On quadrouple periodic functions of two variables on which the theory of the Abelian Transcendents is based *

Carl Gustav Jacob Jacobi

1.

In the *Fundamenta nova theoriae functionum ellipticarum* I noted (§ 19.) that two periods contain every periodicity possible in analysis. Let us examine this subject more accurately in the following.
I call the function $\lambda(u)$ *periodic*, if there is a constant i of such a kind that for every arbitrary value of u

$$\lambda(u+i) = \lambda(u).$$

I call the constant i the *index* of the function. But it is plain that from one index innumerable others result, since any positive or negative multiple of it is also an index. From those I call the one, of which no part is the index of a function, the index *proper* index of the function. In the elements one considers the periodic function $\sin(u)$, e^u, whose proper indices are 2π, $2\pi\sqrt{-1}$, respectively.
Now let us put, a first example for which is shown in the elliptic functions,

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that the function $\lambda(u)$ enjoys two periods, which can not be reduced to one. Let their indices be i, i', whence these two equations hold at the same time:

$$\lambda(u+i) = \lambda(u), \qquad \lambda(u+i') = \lambda(u),$$

from which, while m, m' denote arbitrary positive or negative numbers, this more general one follows:

$$\lambda(u + mi + m'i) = \lambda(u)$$

or $mi + m'i'$ will also be an index. And first it is plain *that the indices i, i' must be incommensurable.* For, if Δ is their greatest common factor, one can put

$$i = m\Delta, \quad i' = m'\Delta,$$

while m, m' are mutually prime integers. Hence we can determine other numbers n, n' of such a kind that

$$mn + m'n' = 1.$$

Having done this, one has the index

$$ni + n'i' = \Delta,$$

since from which one index the indices i, i', as its multiples, result, we see, *if the indices of the two periods the function enjoys are commensurable, that the two periods reduce to one, whose index is their greatest common divisor.*

Since it is clear from the preceding that *the quotient of two indices*, which do not result from a single one, cannot be a rational quantity, it is easily seen that it *can also not be a real quantity.* For, let

$$i = z\Delta, \quad i' = z'\Delta,$$

while z, z' denote real incommensurable quantities; it is possible to find positive or negative integer numbers m, m' of such a kind that

$$mz + m'z' = z''$$

becomes smaller than a given quantity. Having constituted all this, it will be

$$\lambda(u + mi + m'i') = \lambda(u + z''\Delta) = \lambda(u),$$

whence the function $\lambda(u)$ would have an index smaller than any given quantity and nevertheless not vanishing. And this is absurd.
From the preceding it follows, if the indices of the periods, which can not be reduced to one, are imaginary quantities,

$$i = a + b\sqrt{-1}, \qquad i' = a' + b\sqrt{-1},$$

while a, b, a', b' denote real quantities, that one can never have:

$$ab' - a'b = 0.$$

For, then the quotient of the indices

$$\frac{a' + b'\sqrt{-1}}{a + b\sqrt{-1}} = \frac{a'}{a} = \frac{b'}{b}$$

would be a real quantity.

2.

Now let us examine, whether a function can enjoy three periods, which can not be constructed from two others. Let the indices of three periods of such a kind be

$$i = a + b\sqrt{-1}, \qquad i' = a' + b'\sqrt{-1}, \qquad i'' = a'' + b''\sqrt{-1},$$

while a, b, a', b', a'', b'' denote real quantities. From the preceding we assume that none of the three quantities

$$a'b'' - a''b', \qquad a''b - ab'', \qquad ab' - a'b$$

vanishes. For, otherwise either two periods would reduce to one, which contradicts the assumption, or the function would have an index smaller than any given quantity which does nevertheless not vanish, what would be absurd.
And first I observe that those quantities can not divide the same number.
For, let us put that

$$a'b'' - a''b' : a''b - ab'' : ab' - a'b = m : m' : m'',$$

while m, m', m'' denote integer numbers, which we assume to have no common factor. It will be:

$$ma + m'a' + m''a'' = 0$$
$$mb + m'b' + m''b'' = 0,$$

and hence also:

$$mi + m'i' + m''i'' = 0.$$

Let f be the greatest common divisor of m', m'', which must be prime to m, since the three numbers m, m', m'' are not divisible by the same number;

$$\frac{mi}{f} = -\left[\frac{m'}{f} \cdot i' + \frac{m''}{f} \cdot i''\right]$$

will also be an index of the function. Now, since the indices i and $\frac{mi}{f}$ are commensurable, and their greatest common divisor is $\frac{i}{f}$, also $\frac{i}{f}$ will also be an index, as it was demonstrated in § 1. Further, choose numbers n', n'' of such a kind that

$$\frac{m'}{f} \cdot n' + \frac{m''}{f} \cdot n'' = 1;$$

I say, that three periods are composed of two, whose indices are

$$\frac{i}{f} \quad \text{and} \quad n'i' - n'i'',$$

from which both the index i and the remaining indices i', i'' are constructed, of course. For, one has

$$-mn' \cdot \frac{i}{f} + \frac{m''}{f}\,(n''i' - n'n'') = n'\left[\frac{m'}{f}i' + \frac{m''}{f}i''\right] + \frac{m''}{f}(n''i' - n'i'') = i'$$
$$-mn'' \cdot \frac{i}{f} + \frac{m'}{f}\,(n''i' - n'n'') = n''\left[\frac{m'}{f}i' + \frac{m''}{f}i''\right] - \frac{m''}{f}(n''i' - n'i'') = i''.$$

Hence, if the three quantities

$$a'b'' - a''b', \qquad a''b - ab'', \qquad ab' - a'b$$

are of the same nature as integer numbers, or, what is the same, *if, while m, m', m'' denote integer numbers, there is a relation of such a kind among the three indices i, i', i'':*

$$mi + m'i' + m''i'' = 0,$$

three periods can be constructed from two, or the function is only double periodic. Next, I observe, *while α, α', α'' denote integer numbers, that an equation on this kind can not hold:*

$$\alpha(a'b'' - a''b') + \alpha'(a''b - ab'') + \alpha''(ab' - a'b) = 0.$$

For, from the arbitrarily taken six integer numbers

$$\beta, \quad \beta', \quad \beta''; \qquad \gamma, \quad \gamma', \quad \gamma'',$$

let us set:

$$\begin{aligned}
u \ &= (\gamma'\alpha'' - \gamma''\alpha')a + (\gamma''\alpha - \gamma\alpha'')a' + (\gamma\alpha' - \gamma'\alpha)a'', \\
v \ &= (\alpha'\beta'' - \alpha''\beta')a + (\alpha''\beta - \alpha\beta'')a' + (\alpha\beta' - \alpha'\beta)a'', \\
u' &= (\gamma'\alpha'' - \gamma''\alpha')b + (\gamma''\alpha - \gamma\alpha'')b' + (\gamma\alpha' - \gamma'\alpha)b'', \\
v' &= (\alpha'\beta'' - \alpha''\beta')b + (\alpha''\beta - \alpha\beta'')b' + (\alpha\beta' - \alpha'\beta)b'';
\end{aligned}$$

hence the following expressions will also be the indices of the propounded function:

$$u + v\sqrt{-1}, \qquad u' + v'\sqrt{-1}.$$

Now, if one sets

$$\varepsilon = \alpha(\beta'\gamma'' - \beta''\gamma') + \alpha'(\beta''\gamma - \beta\gamma'') + \alpha''(\beta\gamma' - \beta'\gamma),$$

one finds:

$$uv'u'v = \varepsilon[\alpha(a'b'' - a''b') + \alpha'(a''b - ab'') + \alpha''(ab' - a'b)].$$

Hence, if the expression in the brackets vanishes, one has:

$$uv' - u'v = 0.$$

But we saw in § 1 that this equation can only hold, if the indices

$$u + v\sqrt{-1}, \quad u' + v'\sqrt{-1},$$

are commensurable or result from one index. In this case, while f, f' denote integers, one can set

$$f(u + v\sqrt{-1}) - f'(u' + v'\sqrt{-1}) = 0,$$

which equation, having substituted the values of u, v, u', v', takes on this form:

$$mi + m'i' + m''i'' = 0,$$

where m, m', m'' are integers; we demonstrated that this can not be true.

3.

