On the continued fraction the integral $\int\limits_x^\infty e^{-xx}dx$ can be expanded into *

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Having put $q = \frac{1}{2xx}$, Laplace (*Traite de Mecanique celeste, T. IV, L. X*) gave the propounded integral, which is used in studies of celestial mechanics and in other questions, as the expansion into the following continued fraction,

$$\int_x^\infty e^{-xx}dx = \frac{e^{-xx}}{2x} \cdot \cfrac{1}{1 + \cfrac{q}{1 + \cfrac{2q}{1 + \cfrac{3q}{1 + 4q}}}} \tag{1}$$
$$\ddots$$

Nevertheless, the proof given by Laplace, since it utilizes divergent series, is hardly accepted by anyone nowadays; that demonstration can be rendered more accurate this way:
Let us set

$$(2) \quad v = e^{xx} \int\limits_x^\infty e^{-xx}dx,$$

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by differentiation we have

$$(3) \quad \frac{dv}{dx} = 2xv - 1.$$

Having differentiated this equation n times, it results

$$(4) \quad \frac{d^{n+1}v}{dx^{n+1}} = 2x\frac{d^n v}{dx^n} + 2n\frac{d^{n-1}v}{dx^{n-1}}$$

or, having put

$$(5) \quad v_n = \frac{1}{1 \cdot 2 \cdot 3 \cdots n}\frac{d^n v}{dx^n},$$

we have

$$(6) \quad (n+1)v_{n+1} = 2xv_n + 2v_{n-1} + 2v_{n-1}, \quad v_1 = 2xv - 1.$$

Further, having put

$$(-1)^n 2x^{n+1} v_n = y_{n+1}, \quad q = \frac{1}{2xx},$$

from (6)

$$(7) \quad y_n = y_{n+1} + (n+1)qy_{n+2}, \quad 1 = y_1 + qy_2.$$

Hence it follows

$$\frac{1}{y_1} = 1 + \frac{qy_2}{y_1} = 1 + \cfrac{q}{1 + \cfrac{2qy_3}{y_2}} = 1 + \cfrac{q}{1 + \cfrac{2q}{1 + \cfrac{3qy_4}{y_3}}}$$

or in general

$$(8) \quad \frac{1}{y_1} = 1 + \cfrac{q}{1 + \cfrac{2q}{1 + \cfrac{3q}{1+ \ddots + \cfrac{nq}{1 + \cfrac{(n+1)qy_{n+2}}{y_{n+1}}}}}},$$

where

$$
(9) \qquad \begin{aligned} y_1 &= 2xe^{xx} \int_x^\infty e^{-xx} = 2xv, \\ y_{n+1} &= (-1)^n \frac{2x^{n+1}}{1 \cdot 2 \cdot 3 \cdots n} \frac{d^n v}{dx^n}. \end{aligned}
$$

Laplace asserted that the value of $\frac{1}{y_1}$ in question is always contained between two subsequent values of the continued fraction; for this to be seen to be true he had to show that the neglected quotients

$$
\frac{(n+1)qy_{n+2}}{y_{n+1}}
$$

have the same sign. To prove this by expansions into series is rather difficult, but you will succeed, if you exhibit the differentials of v by definite integrals. For, from (2) one has

$$
(10) \qquad v = e^{xx} \int_x^\infty e^{-tt} dt = e^{tt} \int_0^\infty e^{-(t+x)^2} dt = \int_0^\infty dt e^{-tt} e^{-2tx},
$$

whence

$$
(11) \qquad \frac{d^n v}{dx^n} = \int_0^\infty dt (-2t)^n e^{-tt} e^{-2tx} = e^x x \int_0^\infty (-2t)^n e^{-(t+x)^2}
$$

or

$$
(12) \qquad \frac{d^n v}{dx^n} = (-2)^n e^{xx} \int_x^\infty dt (t-x)^n e^{-tt},
$$

whence also

$$
(13) \qquad y_{n+1} = \frac{(2x)^{n+1} e^{xx}}{1 \cdot 2 \cdot 3 \cdots n} \int_x^\infty dt (t-x)^n e^{-tt}
$$

which is always a positive quantity. Therefore, it immediately follows, what he had to prove, that the neglected quotient is always positive.

Koenigsberg, 30 June 1834