-\gamma, \alpha+\beta+1-\gamma, 1-x)$ for $\gamma > 1$ is immediately reduced to the first case.

47.

We go over to other transformations, beginning with the substitution $x = \frac{y}{y-1}$. Hence $dx = -\frac{dy}{(y-1)^2}$, and hence

$$\begin{aligned}
\frac{dP}{dx} &= -\frac{dP}{dy}(1-y)^2, \quad \text{differentiating again} \\
d\frac{dP}{dx} &= -(1-y)^2 d\frac{dP}{dy} + 2(1-y)dP, \quad \text{and hence} \\
\frac{ddP}{dx^2} &= +(1-y)^4\frac{ddP}{dy^4} - 2(1-y)^3\frac{dP}{dy}
\end{aligned}$$

Having substituted these values equation 80 goes over into this one

$$0 = \alpha\beta P + (1-y)(\gamma + (\alpha+\beta-1-\gamma)y)\frac{dP}{dy} + (1-y)(y-yy)\frac{ddP}{dy^2}$$

But to obtain an equation similar to 80, let us set $P = (1-y)^{\mu}P$, whence

$$\begin{aligned}
\frac{dP}{dy} &= -\mu(1-y)^{\mu-1}P' + (1-y)^{\mu}\frac{dP'}{dy} \\
\frac{ddP}{dy^2} &= (\mu\mu-\mu)(1-y)^{\mu-2}P' - 2\mu(1-y)^{\mu-1}\frac{dP'}{dy} + (1-y)^{\mu}\frac{ddP'}{dy^2}
\end{aligned}$$

Having substituted these, after division by $(1-y)^{\mu}$,

$$0 = P' \{\alpha\beta - \mu(\gamma + (\alpha + \beta - 1\gamma)y) + y(\mu\mu - \mu)\}$$

$$+ \frac{dP'}{dy} \{\gamma + (\alpha + \beta - 1\gamma)y - 2\mu y\} (1 - y)$$

$$+ \frac{ddP'}{dy^2} \{y - yy\} (1 - y)$$

Let us determine μ in such a way that the multiplicator of P' becomes divisible by $1 - y$, what will happen by setting either $\mu = \alpha$ or $\mu = \beta$. The first value changes the preceding equation into this one

$$0 = \alpha(\beta - \gamma)P' + (\gamma - (\gamma + \alpha + 1 - \beta)y)\frac{dP'}{dy} + (y - yy)\frac{ddP'}{dy^2}$$

or

$$0 = \alpha(\gamma - \beta)P' - (\gamma - (\gamma - \beta + \alpha + 1)y)\frac{dP'}{dy} - (y - yy)\frac{ddP'}{dy^2}$$

which is to be satisfied in such a way that for $y = 0$ we have $P' = 1$ and $\frac{dP'}{dy} = \frac{\alpha(\gamma-\beta)}{\gamma}$. But hence one deduces $P' = F(\alpha, \gamma - \beta, \gamma, y)$ and one has

$$[91] \quad F(\alpha, \beta, \gamma, x) = (1 - y)^{\alpha} F(\alpha, \gamma - \beta, \gamma, y) = (1 - x)^{-\alpha} F\left(\alpha, \gamma - \beta, \gamma, -\frac{x}{1 - x}\right)$$

If we would have taken the other value β for μ, in completely the same way it would have resulted

$$[92] \quad F(\alpha, \beta, \gamma, x) = (1 - x)^{-\beta} F\left(\beta, \gamma - \alpha, \gamma, -\frac{x}{1 - x}\right)$$

which formula also follows from the preceding by permutation of the elements α, β immediately. By means of the formula just found the values of our series for negative values of the fourth element are always reduced to values of a similar series for positive values of the fourth element between 0 and 1, since

$$F(\alpha, \beta, \gamma, -x) = (1 + x)^{-\alpha} F\left(\alpha, \gamma - \beta, \gamma, \frac{x}{1 + x}\right)$$

48.