Having prepared these things, I will now demonstrate, *if three periods can not be reduced to two, that one can always determine integer numbers m, m', m'' of such a kind that each of both expressions*

$$ma + m'a' + m''a'',$$
$$mb + m'b' + m''b''$$

become smaller than any given quantity at the same time, or the propounded function has an index smaller than any given quantity but nevertheless not vanishing.
For the sake of brevity, I put:

$$a'b'' - a''b' = A, \qquad a''b - ab'' = A', \qquad ab' - ab = A'',$$

whence

$$aA + a'A' + a''A'' = 0, \quad bA + b'A' + b''A'' = 0.$$

Further, while α, α', α'' denote integer numbers, I put:

$$\frac{\alpha A'}{A} - \alpha' = \Delta, \quad \frac{\alpha A''}{A} - \alpha'' = \Delta',$$

whence

$$\alpha a + \alpha' a' + \alpha'' a'' = -[a'\Delta + a''\Delta'],$$
$$\alpha b + \alpha' b' + \alpha'' b'' = -[b'\Delta + b''\Delta'].$$

Now the numbers α, α' can be determined in such a way that Δ becomes smaller than any given quantity. Further, having determined α, α', a third number α'' can be assumed in such a way that, not having taken into account the signs,

$$\Delta' < \frac{1}{2}.$$

Having determined α, α', α'' this way, the preceding expressions become absolutely smaller than $\frac{1}{2}a''$ and $\frac{1}{2}b''$, respectively. Hence *given the quantities a, a', a'' and b, b', b'', it is always possible to determine integer numbers α, α', α'' of such a kind, that, having put*

$$a''' = \alpha a + \alpha' a' + \alpha'' a'', \qquad b''' = \alpha b + \alpha' b' + \alpha'' b'',$$

it simultaneously is

$$a''' < \frac{1}{2}a'', \qquad b''' < \frac{1}{2}b'',$$

not having taken into account the signs.
Now successively put:

$$\begin{aligned}
\beta a' + \beta' a'' + \beta'' a''' &= a^{\mathrm{IV}}, & \beta b' + \beta' b'' + \beta'' b''' &= b^{\mathrm{IV}},\\
\gamma a'' + \gamma' a''' + \gamma'' a^{\mathrm{IV}} &= a^{\mathrm{V}}, & \gamma b'' + \gamma' b''' + \gamma'' b^{\mathrm{IV}} &= b^{\mathrm{V}},\\
\delta a''' + \delta' a^{\mathrm{IV}} + \delta'' a^{\mathrm{V}} &= a^{\mathrm{VI}}, & \delta b''' + \delta' b^{\mathrm{IV}} + \delta'' b^{\mathrm{V}} &= b^{\mathrm{VI}},\\
&\dots & &\dots
\end{aligned}$$

The coefficients of these equations β, γ etc. β', γ' etc. β'', γ'' etc. can be assumed as numbers of such a kind from the preceding that, not taking into account the signs, we simultaneously have:

$$\begin{aligned}
&a^{\mathrm{IV}} < \tfrac{1}{2}a''', \quad a^{\mathrm{V}} < \tfrac{1}{2}a^{\mathrm{IV}}, \quad a^{\mathrm{VI}} < \tfrac{1}{2}a^{\mathrm{V}} \quad \text{etc.}\\
&b^{\mathrm{IV}} < \tfrac{1}{2}b''', \quad b^{\mathrm{V}} < \tfrac{1}{2}b^{\mathrm{IV}}, \quad b^{\mathrm{VI}} < \tfrac{1}{2}b^{\mathrm{V}} \quad \text{etc.}
\end{aligned}$$

Hence it is clear that the terms of the two series

$$\begin{aligned}
&a'', \quad a''', \quad a^{\mathrm{IV}}, \quad a^{\mathrm{V}}, \quad a^{\mathrm{VI}}, \quad \dots\\
&b'', \quad b''', \quad b^{\mathrm{IV}}, \quad b^{\mathrm{V}}, \quad b^{\mathrm{VI}}, \quad \dots
\end{aligned}$$

if they are continued sufficiently far, become smaller than any given quantity. Let two terms corresponding to each other of these two series $a^{(n)}$, $b^{(n)}$ be smaller than a given quantity. If you consider the formation of the equations and how those quantities depend on the preceding ones, it is easily clear, that they can be expressed in terms of a, a', a'' and b, b', b'' by means of the equations

$$a^{(n)} = ma + m'a' + m''a'',$$
$$b^{(n)} = mb + m'b' + m''b'',$$

in which the coefficients m, m', m'' are integer numbers. Further, it is plain that these coefficient are the same in both equations, since the quantities $a^{(n)}$, $b^{(n)}$ depend on the preceding terms in the same way. Hence it is proved, what was propounded, that one can determine integer numbers (positive or negative) m, m', m'' of such a kind that both expressions

$$ma + m'a' + m''a'',$$
$$mb + m'b' + m''b''$$

become smaller than any given quantity.
The given algorithm, by which the terms of the two series are found one after the other, still works, if in the one series a term vanishes. For, then the next term does certainly not become smaller than the half of the preceding, since a term smaller than a vanishing term does not exist, if we consider only the absolute values. But it is easily seen, while a term of the one series vanishes, the next term can be rendered smaller than any given quantity. For the sake of an example let $a'' = 0$, one finds the next term

$$a''' = -a'\Delta,$$

where Δ can be rendered smaller than any certain given quantity. Therefore, using this term smaller than any given quantity, you will continue the algorithm, while also the terms of the other series become smaller than the given quantity. And the terms corresponding to each other can not vanish at the same time. For, if one has

$$a^{(n)} = 0, \quad \text{and} \quad b^{(n)} = 0$$

at the same time, there would be numbers m, m', m'', for which

$$a^{(n)} = ma + m'a' + m''a'' = 0$$
$$b^{(n)} = mb + m'b' + m''b'' = 0,$$

and hence also

$$mi + m'i' + m''i'' = 0.$$

We saw in § 2 that this is impossible, if one can not construct three periods from two.
Further, the given algorithm supposes that one never has

$$a^{(n)}b^{(n)} - a^{(n+1)}b^{(n)} = 0,$$

which is seen this way. For, let

$$a^{(n+1)} = pa + p'a' + p''a'', \quad b^{(n+1)} = pb + p'b' + p''b''.$$

It will be

$$0 = a^{(n)}b^{(n+1)} - a^{(n+1)}b^{(n)}$$
$$= (m'p'' - m''p')(a'b'' - a''b') + (m''p - mp'')(a''b - ab'') + (mp' - m'p)(ab' - a'b).$$

It was demonstrated in § 2 that this equation cannot hold.

4.

If we set

$$a^{(n)} + b^{(n)}\sqrt{-1} = i^{(n)},$$

it is plain that i''', i^{IV}, i^{V} will be indices of the propounded function. Therefore, we indicated a certain algorithm, by which, given three imaginary indices, an infinite series of indices is formed, whose real and imaginary part become smaller than any given quantity at the same time and nevertheless do not vanish. Hence it is shown for all cases, *if the propounded function enjoys three periods, that they can either be constructed from two or they have an index smaller than every given quantity.* Since this is absurd, *a triple periodic function does not exist.*

Therefore, in other words, we said that the whole periodicity can be constructed from two periods. But this only holds for functions of one variable. If you consider functions of several variables, two periods do not suffice at all. Examples of functions of several variables enjoying more than two periods are given by those functions, which I first considered in the short paper *de transcendentibus Abelianis* (Crelle Journal Bd. 9 p. 394). But this most important subject is to be considered with more attention here.
Let

$$u = C + \frac{2A}{\pi} \cdot \varphi + A_1 \sin 2\varphi + A_2 \sin 4\varphi + A_3 \sin 6\varphi + \cdots$$

be a series, convergent for all real values of φ. Let us set that x is a completely determined function of $\sin^2 \varphi$, e.g., a rational function. Having changed φ into $\varphi + \pi$, x will not be changed, but u will be changed into $u + 2A$. Hence, having put

$$x = \lambda(u),$$

we have

$$\lambda(u + 2A) = \lambda(u).$$

Therefore, $\lambda(u)$ will be a periodic function, and its index will be $2A$.
Now let us consider an integral of this kind:

$$u = \int\limits_0^x \frac{(\alpha + \beta x)dx}{\sqrt{x(1-x)(1-\varkappa^2 x)(1-\lambda^2 x)(1-\mu^2 x)}} = \int\limits_0^x \frac{(\alpha + \beta x)dx}{\sqrt{X}},$$

while $\varkappa^2$, λ^2, μ^2 are real, positive quantities smaller than 1. Let us examine the values of this integral it takes on, while x grows through the real values from $-\infty$ to $+\infty$. Let $\varkappa^2 < \lambda^2 < \mu^2$: Let us distinguish six intervals, in which x can lie:

$$1.\quad -\infty \cdots 0 \qquad 2.\quad 0 \cdots 1, \qquad 3.\quad 1 \cdots \frac{1}{\varkappa^2},$$

$$4.\quad -\frac{1}{\varkappa^2} \cdots \frac{1}{\lambda^2}, \qquad 5.\quad \frac{1}{\lambda^2} \cdots \frac{1}{\mu^2}, \qquad 6.\quad \frac{1}{\mu^2} \cdots \infty.$$

