On a certain singular transformation of a double integral *

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1.

That celebrated dissertation written by Gauss with the title "*Determinatio attractionis etc.*", read in the Commentariae Soc. Gott., mainly consisted in this that the given expression

$$\frac{dE}{\sqrt{(A-a\cos E)^2+(B-b\sin E)^2+CC}}$$

is reduced to this simpler form:

$$\frac{dP}{\sqrt{G+G'\cos^2 P+G''\sin^2 P}};$$

this was demonstrated to happen by the author by the substitution:

$$\cos E = \frac{\alpha+\alpha'\cos P+\alpha''\sin P}{\gamma+\gamma'\cos P+\gamma''\sin P},$$

$$\sin E = \frac{\beta+\beta'\cos P+\beta''\sin P}{\gamma+\gamma'\cos P+\gamma''\sin P},$$

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having determined the nine coefficients appropriately. While I studied that extraordinary paper again and again, it did not escape me that almost the same analysis can be applied to a remarkable transformation of a certain double integral, which result I wanted to publish even more, since the theory of double integrals has still hardly been begun to be developed.

2.

For, put

$$\begin{aligned} e &= a + a' \cos^2 \psi + a'' \sin^2 \psi \cos^2 \varphi + a''' \sin \psi \sin^2 \varphi \\ &+ 2b' \cos \psi + 2b'' \sin \psi \cos \varphi + 2b''' \sin \psi \sin \varphi \\ &+ 2c' \sin^2 \sin \psi \cos \varphi \sin \varphi + 2c'' \cos \psi \sin \psi \sin \varphi + 2c''' \cos \psi \sin \psi \cos \varphi, \end{aligned}$$

which expression we see, except the constant term, to involve the terms $\cos \psi$, $\sin \psi \cos \varphi$, $\sin \psi \sin \varphi$, their squares and the products of two of them. Now, I will prove that the expression

$$\int \int \frac{\sin \psi d\psi d\varphi}{e}$$

can be transformed into this simpler one:

$$\int \int \frac{\sin P dP d\vartheta}{G + G' \cos^2 P + G'' \sin^2 P \cos^2 \vartheta + G''' \sin^2 P \sin^2 \vartheta},$$

and this is achieved by the substitution:

$$\begin{aligned} \cos P &= \frac{\alpha + \alpha' \cos \psi + \alpha'' \sin \psi \cos \varphi + \alpha''' \sin \psi \sin \varphi}{\delta + \delta' \cos \psi + \delta'' \sin \psi \cos \varphi + \delta''' \sin \psi \sin \varphi}, \\ \sin P \cos \vartheta &= \frac{\beta + \beta' \cos \psi + \alpha'' \sin \psi \cos \varphi + \alpha''' \sin \psi \sin \varphi}{\delta + \delta' \cos \psi + \delta'' \sin \psi \cos \varphi + \delta''' \sin \psi \sin \varphi}, \\ \sin P \sin \vartheta &= \frac{\gamma + \gamma' \cos \psi + \gamma'' \sin \psi \cos \varphi + \gamma''' \sin \psi \sin \varphi}{\delta + \delta' \cos \psi + \delta'' \sin \psi \cos \varphi + \delta''' \sin \psi \sin \varphi}, \end{aligned}$$

having determined the sixteen coefficients in the correct way. Let us now tackle their determination and the determination of the quantities G, G', G'', G''' etc.

3.