It will be worth one's while to show how by means of the transformations 82, 91 all formulas collected in art. 5 can easily be deduced only from the binomial theorem. For, hence formulas I-IV follow immediately. Formulas VI-IX follow from this immediately, if e^x is considered as the limit of the power $\left(1+\frac{x}{i}\right)^i$ or $\left(1-\frac{x}{i}\right)^{-i}$, and $\log x$ as the limit of $i\left(x^{\frac{1}{i}}-1\right)$, while i grows to infinity. Further, from

$$\cos n\varphi + \sqrt{-1}\cdot\sin n\varphi = (\cos\varphi + \sqrt{-1}\cdot\sin\varphi)^n$$
$$\cos n\varphi - \sqrt{-1}\cdot\sin n\varphi = (\cos\varphi - \sqrt{-1}\cdot\sin\varphi)^n$$

by subtraction and addition formula XVIII and XXII and hence by formula 82 XIX and XXIII follow immediately; hence by formula 91 XVI, XVII, XX and XXI. Setting t for nt and assuming n to be infinite, from XVI and XX the equations XI and XII follow; but assuming n to be infinitely small, from XVI-XVIII XIII-XV follow.

49.

From the substitution $x=\frac{1}{y}$ in the same way we find

$$0=\alpha\beta P-(\alpha+\beta-1-(\gamma-2)y)y\frac{dP}{dy}+(yy-y^3)\frac{ddP}{dy^2}$$

Further, setting $P=y^\mu P'$,

$$\text{I.}\qquad 0=P'(\alpha\beta-\mu(\alpha+\beta-1)+\mu(\gamma-2)y+(\mu\mu-\mu)(1-y))$$
$$-\frac{dP'}{dy}(\alpha+\beta-1-(\gamma-2)y-2\mu(1-y))y$$
$$+(yy-y^3)\frac{ddP'}{dy^2}$$

For the multiplicator of P' to become divisible by y, one has to set either $\mu=\alpha$ or $\mu=\beta$; the first value produces

$$\text{II}:\quad 0=P'\alpha(\gamma-\alpha-1)-\frac{dP'}{dy}(\beta-\alpha-1-(\gamma-2\alpha-2)y)+(y-yy)\frac{ddP'}{dy^2}$$

whose *particular integral* becomes

$$P' = F(\alpha, \alpha + 1 - \gamma, \alpha + 1 - \beta, y)$$

Therefore, equation I is satisfied by the particular integral

$$P = y^{\alpha} F(\alpha, \alpha + 1 - \gamma, \alpha + 1 - \beta, y)$$

and hence the other value $\mu = \beta$ yields another particular integral

$$P = y^{\beta} F(\beta, \beta + 1 - \gamma, \beta + 1 - \alpha, y)$$

whence one has the complete integral

$$P = Ay^{\alpha} F(\alpha, \alpha + 1 - \gamma, \alpha + 1 - \beta, y) + By^{\beta} F(\beta, \beta + 1 - \gamma, \beta + 1 - \alpha, y)$$

while A, B denote constants, which are not arbitrary but completely determined, since P is not the complete integral of equation 80, but only a particular integral. But for the determination of the values of the constants A, B not to lead us to wrong ways, we will deduce the same equation in another way by means of the things we just derived.
In 91 setting $-\frac{x}{1-x} = 1 - z$ and changing β into $\gamma - \beta$, γ into $\alpha + 1 - \beta$, x into z in eq. 86, it will be concluded

$$\begin{aligned}(1-x)^{\alpha} F(\alpha, \beta, \gamma, x) = {} & \frac{\Pi(\gamma-1)\Pi(\beta-\alpha-1)}{\Pi(\gamma-\alpha-1)\Pi(\beta-1)} F(\alpha, \gamma-\beta, \alpha+1-\beta, z) \\ & + \frac{\Pi(\gamma-1)\pi(\alpha-\beta-1)}{\Pi(\alpha-1)\Pi(\gamma-\beta-1)} z^{\beta-\alpha} F(\beta, \gamma-\alpha, \beta+1-\alpha, z)\end{aligned}$$

But by eqs. 91, 92

$$F(\alpha, \gamma-\beta, \alpha+1-\beta, z) = (1-z)^{-\alpha} F\left(\alpha, \alpha+1-\gamma, \alpha+1-\beta, -\frac{z}{1-z}\right)$$

$$F(\beta, \gamma-\alpha, \beta+1-\alpha, z) = (1-z)^{-\beta} F\left(\beta, \beta+1-\gamma, \beta+1-\alpha, -\frac{z}{1-z}\right)$$