For the first, third and fifth interval the value of x will be negative, for the second, fourth and sixth it will be positive. Now let us ask, how for those single intervals x is expressed in terms of $\sin^2\varphi$ in such a way that the propounded integral u can be expanded into a convergent infinite series of the assumed form

$$u = C + \frac{2A}{\pi}\cdot\varphi + A_1\sin 2\varphi + A_2\sin 4\varphi + A_3\sin 6\varphi + \cdots.$$

After this and having put $x = \lambda(u)$, $\lambda(u)$ will be a periodic function with index $2A$, where

$$A = \int\limits_0^{\frac{\pi}{2}} \frac{du}{d\varphi}\cdot d\varphi.$$

1°. If a negative value corresponds to x, put

$$(1.)\qquad x = \frac{-1}{\mu^2\tan^2\varphi};$$

while x increases through the negative values from $-\infty$ to 0, φ grows from 0 to $\frac{\pi}{2}$. After the substitution and having, for the sake of brevity, put

$$\varkappa'^2 = 1-\varkappa,\quad \lambda'^2 = 1-\lambda^2,\quad \mu'^2 = 1-\mu^2,$$

one finds:

$$\int\limits_{-\infty}^{x}\frac{(\alpha+\beta x)dx}{\sqrt{-X}} = \frac{2}{\varkappa\lambda}\int\limits_0^{\varphi}\frac{[(\alpha\mu^2+\beta)\sin^2\varphi-\beta]d\varphi}{\sqrt{(1-\mu'^2\sin^2\varphi)\left(1-\frac{\varkappa^2-\mu^2}{\varkappa^2}\sin^2\varphi\right)\left(1-\frac{\lambda^2-\mu^2}{\lambda^2}\sin^2\varphi\right)}}.$$

Hence having put

$$u_1 = \int\limits_{-\infty}^{0}\frac{(\alpha+\beta x)dx}{\sqrt{-X}} = \frac{2}{\varkappa\lambda}\int\limits_0^{\frac{\pi}{2}}\frac{[(\alpha\mu^2+\beta)\sin^2\varphi-\beta]d\varphi}{\sqrt{(1-\mu'^2\sin^2\varphi)\left(1-\frac{\varkappa^2-\mu^2}{\varkappa^2}\sin^2\varphi\right)\left(1-\frac{\lambda^2-\mu^2}{\lambda^2}\sin^2\varphi\right)}},$$

it will be

$$u = \frac{u_1}{\sqrt{-1}} + \frac{2\sqrt{-1}}{\varkappa\lambda} \int\limits_0^\varphi \frac{[(\alpha\mu^2 + \beta)\sin^2\varphi - \beta]d\varphi}{\sqrt{(1 - \mu'^2 \sin^2\varphi)\left(1 - \frac{\varkappa^2 - \mu^2}{\varkappa^2}\sin^2\varphi\right)\left(1 - \frac{\lambda^2 - \mu^2}{\lambda^2}\sin^2\varphi\right)}}.$$

Now I observe that an integral of this kind

$$\int\limits_0^\varphi \frac{(m + n\sin^2\varphi)d\varphi}{\sqrt{(1 - p^2\sin^2\varphi)(1 - q^2\sin^2\varphi)(1 - r^2\sin^2\varphi)}},$$

if p^2, q^2, r^2 are real and smaller than 1, can always be expanded into a convergent series of the form

$$\frac{2A}{\pi} \cdot \varphi + A_1 \sin 2\varphi + A_2 \sin 4\varphi + A_3 \sin 6\varphi + \cdots,$$

where

$$A = \int\limits_0^{\frac{\pi}{2}} \frac{(m + n\sin^2\varphi)d\varphi}{\sqrt{(1 - p^2\sin^2\varphi)(1 - q^2\sin^2\varphi)(1 - r^2\sin^2\varphi)}}.$$

Hence u can be expanded into the propounded form; and

$$C = \frac{u_1}{\sqrt{-1}}, \quad A = u_1\sqrt{-1}.$$

Therefore, having put $x = \lambda(u)$, $\lambda(u)$ will be a periodic function, whose index is $2u_1\sqrt{-1}$, or it will be

$$\lambda(u + 2u_1\sqrt{-1}) = \lambda(u).$$

2°. If x lies between 0 and 1, I put:

$$(2.) \qquad x = \sin^2\varphi;$$

we have

$$u = 2\int\limits_0^\varphi \frac{[\alpha + \beta\sin^2\varphi]d\varphi}{\sqrt{(1 - \varkappa^2\sin^2\varphi)(1 - \lambda^2\sin^2\varphi)(1 - \mu^2\sin^2\varphi)}}.$$

Hence having put

$$u_2 = \int_0^1 \frac{(\alpha + \beta x)dx}{\sqrt{X}} = \int_0^{\frac{\pi}{2}} \frac{[\alpha + \beta \sin^2 \varphi]d\varphi}{\sqrt{(1 - \varkappa^2 \sin^2 \varphi)(1 - \lambda^2 \sin^2 \varphi)(1 - \mu^2 \sin^2 \varphi)}},$$

u can be expanded into the assigned form, whose first coefficients will be

$$C = 0, \qquad A = u_2.$$

Therefore, $2u_2$ will be another index of the function $x = \lambda(u)$, or it will also be

$$\lambda(u + 2u_2) = \lambda(u).$$

3°. Let x lie between 1 and $\frac{1}{\varkappa^2}$, I put

$$(3.) \qquad x = \frac{1}{\cos^2 \varphi + \varkappa^2 \sin^2 \varphi} = \frac{1}{1 - \varkappa'^2 \sin^2 \varphi}.$$

Since the integral is taken from 0 to x, I divide the interval into two others, the one from 0 to 1, the other from 1 to x. Having done this, after the substitution we find:

$$u = u_2 + \frac{2\sqrt{-1}}{\lambda' \mu'} \int_0^{\varphi} \frac{[\alpha + \beta - \alpha \varkappa'^2 \sin^2 \varphi]d\varphi}{\sqrt{(1 - \varkappa'^2 \sin^2 \varphi)\left(1 - \frac{\varkappa'^2}{\lambda'^2} \sin^2 \varphi\right)\left(1 - \frac{\varkappa'^2}{\mu'^2} \sin^2 \varphi\right)}}.$$

Hence having put

$$u_3 = \int_0^{\frac{1}{\varkappa^2}} = \frac{(\alpha + \beta x)dx}{\sqrt{-X}} = \frac{2}{\lambda' \mu'} \int_0^{\frac{\pi}{2}} \frac{[\alpha + \beta - \alpha \varkappa'^2 \sin^2 \varphi]d\varphi}{\sqrt{(1 - \varkappa'^2 \sin^2 \varphi)\left(1 - \frac{\varkappa'^2}{\lambda'^2} \sin^2 \varphi\right)\left(1 - \frac{\varkappa'^2}{\mu'^2} \sin^2 \varphi\right)}},$$

it is possible to expand u into a series of the propounded form, whose first coefficients will be

$$C = u_2, \qquad A = u_3\sqrt{-1}.$$

Hence the function $x = \lambda(u)$ also enjoys the index $2u_2\sqrt{-1}$, or it will also be

$$\lambda(u + 2u_3\sqrt{-1}) = \lambda(u).$$

4°. We proceed to the fourth interval, from $\frac{1}{\varkappa^2}$ to $\frac{1}{\lambda^2}$; if x lies in this interval, I put

$$(4.) \qquad x = \frac{\lambda'^2 \cos^2\varphi + \varkappa'^2 \sin^2\varphi}{\varkappa^2\lambda'^2 \cos^2\varphi + \lambda^2\varkappa'^2 \sin^2\varphi} = \frac{\lambda'^2 - (\varkappa^2 - \lambda^2)\sin^2\varphi}{\varkappa^2\lambda'^2 - (\varkappa^2 - \lambda^2)\sin^2\varphi},$$

having done which, while x grows from $\frac{1}{\varkappa^2}$ to $\frac{1}{\lambda^2}$, φ grows from 0 to $\frac{\pi}{2}$. After the substitution, the propounded integral goes over into this one

$$u = \int_0^x \frac{(\alpha + \beta x)dx}{\sqrt{X}} =$$

$$u_2 + u_3\sqrt{-1} - \frac{2}{\varkappa\lambda'^2\sqrt{\varkappa^2 - \mu^2}} \int_0^\varphi \frac{[\lambda'^2(\alpha x^2 + \beta) - (\varkappa^2 - \lambda^2)(\alpha + \beta)\sin^2\varphi]d\varphi}{\sqrt{\left(1 - \frac{\varkappa^2-\lambda^2}{\lambda'^2}\sin^2\varphi\right)\left(1 - \frac{\varkappa^2-\lambda^2}{\varkappa^2\lambda'^2}\sin^2\varphi\right)\left(1 - \frac{\mu'^2(\varkappa^2-\lambda^2)}{\lambda'^2(\varkappa^2-\mu^2)\sin^2\varphi}\right)}}.$$