Since

$$\cos^2 P + \sin^2 P \cos^2 \vartheta + \sin^2 P \sin^2 \vartheta = 1,$$

the expression

$$\begin{aligned}(\alpha\ +\alpha' \cos\psi\ +\alpha'' \sin\psi\cos\varphi\ +\alpha''' \sin\psi\sin\varphi)^2 \\ +\ (\beta+\beta'\cos\psi\ +\beta''\sin\psi\cos\varphi\ +\beta'''\sin\psi\sin\varphi)^2\end{aligned}$$

necessarily vanishes, whence, since

$$\cos^2\psi + \sin^2\psi\cos^2\varphi + \sin^2\psi\sin^2\varphi = 1,$$

to reason as Gauss, it has to take on this form:

$$k(\cos^2\psi + \sin^2\psi\cos^2\varphi + \sin^2\psi\sin^2\varphi - 1).$$

Hence we obtain these ten conditions:

$$(\text{I.})\qquad \begin{cases} \alpha\alpha + \beta\beta + \gamma\gamma - \delta\delta = -k, \\ \alpha'\alpha' + \beta'\beta' + \gamma'\gamma' - \delta'\delta' = k, \\ \alpha''\alpha'' + \beta''\beta'' + \gamma''\gamma'' - \delta''\delta'' = k, \\ \alpha'''\alpha''' + \beta'''\beta''' + \gamma'''\gamma''' - \delta'''\delta''' = k, \\ \\ \alpha\alpha' + \beta\beta' + \gamma\gamma' - \delta\delta' = 0, \\ \alpha\alpha'' + \beta\beta'' + \gamma\gamma'' - \delta\delta'' = 0, \\ \alpha\alpha''' + \beta\beta''' + \gamma\gamma''' - \delta\delta''' = 0, \\ \\ \alpha''\alpha''' + \beta''\beta''' + \gamma''\gamma''' - \delta''\delta''' = 0, \\ \alpha'''\alpha'' + \beta'''\beta' + \gamma'''\gamma' - \delta'''\delta' = 0, \\ \alpha'\alpha'' + \beta'\beta''' + \gamma'\gamma'' - \delta'\delta' = 0, \end{cases}$$

Since the sixteen coefficients can be multiplied by an arbitrary quantity, it is possible to assume k arbitrarily.

4.

Further let us put that the expression:

$$\begin{aligned}&G'\ \ (\alpha + \alpha'\cos\psi + \alpha''\sin\psi\cos\varphi + \alpha'''\sin\psi\sin\varphi)^2\\
+\,&G''\ (\beta + \beta'\cos\psi + \beta''\sin\psi\cos\varphi + \beta'''\sin\psi\sin\varphi)^2\\
+\,&G'''(\gamma + \gamma'\cos\psi + \gamma''\sin\psi\cos\varphi + \gamma'''\sin\psi\sin\varphi)^2\\
+\,&G\ \ \ (\delta + \delta'\cos\psi + \delta''\sin\psi\cos\varphi + \delta'''\sin\psi\sin\varphi)^2\\
=\,&(G + G'\cos^2 P + G''\sin^2 P\cos^2\vartheta + G'''\sin^2 P\sin^2\vartheta)\\
&\times(\delta + \delta'\cos\psi + \delta''\sin\psi\cos\varphi + \delta'''\sin\psi\sin\varphi)^2\end{aligned}$$

goes over into *ke*.

Hence these ten equations must be satisfied:

$$\text{(II.)}\quad\left\{\begin{aligned}
&G'\alpha\alpha + G''\beta\beta + G'''\gamma\gamma + G\delta\delta = ak,\\
&G'\alpha'\alpha' + G''\beta'\beta' + G'''\gamma'\gamma' + G\delta'\delta' = a'k,\\
&G'\alpha''\alpha'' + G''\beta''\beta'' + G'''\gamma''\gamma'' + G\delta''\delta'' = a''k,\\
&G'\alpha'''\alpha''' + G''\beta'''\beta''' + G'''\gamma'''\gamma''' + G\delta'''\delta''' = a'''k,\\[1ex]
&G'\alpha\alpha' + G''\beta\beta' + G'''\gamma\gamma' + G\delta\delta' = b'k,\\
&G'\alpha\alpha'' + G''\beta\beta'' + G'''\gamma\gamma'' + G\delta\delta'' = b''k,\\
&G'\alpha\alpha''' + G''\beta\beta''' + G'''\gamma\gamma''' + G\delta\delta''' = b'''k,\\[1ex]
&G'\alpha''\alpha''' + G''\beta''\beta''' + G'''\gamma''\gamma''' + G\delta''\delta''' = c'k,\\
&G'\alpha'''\alpha' + G''\beta'''\beta' + G'''\gamma'''\gamma'' + G\delta'''\delta' = c''k,\\
&G'\alpha'\alpha'' + G''\beta'\beta'' + G'''\gamma'\gamma'' + G\delta'\delta'' = c'''k,
\end{aligned}\right.$$