Having substituted these and having set $z = \frac{1}{1-x}$, $1 - z = -\frac{x}{1-x}$, $-\frac{z}{1-z} = \frac{1}{x}$, we have

$$[93] \quad F(\alpha, \beta, \gamma, x) = \frac{\Pi(\gamma-1)\Pi(\beta-\alpha-1)}{\Pi(\gamma-\alpha-1)\Pi(\beta-1)}(-x)^{-\alpha} F\left(\alpha, \alpha+1-\gamma, \alpha+1-\beta, \frac{1}{x}\right)$$
$$+\frac{\Pi(\gamma-1)\Pi(\alpha-\beta-1)}{\Pi(\alpha-1)\Pi(\gamma-\beta-1)}(-x)^{-\beta} F\left(\beta, \beta+1-\gamma, \beta+1-\alpha, \frac{1}{x}\right)$$

which agrees to the equation found above, if one sets

$$A = \frac{\Pi(\gamma-1)\Pi(\beta-\alpha-1)}{\Pi(\gamma-\alpha-1)\Pi(\beta-1)}(-1)^{\alpha}$$
$$B = \frac{\Pi(\gamma-1)\Pi(\alpha-\beta-1)}{\Pi(\alpha-1)\Pi(\gamma-\beta-1)}(-1)^{\beta}$$

where it is to be noted that

$$(-1)^{\alpha} = \cos\alpha k\pi + \sqrt{-1}\cdot\sin\alpha k\pi$$
$$(-1)^{\beta} = \cos\beta k\pi + \sqrt{-1}\cdot\sin\beta k\pi$$

while k denotes an arbitrary integer number.

50.

By equation 93 the value of our function for values of the fourth element larger than 1 is reduced to the case, where the fourth element is smaller than 1. At the same time it is plain that to *negative* values of the fourth element, larger than 1, always one real value of the function F corresponds, but to the positive values on the other hand only then one real value can correspond, if α and β are either integers or rational fractions, whose denominators are odd; in the remaining cases $F(\alpha, \beta, \gamma, x)$ admits only imaginary values for a positive value, smaller than 1, of x.

51.

The relations among the many functions F expanded up to this point were all linear: Now, we add another one of a different kind. Let

$$\begin{aligned} P &= F(\alpha, \beta, \gamma, x) \\ Q &= x^{1-\gamma} F(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma, x) \\ R &= F(\alpha, \beta, \alpha+\beta+1-\gamma, 1-x) \end{aligned}$$

so that P, Q, R are three particular integrals of equation 80, or that

$$\text{I.} \qquad 0 = \alpha\beta P - (\gamma - (\alpha+\beta+1)x)\frac{dP}{dx} - (x - xx)\frac{ddP}{dx^2}$$

$$\text{II.} \qquad 0 = \alpha\beta Q - (\gamma - (\alpha+\beta+1)x)\frac{dQ}{dx} - (x - xx)\frac{ddQ}{dx^2}$$

$$\text{I.} \qquad 0 = \alpha\beta R - (\gamma - (\alpha+\beta+1)x)\frac{dR}{dx} - (x - xx)\frac{ddR}{dx^2}$$

Multiplying the first equation by Q, the second by P, by subtracting we find

$$0 = (\gamma - (\alpha+\beta+1)x)\frac{QdP - PdQ}{dx} + (x - xx)\frac{QddP - PddQ}{dx^2}$$

But this equation multiplied by $x^{\gamma-1}(1-x)^{\alpha+\beta-\gamma}$ becomes integrable and yields

$$[94] \qquad x^{\gamma}(1-x)^{\alpha+\beta+1-\gamma}\frac{QdP - PdQ}{dx}$$

In like manner one has

$$[95] \qquad B = x^{\gamma}(1-x)^{\alpha+\beta+1-\gamma}\frac{RdQ - QdR}{dx}$$

$$[96] \qquad C = x^{\gamma}(1-x)^{\alpha+\beta+1-\gamma}\frac{RdP - PdR}{dx}$$

The constants A, B, C are easily determined by the following method.

For $x = 0$ we find $P = 1$; further, $x^{\gamma} Q = xF(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma, x) = 0$ for $x = 0$; but its differential divided by dx, namely $\gamma x^{\gamma-1} Q + x^{\gamma}\frac{dQ}{dx}$, becomes $= 1$; hence one concludes $\frac{x^{\gamma} dQ}{dx} = 1 - \gamma$ for $x = 0$, and hence

$$A = \gamma - 1$$

But to determine B and C, let us recall the equation

$$R = f(\alpha, \beta, \gamma)P + f(\alpha + 1 - \gamma, \beta + 1 - \gamma, 2 - \gamma)Q$$

which, having differentiated it, gives

$$\frac{dR}{dx} = f(\alpha, \beta, \gamma)\frac{dP}{dx} + f(\alpha + 1 - \gamma, \beta + 1 - \gamma, 2 - \gamma)\frac{dQ}{dx}$$