Since this can again be expanded into the assigned form, having put

$$u_4 = \int_{\frac{1}{\varkappa^2}}^{\frac{1}{\lambda^2}} \frac{(\alpha + \beta x)dx}{\sqrt{X}} =$$

$$\frac{2}{\varkappa\lambda'^3\sqrt{\varkappa^2 - \mu^2}} \int_0^{\frac{\pi}{2}} \frac{[\lambda'^2(\alpha x^2 + \beta) - (\varkappa^2 - \lambda^2)(\alpha + \beta)\sin^2\varphi]d\varphi}{\sqrt{\left(1 - \frac{\varkappa^2-\lambda^2}{\lambda'^2}\sin^2\varphi\right)\left(1 - \frac{\varkappa^2-\lambda^2}{\varkappa^2\lambda'^2}\sin^2\varphi\right)\left(1 - \frac{\mu'^2(\varkappa^2-\lambda^2)}{\lambda'^2(\varkappa^2-\mu^2)\sin^2\varphi}\right)}},$$

one has the following first coefficients of the expansion:

$$C = u_2 + u_3\sqrt{-1}, \quad A = u_4;$$

hence the function $x = \lambda(u)$ has the index $2u_4$, or we also have:

$$\lambda(u + 2u_4) = \lambda(u).$$

5°. Fifthly, let x lie between $\frac{1}{\lambda^2}$ and $\frac{1}{\mu^2}$, in which case we set:

$$(5.)\qquad x=\frac{(\varkappa^2-\mu^2)\cos^2\varphi+(\varkappa^2-\lambda^2)\sin^2\varphi}{\lambda^2(\varkappa^2-\mu^2)\cos^2\varphi+\mu^2(\varkappa^2-\lambda^2)\sin^2\varphi}=\frac{\varkappa^2-\mu^2-(\lambda^2-\mu^2)\sin^2\varphi}{\lambda^2(\varkappa^2-\mu^2)-\varkappa^2(\lambda^2-\mu^2)\sin^2\varphi},$$

having done which, while x increases from $\frac{1}{\lambda^2}$ to $\frac{1}{\mu^2}$, φ grows from 0 to $\frac{\pi}{2}$. After the substitution we obtain

$$u=u_2+u_3\sqrt{-1}-u_4$$

$$-\frac{2\sqrt{-1}}{\lambda\lambda'\sqrt{(\varkappa^2-\mu^2)^3}}\int\limits_0^{\varphi}\frac{[(\varkappa^2-\mu^2)(\alpha\lambda+\beta)-(\lambda^2-\mu^2)(\alpha\varkappa^2+\beta)\sin^2\varphi]d\varphi}{\sqrt{\left(1-\frac{\varkappa^2(\lambda^2-\mu^2)}{\lambda^2(\varkappa^2-\mu^2)\sin^2\varphi}\right)\left(1-\frac{\lambda^2-\mu^2}{\varkappa^2-\mu^2}\sin^2\varphi\right)\left(1-\frac{\varkappa'^2(\lambda^2-\mu^2)}{\lambda'^2(\varkappa^2-\mu^2)}\sin^2\varphi\right)}}.$$

It is again plain that this expression can be expanded into the assigned from, and, having put

$$u_5=\int\limits_{\frac{1}{\lambda^2}}^{\frac{1}{\mu^2}}\frac{(\alpha+\beta x)dx}{\sqrt{-X}}$$

$$=\frac{2}{\lambda\lambda'\sqrt{(\varkappa^2-\mu^2)^3}}\int\limits_0^{\frac{\pi}{2}}\frac{[(\varkappa^2-\mu^2)(\alpha\lambda+\beta)-(\lambda^2-\mu^2)(\alpha\varkappa^2+\beta)\sin^2\varphi]d\varphi}{\sqrt{\left(1-\frac{\varkappa^2(\lambda^2-\mu^2)}{\lambda^2(\varkappa^2-\mu^2)\sin^2\varphi}\right)\left(1-\frac{\lambda^2-\mu^2}{\varkappa^2-\mu^2}\sin^2\varphi\right)\left(1-\frac{\varkappa'^2(\lambda^2-\mu^2)}{\lambda'^2(\varkappa^2-\mu^2)}\sin^2\varphi\right)}}$$

the first coefficients of the expansion will be

$$C=u_2+u_3\sqrt{-1}-u_4,\quad A=-u_5\sqrt{-1}.$$

Hence the function $x=\lambda(u)$ will also have the index $2u_5\sqrt{-1}$, or it will be

$$\lambda(u+2u_5\sqrt{-1})=\lambda(u).$$

6°. Finally, if x lies in the sixth interval, between $\frac{1}{\mu^2}$ and ∞, I put

$$(6.)\qquad x=\frac{1}{\mu^2}+\frac{\lambda^2-\mu^2}{\lambda^2\mu^2}\tan^2\varphi,$$

it results

$$u=u_2+u_3\sqrt{-1}-u_4-u_5\sqrt{-1}$$

$$+\frac{2}{\lambda^3\mu'\sqrt{\varkappa^2-\mu^2}}\int\limits_0^{\varphi}\frac{[\lambda^2(\alpha\mu^2+\beta)-\mu^2(\alpha\lambda^2+\beta)\sin^2\varphi]d\varphi}{\sqrt{\left(1-\frac{\mu^2}{\lambda^2}\sin^2\varphi\right)\left(1-\frac{\lambda'^2\mu^2}{\mu'^2\lambda^2}\sin^2\varphi\right)\left(1-\frac{(\varkappa^2-\lambda^2)\mu^2}{(\varkappa^2-\mu^2)\lambda^2}\sin^2\varphi\right)}}.$$

Having expanded this expression into the assigned form, what is possible, and having put

$$u_6=\int\limits_{\frac{1}{\mu^2}}^{\infty}\frac{(\alpha+\beta x)dx}{\sqrt{X}}$$

$$=\frac{2}{\lambda^3\mu'\sqrt{\varkappa^2-\mu^2}}\int\limits_0^{\frac{\pi}{2}}\frac{[\lambda^2(\alpha\mu^2+\beta)-\mu^2(\alpha\lambda^2+\beta)\sin^2\varphi]d\varphi}{\sqrt{\left(1-\frac{\mu^2}{\lambda^2}\sin^2\varphi\right)\left(1-\frac{\lambda'^2\mu^2}{\mu'^2\lambda^2}\sin^2\varphi\right)\left(1-\frac{(\varkappa^2-\lambda^2)\mu^2}{(\varkappa^2-\mu^2)\lambda^2}\sin^2\varphi\right)}},$$

one has the first coefficients of the expansion

$$C=u_2+u_3\sqrt{-1}-u_4-u_5\sqrt{-1},\quad A=u_6,$$

whence the function $x=\lambda(u)$ also has the index $2u_6$, or it will be

$$\lambda(u+2u_6)=\lambda(u).$$

Hence we now demonstrated, what was propounded, how for all real values of x the propounded integral

$$u=\int\limits_0^{x}\frac{(\alpha+\beta x)dx}{\sqrt{X}},$$

can be expanded into a convergent series of the form

$$u=C+\frac{2A}{\pi}\cdot\varphi+A_1\sin 2\varphi+A_2\sin 4\varphi+A_3\sin 6\varphi+\cdots.$$

And the six different expansions we assigned for the six intervals, in which x can lie, suggested as many indices of the periodic function $x=\lambda(u)$.

5.