By means of the twenty equations (I.) and (II.) both the sixteen coefficients α, β, γ etc. and the four quantities G, G', G'', G''' are determined. Additionally, one has to note that from the equations (I.) the equations (II.) follow putting

$$G' = G'' = G''' = 1, \quad G = -1;$$
$$a' = a'' = a''' = 1, \quad a = -1; \quad b' = b'' = b''' = c' = c'' = c''' = 0.$$

Having constituted these things concerning the formulas, whatever follows from the equations (II.), it is possible to derive results similar to it. The same, on a similar subject, was mentioned in the paper on the principal axes of surfaces of second order.

5.

Given a system of equations:

$$\begin{aligned}
&\alpha u + \beta x + \gamma y + \delta z + m, \\
&\alpha' u + \beta' x + \gamma' y + \delta' z + m', \\
&\alpha'' u + \beta'' x + \gamma'' y + \delta'' z + m'', \\
&\alpha''' u + \beta''' x + \gamma''' y + \delta''' z + m''',
\end{aligned}$$

let us put that their resolution is found to be:

$$\begin{aligned}
Am + A'm' + A''m'' + A'''m''' &= u, \\
Bm + B'm' + B''m'' + B'''m''' &= x, \\
Cm + C'm' + C''m'' + C'''m''' &= y, \\
Dm + D'm' + D''m'' + D'''m''' &= z.
\end{aligned}$$

We suppressed the sixteen values of the quantities A, B etc. for the sake of brevity; they are given in books on Algebra, and the algorithms, by means of which they are formed, are well understood today. Having constituted all this, after we had selected the following equations from the equations (II.):

$$\begin{aligned}
\alpha \cdot G'\alpha + \beta \cdot G''\beta + \gamma \cdot G'''\gamma + \delta \cdot G\delta &= ak, \\
\alpha' \cdot G'\alpha + \beta' \cdot G''\beta + \gamma' \cdot G'''\gamma + \delta' \cdot G\delta &= b'k, \\
\alpha'' \cdot G'\alpha + \beta'' \cdot G''\beta + \gamma'' \cdot G'''\gamma + \delta'' \cdot G\delta &= b''k, \\
\alpha''' \cdot G'\alpha + \beta''' \cdot G''\beta + \gamma''' \cdot G'''\gamma + \delta''' \cdot G\delta &= b'''k,
\end{aligned}$$

by their resolution we obtain:

$$
\text{(III.)}\quad
\begin{cases}
k(Aa + A'b' + A''b'' + A'''b''') = G'\alpha, \\
k(Ba + B'b' + B''b'' + B'''b''') = G''\beta, \\
k(Ca + C'b' + C''b'' + C'''b''') = G'''\gamma, \\
k(Da + D'b' + D''b'' + D'''b''') = G\delta. \\
\qquad \text{The same way one obtains} \\
k(Ab' + A'a' + A''c''' + A'''c'') = G'\alpha', \\
k(Bb' + B'a' + B''c''' + B'''c'') = G''\beta', \\
k(Cb' + C'a' + C''c''' + C'''c'') = G'''\gamma', \\
k(Db' + D'a' + D''c''' + D'''c'') = G\delta'. \\
k(Ab'' + A'c''' + A''a'' + A'''c') = G'\alpha'', \\
k(Bb'' + B'c''' + B''a'' + B'''c') = G''\beta'', \\
k(Cb'' + C'c''' + C''a'' + C'''c') = G'''\gamma'', \\
k(Db'' + D'c''' + D''a'' + D'''c') = G\delta''. \\
k(Ab''' + A'c'' + A''c' + A'''a''') = G'\alpha''', \\
k(Bb''' + B'c'' + B''c' + B'''a''') = G''\beta''', \\
k(Cb''' + C'c'' + C''c' + C'''a''') = G'''\gamma''', \\
k(Db''' + D'c'' + D''c' + D'''a''') = G\delta'''.
\end{cases}
$$