Multiplying the first by $\frac{dQ}{dx}$, the second by Q, by subtracting the resulting equations, we find

$$\frac{QdR - RdQ}{dx} = f(\alpha, \beta, \gamma)\frac{QdP - PdQ}{dx} \quad \text{and hence}$$

$$B = (1 - \gamma)f(\alpha, \beta, \gamma) = \frac{\Pi(\alpha + \beta - \gamma)\Pi(1 - \gamma)}{\Pi(\alpha - \gamma)\Pi(\beta - \gamma)}$$

Similarly, having multiplied the first equation by $\frac{dP}{dx}$, the second by P, the subtraction gives

$$\frac{RdP - PdR}{dx} = f(\alpha + 1 - \gamma, \beta + 1 - \gamma, 2 - \gamma)\frac{QdP - PdQ}{dx}$$

and hence

$$C = (\gamma - 1)f(\alpha + 1 - \gamma, \beta + 1 - \gamma, 2 - \gamma) = \frac{\Pi(\alpha + \beta - \gamma)\Pi(\gamma - 1)}{\Pi(\alpha - 1)\Pi(\beta - 1)}$$

If you like it more, these three equations can also be exhibited in such a way that the function

[97] $$\frac{1}{\gamma}F(\alpha, \beta, \alpha + \beta + 1 - \gamma, 1 - x)F(\alpha + 1, \beta + 1, \gamma + 1, x)$$
$$+\frac{1}{\alpha + \beta + 1 - \gamma}F(\alpha + 1, \beta + 1, \alpha + \beta + 2 - \gamma, 1 - x)F(\alpha, \beta, \gamma, x)$$
$$= \frac{\Pi(\alpha + \beta - \gamma)\Pi(\gamma - 1)}{\Pi\alpha\Pi\beta}x^{-\gamma}(1 - x)^{\gamma - \alpha - \beta - 1}$$

52.

Denoting the function $F(-\alpha, -\beta, 1-\gamma, x)$ by S, it will be

$$0 = \alpha\beta S - (1 - \gamma + (\alpha + \beta - 1)x)\frac{dS}{dx} - (x - xx)\frac{ddS}{dx^2}$$

Having combined this equation with I. of the preceding art.,

$$0 = \alpha\beta\left(\frac{SdP + PdS}{dx}\right) - (1 - 2x)\frac{dP}{dx}\cdot\frac{dS}{dx} - (x - xx)\frac{dSddP + dPddS}{dx^2}$$

which is integrable and yields

$$\text{Const.} = \alpha\beta PS - (x - xx)\frac{dP}{dx}\cdot\frac{dS}{dx}$$

The value of the constant follows immediately from $x = 0$ to be $= \alpha\beta$. If you prefer a finite formula, you have

$$[98] \quad F(\alpha, \beta, \gamma, x)F(-\alpha, -\beta, 1-\gamma, x) - \frac{\alpha\beta}{\gamma - \gamma\gamma}F(\alpha+1, \beta+1, \gamma+1, x)F(1-\alpha, 1-\beta, 2-\gamma, x) = 1$$

Transforming the four functions according to formula 82 here and then writing $\gamma - \alpha$, $\gamma - \beta$ for α, β, you will have

$$[99] \quad (1-x)F(\alpha, \beta, \gamma, x)F(1-\alpha, 1-\beta, 1-\gamma, x) - \frac{(\gamma-\alpha)\cdot(\gamma-\beta)}{\gamma - \gamma\gamma}xF(\alpha, \beta, \gamma+1, x)F(1-\alpha, 1-\beta, 2-\gamma, x) = 1$$

Certain special Theorems

53.

All relations found up to this point are most general in that regard, that the elements α, β, γ are not restricted by any conditions. But furthermore we

find many others, which require special relations among the elements α, β, γ: Without any doubt still many of them are hidden, and those, we will give here, can maybe be derived from higher principles in the future.
First, in equation 80 let us $x = \frac{4y}{(1+y)^2}$, whence

$$dx = dy \cdot \frac{4(1-y)}{(1+y)^3}$$

and hence

$$\frac{dP}{dx} = \frac{dP}{dy} \cdot \frac{(1+y)^3}{4(1-y)}$$
$$\frac{ddP}{dx} = d\frac{dP}{dy} \cdots \frac{(1+y)^3}{4(1-y)} + \frac{dP}{dy} \cdot \frac{(2-y)(1+y)^2}{2(1-y)^2} dy$$
$$\frac{ddP}{dx^2} = \frac{ddP}{dy^2} \cdot \frac{(1+y)^6}{16(1-y)^2} + \frac{(2-y)(1+y)^5}{8(1-y)^3} \cdot \frac{dP}{dy}$$