In the preceding, we used those substitutions for the single intervals, by which the propounded integrals always goes over into the form

$$C + \int_0^\varphi \frac{(m + n\sin^2\varphi)d\varphi}{\sqrt{(1 - p^2\sin^2\varphi)(1 - q^2\sin^2\varphi)(1 - r^2\sin^2\varphi}},$$

where p^2, q^2, r^2 are smaller than 1, and, while x grows from the lower limit to the upper limit, at the same time φ grows from 0 to $\frac{\pi}{2}$. The same can also be done for the single intervals by another substitution of the same form

$$x = \frac{d + e\sin^2\varphi}{f + g\sin^2\varphi},$$

so that, while the variable x increases from the lower limit to the upper limit, at the same time φ decreases from $\frac{\pi}{2}$ to 0. Richelot demonstrated more generally in his paper, which will be published soon, while X denotes an arbitrary polynomial function of sixth order, which can be resolved into its linear factors, that the integral

$$u = \int \frac{(\alpha + \beta x)dx}{\sqrt{X}},$$

applying twelve real substitutions of the form

$$x = \frac{d + e\sin^2\varphi}{f + g\sin^2\varphi},$$

can be reduced to the form

$$\int \frac{(m + n\sin^2\varphi)d\varphi}{\sqrt{(1 - p^2\sin^2\varphi)(1 - q^2\sin^2\varphi)(1 - r^2\sin^2\varphi)}},$$

where p^2, q^2, r^2 are real, positive and smaller than 1. He applied the same ideas to the general case, in which X is of arbitrary order $2n$. Therefore, it was possible to extend the preceding considerations to this case. Furthermore, by innumerable other substitutions, he was even able to get to a form of the integral, which admitted the expansion into a convergent series of sines and cosines of multiples of the same angle, of which just one was necessary here. And nevertheless one cannot not get to other indices than those, which are constructed by those, we assigned, by means of other substitutions.

But the indices we assigned, three real ones and three imaginary ones, are of such a nature that it is possible to construct one real one from the two

remaining real ones and one imaginary one from the two remaining imaginary ones. We will prove this in the following.
The propounded integral

$$u = \int_0^x \frac{(\alpha + \beta x)dx}{\sqrt{X}},$$

is only determined, if for the single intervals the correct sign of the square root is chosen. Since for the next interval a new factor of the expression under the square root sign also changes the sign, we set that hence by the multiplication by that factor always the quantity $\sqrt{-1}$ results, so that to the expressions $\frac{1}{\sqrt{X}}$ in the assigned intervals these signs correspond

$$-\sqrt{-1}, \quad +, \quad +\sqrt{-1}, \quad -, \quad -\sqrt{-1}, \quad +,$$

respectively, since also the prefix $\pm\sqrt{-1}$ can be ascribed to the sign. Having constituted these and having used the values of u_1, u_2 etc. we assigned above, we obtain

$$\int_{-\infty}^{\infty} \frac{(\alpha + \beta x)dx}{\sqrt{X}} = -u_1\sqrt{-1} + u_2 + u_3\sqrt{-1} - u_4 - u_5\sqrt{-1} + u_6.$$

Since, having put $\frac{1}{x}$ instead of x, the two limits coincide, I mention, that it will be

$$0 = -u_1\sqrt{-1} + u_2 + u_3\sqrt{-1} - u_4 - u_5\sqrt{-1} + u_6;$$

or

$$u_1 + u_5 = u_3, \qquad u_2 + u_6 = u_4,$$

or, what it the same,

$$\int_{-\infty}^{0} \frac{(\alpha + \beta x)dx}{\sqrt{-X}} + \int_{\frac{1}{\lambda^2}}^{\frac{1}{\mu^2}} \frac{(\alpha + \beta x)dx}{\sqrt{-X}} = \int_{1}^{\frac{1}{\varkappa^2}} \frac{(\alpha + \beta x)dx}{\sqrt{-X}},$$

$$\int_{0}^{1} \frac{(\alpha + \beta x)dx}{\sqrt{X}} + \int_{\frac{1}{\mu^2}}^{\infty} \frac{(\alpha + \beta x)dx}{\sqrt{X}} = \int_{\frac{1}{\varkappa^2}}^{\frac{1}{\lambda^2}} \frac{(\alpha + \beta x)dx}{\sqrt{X}},$$

in which equations $\sqrt{-X}$ and $\sqrt{X}$ are always taken positively. But since because of the ambiguity of the radical $\sqrt{X}$ we desire another proof of the memorable preceding formulas, we will deduce the same from a special case of the Abelian theorem. We will explain this here more accurately.

6.

Let us consider the following cubic equation

$$f(x) = x(1 - \varkappa^2 x)(1 - \mu^2 x) - h(1 - x)(1 - \lambda^2 x) = 0,$$

whose three roots we want to consider as functions of h. Let

$$1 > \varkappa^2 > \lambda^2 > \mu^2;$$

if h is positive, having put

$$x = -\infty, \quad 0, \quad 1, \quad \frac{1}{\varkappa^2}, \quad \frac{1}{\lambda^2}, \quad \frac{1}{\mu^2}, \quad +\infty$$

the function $f(x)$ has the signs:

$$-, \quad -, \quad +, \quad +, \quad -, \quad -, \quad +.$$

Hence the three roots of the cubic equations are real, one between 0 and 1, the second between $\frac{1}{\varkappa^2}$ and $\frac{1}{\lambda^2}$, the third between $\frac{1}{\mu^2}$ and $+\infty$. If h is negative, for the same values of x, the function $f(x)$ has these signs, respectively:

$$-, \quad +, \quad +, \quad -, \quad -, \quad ,+, \quad +.$$

Hence even in this case the three roots of the cubic equation are real, the first negative, the others positive, and the second lies between 1 and $\frac{1}{\varkappa^2}$, the third between $\frac{1}{\lambda^2}$ and $\frac{1}{\mu^2}$.
Having differentiated the propounded equation, it immediately results:

$$\begin{aligned}\frac{dh}{hdx} &= \frac{1}{x} + \frac{1 - \varkappa^2}{(1 - x)(1 - \varkappa^2 x)} + \frac{\lambda^2 - \mu^2}{(1 - \lambda^2 x)(1 - \mu^2 x)} \\ &= \frac{1}{x(1 - x)} - \frac{x^2 - \lambda^2}{(1 - \varkappa^2 x)(1 - \lambda^2 x)} - \frac{\mu^2}{1 - \mu^2 x}.\end{aligned}$$

This formula teaches,

1) if h is positive, and x lies either between 0 and 1 or between $\frac{1}{\varkappa^2}$ and $\frac{1}{\lambda^2}$ or $\frac{1}{\mu^2}$ and $+\infty$

2) if h is negative, and x is either negative, or lies between 1 and $\frac{1}{\varkappa^2}$ or between $\frac{1}{\lambda^2}$ and $\frac{1}{\mu^2}$, that the expression $\frac{dx}{dh}$ is always positive. Hence in each of both cases, in which h is either positive or negative, all three roots of the cubic equations continuously increase or decrease together with h. Now

$$\begin{array}{lllll} \text{having put} & h = -\infty, & \text{the roots become} & -\infty, \quad 1, & \frac{1}{\lambda^2}, \\ -- & h = \quad 0, & -- & 0, \quad \frac{1}{\varkappa^2}, & \frac{1}{\mu^2}, \\ -- & h = +\infty, & -- & 1, \quad \frac{1}{\lambda^2}, & +\infty. \end{array}$$

Hence if we call the roots, which alternate in size and which we called the first, second, third, a, b, c in each case, we see, that these increase simultaneously:

$$\begin{array}{llll} h \text{ from } 0 \text{ to } +\infty, & a \text{ from } 0 \text{ to } 1, & b \text{ from } \frac{1}{\varkappa^2} \text{ to } \frac{1}{\lambda^2}, & c \text{ from } \frac{1}{\mu^2} \text{ to } +\infty \\ h \text{ from } -\infty \text{ to } 0, & a \text{ from } -\infty \text{ to } 0, & b \text{ from } 1 \text{ to } \frac{1}{\varkappa^2}, & c \text{ from } \frac{1}{\lambda^2} \text{ to } +\frac{1}{\mu^2}. \end{array}$$

Hence also these increase continuously at the same time:

$$h \text{ from } -\infty \text{ to } 0, \quad a \text{ from } -\infty \text{ to } 1, \quad b \text{ from } 1 \text{ to } \frac{1}{\lambda^2}, \quad c \text{ from } \frac{1}{\lambda^2} \text{ to } +\infty.$$

Now, while h grows from h_0 to h_1, let us set that a increases from a_0 to a_1, b from b_0 to b_1, c from c_0 to c_1 simultaneously. Having put

$$\frac{df(x)}{dx} = f'(x),$$

differentiating the propounded equation one has:

$$f'(x)dx - (1-x)(1-\lambda^2 x)dh = 0,$$

or if one substitutes

$$(1-x)(1-\lambda^2 x)\sqrt{h} = \sqrt{x(1-x)(1-\varkappa^2 x)(1-\lambda^2 x)(1-\mu^2 x)} = \sqrt{X},$$

having multiplied by $\alpha + \beta x$, one has:

$$\frac{(\alpha+\beta x)dx}{\sqrt{X}} = \frac{(\alpha+\beta x)dh}{f'(x)\cdot\sqrt{h}}.$$