From these equation in the mentioned way others are derived, which corresponds to the equations (I.):

$$
\text{(IV.)}\quad
\begin{cases}
\alpha = -kA, & \alpha' = kA', & \alpha'' = kA'', & \alpha''' = kA''', \\
\beta = -kB, & \beta' = kB', & \beta'' = kB'', & \beta''' = kB''', \\
\gamma = -kC, & \gamma' = kC', & \gamma'' = kC'', & \gamma''' = kC''', \\
\delta = kD, & \delta' = -kD', & \delta'' = -kD'', & \delta''' = -DC'''.
\end{cases}
$$

6.

Having combined each two equations, immediately these four systems of equations result:

$$
\text{(V.)}\quad
\begin{cases}
1) & 0 = A(a+G') + A'b' + A''b'' + A'''b''', \\
 & 0 = Ab' + A'(a'-G') + A''c''' + A'''c'', \\
 & 0 = Ab'' + A'c''' + A''(a''-G') + A'''c', \\
 & 0 = Ab''' + A'c'' + A''c' + A'''(a'''-G'), \\[1em]
2) & 0 = B(a+G'') + B'b' + B''b'' + B'''b''', \\
 & 0 = Bb' + B'(a'-G') + B''c''' + B'''c'', \\
 & 0 = Bb'' + B'c''' + B''(a''-G') + B'''c', \\
 & 0 = Bb''' + B'c'' + B''c' + B'''(a'''-G'), \\[1em]
3) & 0 = C(a+G''') + C'b' + C''b'' + C'''b''', \\
 & 0 = Cb' + C'(a'-G') + C''c''' + C'''c'', \\
 & 0 = Cb'' + C'c''' + C''(a''-G') + C'''c', \\
 & 0 = Cb''' + C'c'' + C''c' + C'''(a'''-G'), \\[1em]
4) & 0 = D(a-G) + D'b' + D''b'' + D'''b''', \\
 & 0 = Db' + D'(a'+G) + D''c''' + D'''c'', \\
 & 0 = Db'' + D'c''' + D''(a''+G') + D'''c', \\
 & 0 = Db''' + D'c'' + D''c' + D'''(a'''+G).
\end{cases}
$$

From the first system by elimination of the quantities A, A', A'', A''' one obtains an equation, by which G' is to be understood to be given; in like manner, from the second, third, fourth system having respectively eliminated B, B', B'', B'''; C, C', C'', C'''; D, D', D'', D''' one obtains equations we see to give the quantities G'', G', G. But considering these four systems it is plain that all the equations with respect to the quantities G, $-G'$, $-G''$, $-G'''$ are completely the same, so that, if one of the quantities G, $-G'$, $-G''$, $-G'''$ is denoted by x, one and the same equation among x and the given quantities will exhibit those four quantities G, $-G'$, $-G''$, $-G'''$ etc.. That equation, after the elimination, becomes:

$$
\begin{aligned}
0 = &(a-x)(a'+x)(a''+x)(a'''+x)\\
&-(a-x)(a'+x)c'c' - (a-x)(a''+x)c''c'' - (a-x)(a'''+x)c'''c'''\\
&-(a''+x)(a'''+x)b'b' - (a'''+x)(a'+x)b''b'' - (a'+x)(a''+x)b'''b'''\\
&+2c'c''c'''(a-x) + 2c'b''b'''(a'+x) + 2c''b'''b'(a''+x) + 2c'''b'b''(a'''+x)\\
&+b'b'c'c' + b''b''c''c'' + b'''b'''c'''c''' - 2b'b''c'c'' - 2b'''c'''c'.
\end{aligned}
$$

We will postpone the most important task of investigating the nature of this biquadratic equation to another disquisition on the subject.