Hence that equation becomes

$$\begin{aligned} 0 = {} & \alpha\beta P \\ & -(\gamma(1+y)^2 - 4(\alpha+\beta+1)y)\frac{1+y}{4(1-y)} \cdot \frac{dP}{dy} \\ & -\frac{ddP}{dy^2} \cdot \frac{y(1+y)^2}{4} - \frac{y(2-y)(1+y)}{2(1-y)} \cdot \frac{dP}{dy} \end{aligned}$$

or

$$0 = 4\alpha\beta(1-y)P$$

$$-(\gamma(1+y)^2 - 4(\alpha+\beta+1)y - 2y(2-y)(1+y))\frac{dP}{dy}$$

$$-(y-yy)(1+y)^2\frac{ddP}{dy^2}$$

$$0 = 4\alpha\beta(1-y)P$$

$$-(1+y)(\gamma - (4\alpha+4\beta-2\gamma)y + (\gamma-2)yy)\frac{dP}{dy}$$

$$-(1+y)^2(y-yy)\frac{ddP}{dy^2}$$

Setting $P = (1+y)^{2\alpha}Q$, hence one deduces

$$\text{I.} \qquad 0 = 2\alpha(2\beta - \gamma + (2\alpha+1-\gamma)y)Q$$

$$-(\gamma - (4\beta-2\gamma)y + (\gamma-4\alpha-2)yy)\frac{dQ}{dy}$$

$$-(y-yy)(1+y)\frac{ddQ}{dy^2}$$

Now assuming that $\beta = \alpha + \frac{1}{2}$, this equation takes on the following form

$$0 = 2\alpha(2\alpha+1-\gamma)Q$$

$$-(\gamma - (4\alpha+2-\gamma)y)\frac{dQ}{dy}$$

$$-(y-yy)\frac{ddQ}{dy^2}$$

whose integral is

$$Q = F(2\alpha, 2\alpha + 1 - \gamma, \gamma, y)$$

so that it results

$$[100] \qquad (1+y)^{2\alpha} F(2\alpha, 2\alpha + 1 - \gamma, \gamma, y) = F\left(\alpha, \alpha + \frac{1}{2}, \gamma, \frac{4y}{(1+y)^2}\right)$$

54.

If instead of the relation $\beta = \alpha + \frac{1}{2}$ we take $\gamma = 2\beta$, equation I of the prec. art. becomes

$$\begin{aligned} 0 = {} & 2\alpha(2\alpha + 1 - 2\beta)yQ \\ & -(2\beta - (4\alpha + 2 - 2\beta)yy)\frac{dQ}{dy} \\ & -y(1-yy)\frac{ddQ}{dy^2} \end{aligned}$$

Now, setting $yy = z$,

$$\begin{aligned} \frac{dQ}{dy} &= 2y\frac{dQ}{dz} \\ \frac{ddQ}{dy^2} &= 4yy\frac{ddQ}{dz^2} + \frac{2dQ}{dz} \quad \text{and hence} \end{aligned}$$

$$\begin{aligned} 0 = {} & \alpha\left(\alpha + \frac{1}{2} - \beta\right)Q \\ & -\left(\beta + \frac{1}{2} - \left(2\alpha + \frac{3}{2} - \beta\right)z\right)\frac{dQ}{dz} \\ & -(z - zz)\frac{ddQ}{dz^2} \end{aligned}$$

since whose integral is

$$Q = F\left(\alpha, \alpha + \frac{1}{2} - \beta, \beta + \frac{1}{2}, z\right)$$

we have

$$[101] \quad (1+y)^{2\alpha} F\left(\alpha, \alpha+\frac{1}{2}-\beta, \beta+\frac{1}{2}, yy\right) = F\left(\alpha, \beta, 2\beta, \frac{4y}{(1+y)^2}\right)$$

55.