If in this formula we substituted its three values a, b, c for x, these three formulas result:

$$\int_{a_0}^{a_1} \frac{(\alpha+\beta x)dx}{\sqrt{X}} = \int_{h_0}^{h_1} \frac{(\alpha+\beta a)dh}{f'(a)\cdot\sqrt{h}},$$

$$\int_{b_0}^{b_1} \frac{(\alpha+\beta x)dx}{\sqrt{X}} = \int_{h_0}^{h_1} \frac{(\alpha+\beta b)dh}{f'(b)\cdot\sqrt{h}},$$

$$\int_{c_0}^{c_1} \frac{(\alpha+\beta x)dx}{\sqrt{X}} = \int_{h_0}^{h_1} \frac{(\alpha+\beta c)dh}{f'(c)\cdot\sqrt{h}}.$$

Now, since from a very well-known algebraic theorem one has:

$$\frac{\alpha+\beta a}{f'(a)} + \frac{\alpha+\beta b}{f'(b)} + \frac{\alpha+\beta a}{f'(c)} = 0,$$

by taking the three propounded formulas, if $\sqrt{h}$ is always assumed with the same sign, it results

$$\varepsilon\int_{a_0}^{a_1} \frac{(\alpha+\beta x)dx}{\sqrt{X}} + \varepsilon_1\int_{b_0}^{b_1} \frac{(\alpha+\beta x)dx}{\sqrt{X}} + \varepsilon_2\int_{c_0}^{c_1} \frac{(\alpha+\beta x)dx}{\sqrt{X}} = 0;$$

while the factors ε, ε_1, ε_2, which were to be added because of the ambiguity of the square root, denote either $+1$ or -1.
In order to determine the factors ε, ε_1, ε_2 I observe, that the root $\sqrt{X}$ was introduced into our calculation instead of the expression

$$\sqrt{X} = \sqrt{h}(1-x)(1-\lambda^2 x),$$

which has the same sign, if x lies between $-\infty$ and 1 and if x lies between $\frac{1}{\lambda^2}$ and $+\infty$, but the opposite sign, if x lies between 1 and $\frac{1}{\lambda^2}$; or for the first and the third root a or c it will have the same sign, for the second b it will have the opposite sign. Hence one must set:

$$\varepsilon = -\varepsilon_1 = \varepsilon.$$

Having done this our equation becomes:

$$\int_{a_0}^{a_1} \frac{(\alpha+\beta x)dx}{\sqrt{X}} + \int_{c_0}^{c_1} \frac{(\alpha+\beta x)dx}{\sqrt{X}} = \int_{b_0}^{b_1} \frac{(\alpha+\beta x)dx}{\sqrt{X}}.$$

In this formula the three radicals $\sqrt{X}$ must have the same sign.
We saw that we have at the same time

$$a_0 = 0, \qquad b_0 = \frac{1}{\varkappa^2}, \qquad c_0 = \frac{1}{\mu^2}$$

and

$$a_1 = 1, \qquad b_0 = \frac{1}{\lambda^2}, \qquad c_1 = +\infty$$

which values correspond to the values

$$h_0 = 0, \qquad h_1 = +\infty.$$

Having substituted these, from the propounded formula it follows:

$$\int_{0}^{1} \frac{(\alpha+\beta x)dx}{\sqrt{X}} + \int_{\frac{1}{\lambda^2}}^{\infty} \frac{(\alpha+\beta x)dx}{\sqrt{X}} = \int_{\frac{1}{\varkappa^2}}^{\frac{1}{\lambda^2}} \frac{(\alpha+\beta x)dx}{\sqrt{X}}.$$

Further, one has

$$a_0 = -\infty, \qquad b_0 = 1, \qquad c_0 = \frac{1}{\lambda^2},$$

$$a_1 = 0, \qquad b_1 = \frac{1}{\varkappa^2}, \qquad c_1 = \frac{1}{\mu^2},$$

simultaneously, which values of h correspond to the values

$$h_0 = -\infty, \quad h_1 = 0.$$

Hence from the propounded formula, after we had divided by $\sqrt{-1}$ at same time, it follows:

$$\int_{-\infty}^{0} \frac{(\alpha + \beta x)dx}{\sqrt{-X}} + \int_{\frac{1}{\lambda^2}}^{\frac{1}{\mu^2}} \frac{(\alpha + \beta x)dx}{\sqrt{-X}} = \int_{0}^{\frac{1}{\varkappa^2}} \frac{(\alpha + \beta x)dx}{\sqrt{-X}}.$$

In these formulas the three radicals $\sqrt{X}$ or $\sqrt{-X}$ must have the same sign. These are the formulas, we wanted to prove, deduced from a more general form propounded on indefinite integrals.
In the preceding question it was assumed that h does not contain x; the theorem, propounded by Abel, was based on the much more general assumption that h is the square of an arbitrary rational function of x.

7.

In the preceding it was proved, that the six indices we found,

$$u_1\sqrt{-1}, \quad u_2, \quad u_3\sqrt{-1}, \quad u_4, \quad u_5\sqrt{-1}, \quad u_6$$

are reduced to four by means of the formulas

$$u_1 + u_5 = u_3, \quad u_2 + u_6 = u_4;$$

Let us put these indices

$$u_2, \quad u_6; \quad u_1\sqrt{-1}, \quad u_5\sqrt{-1}.$$

And in general neither u_2 nor u_6 nor $u_1\sqrt{-1}$ and $u_5\sqrt{-1}$ can be reduced to the same index, or u_2 and u_6 and u_1 and u_5 will be incommensurable. Hence the function $x = \lambda(u)$ will have four indices, which can not be reduced to a smaller one, or that function will be quadruple periodic. But that a triple periodic function does not exist was already proven above. But that this case, in which two incommensurable indices are real and two imaginary incommensurable indices are of the form $u_1\sqrt{-1}$, $u_5\sqrt{-1}$, is absurd, is already known from § 1. From the results, which mentioned there, indices Δ and $\Delta'\sqrt{-1}$ of the function

$x = \lambda(u)$ would exist, where Δ and Δ' are real quantities smaller than an arbitrarily small quantity. Hence, while the function $\lambda(u)$ remains unchanged, u could take on all real or imaginary values, or among total number of values, which u could take on, while the function $\lambda(u)$ does not change, there would always exist some, which differ from an arbitrary given real or imaginary quantity less than another given arbitrarily small quantity. This would be absurd.

We would have discovered even more periods, if the function X under the square root is of higher than fifth or sixth order. For, generally, if X is of $2n$-th or $(2n-1)$-th order and, having put

$$u = \int \frac{f(x)dx}{\sqrt{X}},$$

where $f(x)$ is an arbitrary given polynomial function, x is considered as a function of u, the function will have $2n-2$ indices, which can in general not be reduced to a smaller number; and, if the coefficients of the function X are real quantities, $n-1$ of them will be real and $n-1$ will be imaginary.

For, the same way as above, for the general case the following results can be proved. Let

$$X = x(1-x)(1-\varkappa^2 x)(1-\varkappa_1^2 x)\cdots(1-\varkappa_{2n-4}^2 x),$$

where

$$1 > \varkappa^2 > \varkappa_1^2 > \cdots > \varkappa_{2n-5}^2 > \varkappa_{2n-4}^2;$$

the equation of n-th order

$$x(1-x)(1-\varkappa^2 x)(1-\varkappa_1^2 x)\cdots(1-\varkappa_{2n-4}^2 x) = h(1-x)(1-x)(1-\varkappa_1^2 x)\cdots(1-\varkappa_{2n-5}^2 x)$$

has n real roots; if we call them $a_1, a_2, \cdots, a_n$ and they increase in magnitude, while h increases from $-\infty$ to 0, and then from 0 from $+\infty$,

a_1	increases from	$-\infty$	and	0,	and further from	0	to	1,
a_2	increases from	$\frac{1}{\varkappa^2}$	and	$\frac{1}{\varkappa^2}$,	and further from	$\frac{1}{\varkappa^2}$	to	$\varkappa_1^2$,

and generally

$$a_m \quad \text{from} \quad \frac{1}{\varkappa^2_{2m-5}} \quad \text{to} \quad \frac{1}{\varkappa^2_{2m-4}} \qquad \text{and further from} \quad \frac{1}{\varkappa^2_{2m-4}} \quad \text{to} \quad \frac{1}{\varkappa^2_{2m-2}},$$

and finally

$$a_n \quad \text{from} \quad \frac{1}{\varkappa^2_{2n-5}} \quad \text{to} \quad \frac{1}{\varkappa^2_{2n-4}} \qquad \text{and further from} \quad \frac{1}{\varkappa^2_{2n-4}} \quad \text{to} \quad +\infty.$$