7.

Among the sixteen quantities α, β etc. and the sixteen quantities, which are derived from them, A, A' etc. many most elegant relations exist, which, since Laplace, Vandermonde in the commentariae academiae Parisiensis in 1772, Gauss in the disquis. arithm. section V, J. Binet in vol. IX. of the journal of the institutum polytechnichum Parisiensis and others published them, are well known; and I will quote only few things, which in our special case are easily derived from their results by means of the equations (IV.). First, I will note the following ten equations similar to the equations (I.):

$$
\text{(VII.)}\quad
\begin{cases}
-\alpha\alpha + \alpha'\alpha' + \alpha''\alpha'' + \alpha'''\alpha''' = k,\\
-\beta\beta + \beta'\beta' + \beta''\beta'' + \beta'''\beta''' = k,\\
-\gamma\gamma + \gamma'\gamma' + \gamma''\gamma'' + \delta'''\delta''' = k,\\
-\delta\delta + \delta'\delta' + \delta''\delta'' + \delta'''\delta''' = -k,\\[1ex]
-\alpha\beta + \alpha'\beta' + \alpha''\beta'' + \alpha'''\beta''' = 0,\\
-\alpha\gamma + \alpha'\gamma' + \alpha''\gamma'' + \alpha'''\gamma''' = 0,\\
-\alpha\delta + \alpha'\delta' + \alpha''\delta'' + \alpha'''\delta''' = 0,\\[1ex]
-\beta\gamma + \beta'\gamma' + \beta''\beta'' + \beta'''\gamma''' = 0,\\
-\gamma\delta + \gamma'\delta' + \gamma''\delta'' + \gamma'''\delta''' = 0,\\
-\delta\beta + \delta'\beta' + \delta''\beta'' + \delta'''\beta''' = 0.
\end{cases}
$$

Further, one can prove the following eighteen equations:

$$
(\text{VIII.})\quad
\begin{cases}
\alpha\beta' - \alpha'\beta = -(\gamma''\delta''' - \gamma'''\delta'')\varepsilon, & \alpha'\beta'' - \alpha''\beta' = (\gamma\delta''' - \gamma'''\delta)\varepsilon, \\
\alpha\beta'' - \alpha''\beta = -(\gamma'''\delta' - \gamma'\delta''')\varepsilon, & \alpha''\beta''' - \alpha'''\beta'' = (\gamma\delta' - \gamma'\delta)\varepsilon, \\
\alpha\beta''' - \alpha'''\beta = -(\gamma'\delta'' - \gamma''\delta')\varepsilon, & \alpha'''\beta' - \alpha'\beta''' = (\gamma\delta'' - \gamma''\delta)\varepsilon, \\
\\
\alpha'\gamma - \alpha'\gamma = -(\delta''\beta''' - \delta'''\beta'')\varepsilon, & \alpha'\gamma'' - \alpha''\gamma' = (\delta\beta''' - \delta'''\beta)\varepsilon, \\
\alpha\gamma'' - \alpha''\gamma = -(\delta'''\beta' - \delta'\beta''')\varepsilon, & \alpha''\gamma''' - \alpha'''\gamma'' = (\delta\beta' - \delta'\beta)\varepsilon, \\
\alpha\gamma''' - \alpha'''\gamma = -(\delta'\beta'' - \delta''\beta')\varepsilon, & \alpha'''\gamma' - \alpha'\gamma''' = (\delta\beta'' - \delta''\beta)\varepsilon, \\
\\
\alpha\delta' - \alpha'\delta = (\beta''\gamma''' - \beta'''\gamma'')\varepsilon, & \alpha'\delta'' - \alpha''\delta' = -(\beta\gamma''' - \beta'''\gamma)\varepsilon, \\
\alpha\delta'' - \alpha''\delta = (\beta'''\gamma' - \beta'\gamma''')\varepsilon, & \alpha''\delta''' - \alpha'''\delta'' = -(\beta\gamma' - \beta'\gamma)\varepsilon, \\
\alpha\delta''' - \alpha'''\delta = (\beta'\gamma'' - \beta''\gamma')\varepsilon, & \alpha\delta''' - \beta'\gamma'' = -(\beta\beta'' - \delta''\gamma)\varepsilon,
\end{cases}
$$