Secondly, let us set $x = 4y - 4yy$, whence

$$dx = 4dy(1-2y)$$

$$\frac{dP}{dx} = \frac{dP}{dy} \cdot \frac{1}{4(1-2y)}$$

$$\frac{ddP}{dx^2} = \frac{ddP}{dy^2} \cdot \frac{1}{16(1-2y)^2} + \frac{dP}{dy} \cdot \frac{1}{8(1-2y)^3}$$

whence equation 80 becomes

$$\begin{aligned} 0 = {} & 4\alpha\beta P \\ & -(\gamma - (4\alpha+4\beta+2)y + (4\alpha+4\beta+2)yy)\frac{1}{(1-2y)} \cdot \frac{dP}{dy} \\ & -(y-yy)\frac{ddP}{dy^2} \end{aligned}$$

For it to be possible to cancel the fraction in the second terms, one has to set $\gamma = \alpha + \beta + \frac{1}{2}$, whence it will result

$$\begin{aligned} 0 = {} & 4\alpha\beta P \\ & -\left(\alpha+\beta+\frac{1}{2}-(2\alpha+2\beta+1)y\right)\frac{dP}{dy} \\ & -(y-yy)\frac{ddP}{dy^2} \end{aligned}$$

whose integral is

$$P = F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}, y\right)$$

whence we have

$$[102] \qquad F\left(\alpha, \beta, \alpha+\beta+\frac{1}{2}, 4y-4yy\right) = F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}, y\right)$$

If we would change y into $1-y$ in this equation, hence it would result

$$F\left(\alpha, \beta, \alpha+\beta+\frac{1}{2}, 4y-4yy\right) = F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}, 1-y\right)$$

whence this paradox seems to follow

$$F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}, y\right) = F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}, 1-y\right)$$

which equation is certainly false. To resolve this paradox, one has to remember that one has to distinguish the two meanings of the letter F carefully, namely that it *either* represents a function, whose nature is expressed by the differential equation 80, *or* only the sum of the series. The second, as long as the fourth element lies between -1 and 1, will always exhibit a completely determined quantity, or one has to be careful not to exceed these limits, since otherwise no meaning remains at all. The first meaning on the other hand represents a general function, which is certainly always changed according to the law of continuity, if the fourth element is changed in a continuous flux, whether you attribute real or imaginary values to it, if you just always avoid 0 and 1. Therefore, it is plain, that in the second sense for equal values of the fourth element (having passed through imaginary quantities) the function can take on different values, from which the one, which the *series* F represents, is just one of possibly many, and hence it is not contradictory, that, while *a certain* value of the function $F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}, 4y-4yy\right)$ is equal to $F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}, y\right)$, *another* value becomes $= F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}, 1-y\right)$ and it would be as absurd to conclude the equality of these values as if you would conclude the $30^\circ = 150^\circ$ from $\arcsin\frac{1}{2} = 30^\circ$ and $\arcsin\frac{1}{2} = 150^\circ$ - If we assume the characteristic F in a less general meaning, of course, that it only represents the sum of the series F, the arguments, by which we found equation 102, necessarily yield that y can only grow from 0 so far until it will be $x = 1$, i. e. until $y = \frac{1}{2}$. But in this point the *continuity* of the series $P = F(\alpha, \beta, \alpha+\beta+\frac{1}{2}, 4y-4yy)$ would be interrupted, since $\frac{dP}{dx}$ obviously instantly jumps from a (finite) positive value to a negative value. Therefore, in this meaning equation 102 does not admit an extension beyond the limits $y = \frac{1}{2} - \sqrt{\frac{1}{2}}$ to $y = \frac{1}{2}$. If you prefer, you can exhibit the same equation also this way

$$[103] \qquad F\left(\alpha, \beta, \alpha+\beta+\frac{1}{2}, x\right) = F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}, \frac{1-\sqrt{1-x}}{2}\right)$$

or this way

$$[104] \quad F\left(\alpha, \beta, \alpha+\beta+\frac{1}{2}, 1-x\right) = F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}, \frac{1-\sqrt{x}}{2}\right)$$

whence it follows as a corollary

$$[105] \qquad F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}, \frac{1}{2}\right) = \frac{\Pi\left(\alpha+\beta-\frac{1}{2}\right)\Pi\left(-\frac{1}{2}\right)}{\Pi\left(\alpha-\frac{1}{2}\right)\Pi\left(\beta-\frac{1}{2}\right)} = \frac{\Pi\left(\alpha+\beta-\frac{1}{2}\right)\sqrt{\pi}}{\Pi\left(\alpha-\frac{1}{2}\right)\Pi\left(\beta-\frac{1}{2}\right)}$$

56.

