Proof of the formula $\int\limits_0^1 w^{a-1}(1-w)^{b-1}dw = \frac{\int\limits_0^\infty e^{-x}x^{a-1}dx \int\limits_0^\infty e^{-x}x^{b-1}dx}{\int\limits_0^\infty e^{-x}x^{a+b-1}dx} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ *

Carl Gustav Jacob Jacobi

If all positive values from 0 to $+\infty$ are attributed to the variables x, y, having put

$$x + y = r, \quad x = rw,$$

the new variable r can obtain all positive values from 0 to $+\infty$, and the variable w can take on all positive values from 0 to 1 . At the same time, we find

$$dxdy = rdrdw.$$

Now, using the established notation

*Original Title: "Demonstratio formulae $\int\limits_0^1 w^{a-1}(1-w)^{b-1}dw = \frac{\int\limits_0^\infty e^{-x}x^{a-1}dx \int\limits_0^\infty e^{-x}x^{b-1}dx}{\int\limits_0^\infty e^{-x}x^{a+b-1}dx} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$", first published in *Crelle Journal für die reine und angewandte Mathematik*, Band 11, p. 307, 1833; reprinted in *C.G.J. Jacobi's Gesammelte Werke, Volume 6*, pp. 62-63, translated by: Alexander Aycock for the "Euler-Kreis Mainz".

$$\Gamma(a) = \int_0^\infty e^{-x} x^{a-1} dx,$$

one has

$$\Gamma(a)\Gamma(b) = \int \int e^{-x-y} x^{a-1} y^{b-1} dxdy,$$

having attributed all positive values from 0 to $+\infty$ to the variables x, y. But having put

$$x + y = r, \quad x = rw,$$

the propounded double integral from the preceding is split into two factors in another way, namely

$$\Gamma(a)\Gamma(b) = \int_0^\infty e^{-r} r^{a+b-1} dr \int_0^1 w^{a-1}(1-w)^{b-1} dw,$$

whence

$$\int_0^1 w^{a-1}(1-w)^{b-1} dw = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

This is the fundamental theorem, by which the one species of Eulerian integrals, as Legendre called them, is exhibited in terms of the other.

23rd of August, 1833