If, while h increases from $h^{(0)}$ to h', a_m grows from $a_m^{(0)}$ to a'_m, it will be:

$$\int_{a_1^{(0)}}^{a'_1} \frac{f(x)dx}{\sqrt{X}} - \int_{a_2^{(0)}}^{a'_2} \frac{f(x)dx}{\sqrt{X}} + \cdots \pm \int_{a_n^{(0)}}^{a'_n} \frac{f(x)dx}{\sqrt{X}} = 0,$$

while $f(x)$ denotes an arbitrary polynomial function of order $n-2$. From this formula these special formulas result:

$$\int_{-\infty}^{0} \frac{f(x)dx}{\sqrt{-X}} - \int_{1}^{\frac{1}{\varkappa^2}} \frac{f(x)dx}{\sqrt{-X}} + \int_{\frac{1}{\varkappa_1^2}}^{\frac{1}{\varkappa_2^2}} \frac{f(x)dx}{\sqrt{-X}} \cdots \pm \int_{\frac{1}{\varkappa^2_{2n-5}}}^{\frac{1}{\varkappa^2_{2n-4}}} \frac{f(x)dx}{\sqrt{-X}} = 0,$$

$$\int_{0}^{1} \frac{f(x)dx}{\sqrt{X}} - \int_{\frac{1}{\varkappa^2}}^{\frac{1}{\varkappa_1^2}} \frac{f(x)dx}{\sqrt{X}} + \int_{\frac{1}{\varkappa_2^2}}^{\frac{1}{\varkappa_3^2}} \frac{f(x)dx}{\sqrt{X}} \cdots \pm \int_{\frac{1}{\varkappa^2_{2n-4}}}^{\infty} \frac{f(x)dx}{\sqrt{X}} = 0.$$

In these formulas the n radicals $\sqrt{X}$ or $\sqrt{-X}$ must have the same sign. And the $2n$ doubled definite integrals, if in the first n we write $\sqrt{X}$ instead of $\sqrt{-X}$, will be the indices of the function $x = \lambda(u)$, and n will be real and n imaginary, which by the two preceding equations are reduced to $n-1$ real and $n-1$ imaginary indices; and they can only be reduced to a smaller number in special cases.

8.

From the given results we conclude:
As circular arcs take on innumerable equidistant values for the same sine, as there are innumerable logarithms, distant from each other by the same imaginary quantity, of the

same number, as the elliptic integrals will enjoy the double infinite amount of values, whose real an imaginary parts simultaneously take one innumerable equidistant values, for the same sine of the amplitude: so the Abelian or hyperelliptic integrals, this means integrals, in which the square root in the integral contains a function of a higher than fourth order, will have such a high multiplicity of values that for arbitrary given limits they take on all arbitrary real or imaginary values, or that among the total amount of values, which the same integral can obtain for the same given limits, there are always some, which differ from an arbitrary real or imaginary value less than a certain arbitrarily small quantity.

From the preceding it is plain, if X is of a higher than fourth order, that x can not be considered as analytic function of u; and therefore, it seems that the general methods, by which once the analytic trigonometry and by which recently the theory of elliptic functions was constructed, can not be applied to the Abelian transcendents. But, which fortunately happens in this desperation, the special way, we, starting from completely different considerations, explained in a preceding paper (Crelle Journal Bd. 9, p. 394), and how, at least in our opinion, the Abelian transcendents should be introduced into analysis, also removes the difficulties, which result from the multiplicity of the values of the integral, here. Having made some small adjustments, I will repeat this here.

9.

Let X be a polynomial function of fifth or sixth order again; let us set:

$$\int_a^x \frac{(\alpha+\beta x)dx}{\sqrt{X}} + \int_b^y \frac{(\alpha+\beta x)dx}{\sqrt{X}} = u,$$
$$\int_a^x \frac{(\alpha'+\beta' x)dx}{\sqrt{X}} + \int_b^y \frac{(\alpha'+\beta' x)dx}{\sqrt{X}} = u',$$

while a, b, α, β, a', β' denote constants. One has to consider x, y as the roots of the quadratic equation

$$Ux^2 - U'x + U'' = 0,$$

in which U, U', U'' are functions of u, u'. If u is given as the sum of several integrals

$$\int \frac{(\alpha + \beta x)dx}{\sqrt{X}},$$

and at the same time u' as the sum of as many integrals

$$\int \frac{(\alpha' + \beta' x)dx}{\sqrt{X}},$$

which have the same limits, respectively: From the Abelian theorem one finds U, U', U'' as *rational* functions of these limits and values, which the radical $\sqrt{X}$ takes on for the same. Therefore, in this case the two transcendental equations are reduced to algebraic equations. And this way the Abelian theorem is usually propounded, if X is of fifth or sixth order.

The simplest example of this broad theorem was given above. For, from the results demonstrated above, if we set:

$$X = x(1-x)(1-\varkappa^2 x)(1-\lambda^2 x)(1-\mu^2 x),$$

where $1 > \varkappa^2 > \lambda^2 > \mu^2$, having propounded two transcendental equations,

$$\int_0^x \frac{(\alpha + \beta x)dx}{\sqrt{X}} + \int_{\frac{1}{\mu^2}}^y \frac{(\alpha + \beta x)dx}{\sqrt{X}} = \int_{\frac{1}{\varkappa^2}}^z \frac{(\alpha + \beta x)dx}{\sqrt{X}},$$

$$\int_0^x \frac{(\alpha' + \beta' x)dx}{\sqrt{X}} + \int_{\frac{1}{\mu^2}}^y \frac{(\alpha' + \beta' x)dx}{\sqrt{X}} = \int_{\frac{1}{\varkappa^2}}^z \frac{(\alpha' + \beta' x)dx}{\sqrt{X}},$$

it follows that x, y and z are determined as the roots of the quadratic equation:

$$\frac{(1-z)(1-\lambda^2 z)\cdot x(1-\varkappa^2 x)(1-\mu^2 x) - z(1-\varkappa^2 z)(1-\mu^2 z)(1-x)(1-\lambda^2 x)}{x-z} = 0,$$

or

$$Ux^2 - U'x + U'' = 0,$$

where

$$\begin{aligned} U \;&= \varkappa^2\mu^2(1-z)(1-\lambda^2 z), \\ U' \;&= \varkappa^2 + \mu^2 + [\lambda^2 - (\varkappa^2 - (\varkappa^2+\mu^2)(1+\lambda^2) - \varkappa^2\mu^2]z + \varkappa^2\mu^2(1+\lambda^2)z^2, \\ U'' &= (1-\varkappa^2 z)(1-\mu^2 z). \end{aligned}$$

For, we demonstrated that another transcendental equation holds, if x, y, z are roots of the cubic equation

$$x(1-\varkappa^2 x)(1-\mu^2 x) = h(1-x)(1-\lambda^2 x);$$

having eliminated h from this equation by means of the formula

$$h = \frac{z(1-\varkappa z)(1-\mu z)}{(1-z)(1-\lambda^2 z)}$$

and having divided by $x-z$, we obtain the two remaining roots x and y as the roots of the propounded quadratic equation. And since that equation is not affected by the constants α, β at all, the same algebraic relations, propounded among x, y and z, also satisfy the other transcendental equation, in which one finds the other constants α', β' instead of α, β. And nothing new would be added, if we would add a third transcendental equation, in which one would find still other constants α'', β'' instead of α, β. For, if the relations constituted among x, y, z satisfied the two propounded transcendental equations, hence another equation

$$\int_0^x \frac{(m+nx)dx}{\sqrt{X}} + \int_{\frac{1}{\mu^2}}^y \frac{(m+nx)dx}{\sqrt{X}} = \int_{\frac{1}{\varkappa^2}}^z \frac{(m+nx)dx}{\sqrt{X}},$$

with arbitrary constants m, n, follows immediately.

10.