From the application of formula 87 to equation 104 it follows

$$[106] \quad F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}, \frac{1-\sqrt{x}}{2}\right) = A\left(\alpha, \beta, \frac{1}{2}, x\right) + B\sqrt{x}\cdot F\left(\alpha+\frac{1}{2}, \beta+\frac{1}{2}, \frac{3}{2}, x\right)$$

whence it is plain that the series

$$F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}, \frac{1-t}{2}\right)$$

can be exhibited by the series

$$A + Bt + A\frac{\alpha\cdot\beta}{1\cdot\frac{1}{2}}\cdot tt + B\frac{\left(\alpha+\frac{1}{2}\right)\cdot\left(\beta+\frac{1}{2}\right)}{1\cdot\frac{3}{2}}t^3 + A + \frac{\alpha\cdot(\alpha+1)\cdot\beta\cdot(\beta+1)}{1\cdot 2\cdot\frac{1}{2}\cdot\frac{3}{2}}t^4 + \text{etc.}$$

for the sake of brevity setting

$$A = \frac{\Pi\left(\alpha+\beta-\frac{1}{2}\right)\Pi\left(-\frac{1}{2}\right)}{\Pi\left(\alpha-\frac{1}{2}\right)\Pi\left(\beta-\frac{1}{2}\right)}, \quad B = \frac{\Pi\left(\alpha+\beta-\frac{1}{2}\right)\Pi\left(-\frac{3}{2}\right)}{\Pi\left(\alpha-1\right)\Pi\left(\beta-1\right)}$$

Hence it is possible to conclude that

$$[107] \quad F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}, \frac{1+\sqrt{x}}{2}\right) = A\left(\alpha, \beta, \frac{1}{2}, x\right) + B\sqrt{x}\cdot F\left(\alpha+\frac{1}{2}, \beta+\frac{1}{2}, \frac{3}{2}, x\right)$$

Hence, if this conclusion does not seem legitimate for anyone, (to deduce which would without a doubt not be difficult) we could get to the same equation in the following way. From equation 87

$$F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}, \frac{1+\sqrt{x}}{2}\right) = CF\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}, \frac{1-\sqrt{x}}{2}\right)$$
$$+D\left(\frac{1-x}{4}\right)^{\frac{1}{2}-\alpha-\beta} F\left(1-2\alpha, 1-2\beta, \frac{3}{2}-\alpha-\beta, \frac{1-\sqrt{x}}{2}\right)$$

for the sake of brevity setting

$$C = \frac{\Pi\left(\alpha+\beta-\frac{1}{2}\right)\Pi\left(-\frac{1}{2}-\alpha-\beta\right)}{\Pi\left(\alpha-\beta-\frac{1}{2}\right)\Pi\left(\beta-\alpha-\frac{1}{2}\right)}, \qquad D = \frac{\Pi\left(\alpha+\beta-\frac{1}{2}\right)\Pi\left(\alpha+\beta-\frac{3}{2}\right)}{\Pi\left(2\alpha-1\right)\Pi\left(2\beta-1\right)}$$

But from equation 104 one easily deduces

$$F(\left(1-2\alpha, 1-2\beta, \frac{3}{2}-\alpha-\beta, \frac{1-\sqrt{x}}{2}\right) = F\left(\frac{1}{2}-\alpha, \frac{1}{2}-\beta, \frac{3}{2}-\alpha-\beta, 1-x\right)$$
$$= EF\left(\frac{1}{2}-\alpha, \frac{1}{2}-\beta, \frac{1}{2}, x\right) + G\sqrt{x}\cdot F\left(1-\alpha, 1-\beta, \frac{3}{2}, x\right)$$

for the sake of brevity setting

$$E = \frac{\Pi\left(\frac{1}{2}-\alpha-\beta\right)\Pi\left(-\frac{1}{2}\right)}{\Pi\left(-\alpha\right)\Pi\left(-\beta\right)}, \quad G = \frac{\Pi\left(\frac{1}{2}-\alpha-\beta\right)\Pi\left(-\frac{3}{2}\right)}{\Pi\left(-\frac{1}{2}-\alpha\right)\Pi\left(-\frac{1}{2}-\beta\right)}$$

Hence by equation 82 it follows again

$$F\left(1-2\alpha, 1-2\beta, \frac{3}{2}-\alpha-\beta, \frac{1-\sqrt{x}}{2}\right)$$
$$E(1-x)^{\alpha+\beta-\frac{1}{2}}F\left(\alpha, \beta, \frac{1}{2}, x\right) + G\sqrt{x}\cdot(1-x)^{\alpha+\beta-\frac{1}{2}}F\left(\alpha+\frac{1}{2}, \beta+\frac{1}{2}, \frac{3}{2}, x\right)$$