But above we gave six substitutions of the form

$$x = \frac{d + e\sin^2\varphi}{f + g\sin^2\varphi},$$

by means of which for the different intervals, in which x is contained, we reduced the integral

$$\int \frac{(\alpha+\beta x)dx}{\sqrt{X}}$$

to a form, which allowed an expansion into a convergent series of this kind:

$$\int\limits_a^x \frac{(\alpha+\beta x)dx}{\sqrt{X}} = C + \frac{2A}{\pi}\cdot\varphi + A'\sin 2\varphi + A''\sin 4\varphi + A'''\sin 6\varphi + \cdots .$$

Since those substitutions do not depend on the constants α, β in any way, by means of the same substitutions, one even finds a convergent expansion for other constant α', β' in the single cases:

$$\int\limits_a^x \frac{(\alpha'+\beta' x)dx}{\sqrt{X}} = C' + \frac{2B}{\pi}\cdot\varphi + B'\sin 2\varphi + B''\sin 4\varphi + B''\sin 6\varphi + \cdots .$$

For another given value y, either by means of the same substitutions or by means of another,

$$y = \frac{d' + e'\sin^2\psi}{f' + g'\sin^2\psi},$$

to be used for the interval, in which y lies, find the following convergent expansions:

$$\begin{aligned}
\int\limits_b^y \frac{(\alpha+\beta x)dx}{\sqrt{X}} &= C_1 + \frac{2A_1}{\pi}\cdot\psi + A_1'\sin 2\psi + A_1''\sin 4\psi + A_1'''\sin 6\psi + \cdots \\
\int\limits_b^y \frac{(\alpha'+\beta' x)dx}{\sqrt{X}} &= C_1' + \frac{2B_1}{\pi}\cdot\psi + B_1'\sin 2\psi + B_1''\sin 4\psi + B_1'''\sin 6\psi + \cdots .
\end{aligned}$$

Hence having put

$$\begin{aligned}
\int\limits_a^x \frac{(\alpha+\beta x)dx}{\sqrt{X}} + \int\limits_b^y \frac{(\alpha+\beta x)dx}{\sqrt{X}} &= u, \\
\int\limits_a^x \frac{(\alpha'+\beta' x)dx}{\sqrt{X}} + \int\limits_b^y \frac{(\alpha'+\beta' x)dx}{\sqrt{X}} &= u',
\end{aligned}$$

we see, having changed φ into $\varphi + m\pi$, $\psi + m'\psi$, while m, m' denote arbitrary positive or negative numbers, that

$$u \quad \text{becomes} \quad u \ + 2mA + 2m'A_1,$$

$$u' \quad \text{becomes} \quad u' + 2mA + 2m'A_1,$$

simultaneously, but at the same x, y are not changed. Here, A, A_1 are either the same or different of the six quantities we called

$$\frac{u_1}{\sqrt{-1}}, \quad u_2, \quad u_3\sqrt{-1}, \quad -u_4, \quad -u_5\sqrt{-1}, \quad u_6$$

above, and B, B_1 are quantities, into which A, A_1 go over, if one puts α', β' instead of α, β. Hence, if those six quantities, after α', β' had been written instead of α, β, go over into these, respectively:

$$\frac{u_1'}{\sqrt{-1}}, \quad u_2', \quad u_3'\sqrt{-1}, \quad -u_4', \quad -u_5'\sqrt{-1}, \quad u_6',$$

it follows, while m, m', m'', m''' denote arbitrary integer numbers, having changed

$$u \quad \text{into} \quad u + \frac{2mu_1}{\sqrt{-1}} + 2m'u_2 + 2m''u_5\sqrt{-1} + 2m'''u_6$$

and at the same time

$$u' \quad \text{into} \quad u' + \frac{2mu_1'}{\sqrt{-1}} + 2m'u_2' + 2m''u_5'\sqrt{-1} + 2m'''u_6',$$

that x, y are not changed. We omitted the indices $2u_3\sqrt{-1}$, $2u_4$ and the corresponding $2u_3'\sqrt{-1}$, $2u_4'$, since they are reduced to the remaining ones. Therefore, we found the following fundamental theorem on the periods of our transcendents.

FUNDAMENTAL THEOREM

Having put

$$X = x(1-x)(1-\varkappa^2 x)(1-\lambda^2 x)(1-\lambda^2 x)(1-\mu^2 x),$$

set:

$$2\int_{-\infty}^{0}\frac{(\alpha+\beta x)}{\sqrt{-X}}=i_1,\qquad 2\int_{0}^{1}\frac{(\alpha+\beta x)dx}{\sqrt{X}}=i_2,$$
$$2\int_{\frac{1}{\lambda^2}}^{\frac{1}{\mu^2}}\frac{(\alpha+\beta x)}{\sqrt{-X}}=i_3,\qquad 2\int_{\frac{1}{\mu^2}}^{\infty}\frac{(\alpha+\beta x)dx}{\sqrt{X}}=i_4,$$

$$2\int_{-\infty}^{0}\frac{(\alpha'+\beta' x)}{\sqrt{-X}}=i'_1,\qquad 2\int_{0}^{1}\frac{(\alpha'+\beta' x)dx}{\sqrt{X}}=i'_2,$$
$$2\int_{\frac{1}{\lambda^2}}^{\frac{1}{\mu^2}}\frac{(\alpha'+\beta' x)}{\sqrt{-X}}=i'_3,\qquad 2\int_{\frac{1}{\mu^2}}^{\infty}\frac{(\alpha'+\beta' x)dx}{\sqrt{X}}=i'_4,$$

consider x, y as functions of u, u',

$$x=\lambda(u,u'),\qquad y=\lambda'(u,u'),$$

given by the equations:

$$\int_a^x\frac{(\alpha+\beta x)}{\sqrt{X}}+\int_b^y\frac{(\alpha+\beta x)}{\sqrt{X}}=u,$$
$$\int_a^x\frac{(\alpha'+\beta' x)}{\sqrt{X}}+\int_b^y\frac{(\alpha'+\beta' x)}{\sqrt{X}}=u',$$

it will be:

$$\lambda\begin{pmatrix}u\ +mi_1\sqrt{-1}+m'i_2+m''i_3\sqrt{-1}+m'''i_4,\\ u'+mi'_1\sqrt{-1}+m'i'_2+m''i'_3\sqrt{-1}+m'''i'_4,\end{pmatrix}=\lambda(u,u')$$
$$\lambda'\begin{pmatrix}u\ +mi_1\sqrt{-1}+m'i_2+m''i_3\sqrt{-1}+m'''i_4,\\ u'+mi'_1\sqrt{-1}+m'i'_2+m''i'_3\sqrt{-1}+m'''i'_4,\end{pmatrix}=\lambda'(u,u')$$

whatever positive or negative numbers m, m', m'', m''' are.

The kind of periodicity, which was explained in the preceding theorem, does not have anything, what is observed in the laws of analytic functions. For, one can always determine the numbers m, m', m'', m''' in such a way that the one of the two expressions

$$u \ + m'i_2 + m'''i_4 + (mi_1 + m''i_3)\sqrt{-1},$$
$$u' + m'i'_2 + m'''i'_4 + (mi'_1 + m''i'_3)\sqrt{-1}$$

differs from a given arbitrary quantity

$$p + q\sqrt{-1}$$

less than any arbitrarily small given quantity; and nevertheless it is not possible in general, if not at the same time the numbers become infinitely large, so that, while the one expression comes infinitely close to a given quantity, the other becomes infinitely large at the same time. Hence we see that in our quadrouple periodic functions of two variables

$$x = \lambda(u, u'), \quad y = \lambda'(u, u')$$

the one argument becomes undetermined just then, when the other becomes infinite. This is not absurd.

We see that for the values of x, y the one argument can not be changed by a certain constant, while the other does not change, but the two arguments always change simultaneously, so that the index of the of argument is always completely determined by the index of the other. This is a characteristic property of this periodicity. Without it periodicity cannot hold.

11.

From the Abelian theorem it is known, having put

$$x = \lambda(u, u') \qquad y = \lambda'(u, u'),$$

that the functions

$$x_n = \lambda(nu, nu'), \quad y_n = \lambda'(nu, nu')$$

are given as the roots of the quadratic equation

$$U_n x^2 - U_n x + U''_n = 0,$$

in which U_n, U'_n, U''_n are rational functions of x, y, $\sqrt{X}$, $\sqrt{Y}$, if Y is the same function of y as X is of x. Hence it is also plain that vice versa x, y can be obtained from x_n, y_n by the resolution of an algebraic equation. You see

already from the fundamental theorem that these equations will be of order n^4. This is easily checked for $n = 2$ applying the Abelian theorem; and the same theorem also immediately yields the resolution of an equation of 16-th order, which is required for the bisection, by means of the extraction of square roots only. We will discuss this on another occasion.
But if there are transformations, from the same theorem you immediately see, that one gets to the multiplication by four transformations of n-th order, applied one after the other; hence the resolution of the equation of n^4-th order, which is required for the division into n parts, is reduced to four equations of n-th order, if you assume the division of the indices to be known. But, if n is a prime number, this is also easily seen to depend on an generally irreducible equation of $(1 + n + n^2 + n^3)-$th order and on another of $\left(\frac{n-1}{2}\right)$-th order, which, having assumed the roots of the other as known, admits a solution. And, if n is prime, $1 + n + n^2 + n^3$ will also be the number of transformation of the same of $n-$th order, from which total amount $2(n + 2)$ will be real.