Having substituted these, setting

$$AC + DE2^{2\alpha+2\beta-1} = M, \qquad BC + DG2^{2\alpha+2\beta-1} = N$$

one concludes

$$F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}, \frac{1+\sqrt{x}}{2}\right) = MF\left(\alpha, \beta, \frac{1}{2}, x\right) + N\sqrt{x}\cdot F\left(\alpha+\frac{1}{2}, \beta+\frac{1}{2}, \frac{3}{2}, x\right)$$

whose *form* agrees with equation 107. We certainly could derive $M = A$, $N = -B$ from the nature of the function Π only, since by eqs. 55, 56 it is easily demonstrated to be

$$C = \frac{\cos(\alpha-\beta)\pi}{\cos(\alpha+\beta)\pi}, \quad \frac{DE2^{2\alpha+2\beta-1}}{A} = -\frac{2\sin\alpha\pi\sin\beta\pi}{\cos(\alpha+\beta)\pi}, \quad \frac{DG2^{2\alpha+2\beta-1}}{B} = -\frac{2\cos\alpha\pi\cos\beta\pi}{\cos(\alpha+\beta)\pi}$$

but this work is not even necessary. For, setting $x = 0$, it is plain, that is has to be

$$M = F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}, \frac{1}{2}\right) = A$$

but differentiating that equation it results

$$x^{-\frac{1}{2}}\frac{\alpha\beta}{\alpha+\beta+\frac{1}{2}}F\left(2\alpha, 2\beta+1, \alpha+\beta+\frac{3}{2}, \frac{1+\sqrt{x}}{2}\right)$$
$$= 2\alpha\beta MF\left(\alpha+1, \beta+1, \frac{3}{2}, x\right)$$
$$+\frac{2}{3}\left(\alpha+\frac{1}{2}\right)\left(\beta+\frac{1}{2}\right)N\sqrt{x}\cdot F\left(\alpha+\frac{3}{2}, \beta+\frac{3}{2}, \frac{5}{2}, x\right) + \frac{1}{2}Nx^{-\frac{1}{2}}F\left(\alpha+\frac{1}{2}, \beta+\frac{1}{2}, \frac{3}{2}, x\right)$$

whence setting $x = 0$ it results

$$
\begin{aligned}
N &= \frac{2\alpha\beta}{\alpha+\beta+\frac{1}{2}} F\left(2\alpha+1, 2\beta+1, \alpha+\beta+\frac{3}{2}, \frac{1}{2}\right) \\
&= \frac{2\alpha\beta}{\alpha+\beta+\frac{1}{2}} \cdot \frac{\Pi\left(\alpha+\beta+\frac{1}{2}\right)\Pi\left(-\frac{1}{2}\right)}{\Pi\alpha\Pi\beta} \\
&= \frac{\Pi\left(\alpha+\beta-\frac{1}{2}\right)\Pi\left(-\frac{1}{2}\right)}{\Pi(\alpha-1)\Pi(\beta-1)} = B
\end{aligned}
$$

57.

From the combination of the equations 106, 107 we therefore have

[108]
$$
\begin{aligned}
&2AF\left(\alpha, \beta, \frac{1}{2}, x\right) \\
&= F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}, \frac{1-\sqrt{x}}{2}\right) + F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}, \frac{1+\sqrt{x}}{2}\right)
\end{aligned}
$$

[109]
$$
\begin{aligned}
&2B\sqrt{x} \cdot F\left(\alpha+\frac{1}{2}, \beta+\frac{1}{2}, \frac{3}{2}, x\right) \\
&= F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}, \frac{1-\sqrt{x}}{2}\right) - F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}, \frac{1+\sqrt{x}}{2}\right)
\end{aligned}
$$

In equation 109 changing α into $\alpha-\frac{1}{2}$, β into $\beta-\frac{1}{2}$, you will easily see, that hence it results

[110]
$$
\begin{aligned}
&\frac{\left(\alpha-\frac{1}{2}\right)\cdot\left(\beta-\frac{1}{2}\right)}{\alpha+\beta-\frac{1}{2}} A\sqrt{x} \cdot F\left(\alpha, \beta, \frac{3}{2}, x\right) \\
&= F\left(2\alpha-1, 2\beta-1, \alpha+\beta-\frac{1}{2}, \frac{1+\sqrt{x}}{2}\right) - F\left(2\alpha-1, 2\beta-1, \alpha+\beta-\frac{1}{2}, \frac{1-\sqrt{x}}{2}\right)
\end{aligned}
$$