8.

From the equations, by means of which we expressed P, ϑ in terms of ψ, φ, namely:

$$
\cos P = \frac{\alpha + \alpha' \cos\psi + \alpha'' \sin\psi\cos\varphi + \alpha''' \sin\psi\sin\varphi}{\delta + \delta' \cos\psi + \delta'' \sin\psi\cos\varphi + \delta''' \sin\psi\sin\varphi},
$$

$$
\sin P\cos\vartheta = \frac{\beta + \beta' \cos\psi + \beta'' \sin\psi\cos\varphi + \beta''' \sin\psi\sin\varphi}{\delta + \delta' \cos\psi + \delta'' \sin\psi\cos\varphi + \delta''' \sin\psi\sin\varphi},
$$

$$
\sin P\sin\vartheta = \frac{\gamma + \gamma' \cos\psi + \gamma'' \sin\psi\cos\varphi + \gamma''' \sin\psi\sin\varphi}{\delta + \delta' \cos\psi + \delta'' \sin\psi\cos\varphi + \delta''' \sin\psi\sin\varphi},
$$

by means of the equations (I.) one easily proves the following ones, by which ψ, φ are vice versa expressed by P, ϑ:

$$
\text{(IX.)}\quad \begin{cases}
\delta - \alpha\cos P - \beta\sin P\cos\vartheta - \gamma\sin P\sin\vartheta \\
\qquad = \dfrac{k}{\delta + \delta'\cos\psi + \delta''\sin\psi\cos\vartheta + \delta'''\sin\psi\sin\varphi'}, \\
\cos\psi = \dfrac{-\delta' + \alpha'\cos P + \beta'\sin P\cos\vartheta + \gamma'\sin P\sin\vartheta}{\delta - \alpha\cos P - \beta\sin P\cos\vartheta - \gamma\sin P\sin\vartheta}, \\
\sin\psi\cos\varphi = \dfrac{-\delta'' + \alpha''\cos P + \beta''\sin P\cos\vartheta + \gamma''\sin P\sin\vartheta}{\delta - \alpha\cos P - \beta\sin P\cos\vartheta - \gamma\sin P\sin\vartheta}, \\
\sin\psi\sin\varphi = \dfrac{-\delta''' + \alpha'''\cos P + \beta'''\sin P\cos\vartheta + \gamma'''\sin P\sin\vartheta}{\delta - \alpha\cos P - \beta\sin P\cos\vartheta - \gamma\sin P\sin\vartheta}.
\end{cases}
$$

9.

It remains to find the values of the sixteen quantities α, β etc. This is achieved by the following formulas:

$$
\text{(X.)}\quad \begin{cases}
\dfrac{\alpha'\alpha'}{k} = \dfrac{(a'-G')(a''-G')(a'''-G') - c'c'(a'-G') - c''c''(a''-G') - c'''c'''(a'''-G') + 2c'c''c'''}{(G'+G)(G'-G'')(G'-G''')}, \\
\dfrac{\alpha'\alpha'}{k} = \dfrac{(a''-G')(a'''-G')(a+G') - c'c'(a+G') - b'''b'''(a''-G') - b''b''(a'''-G') + 2b''b'''c'}{(G'+G)(G'-G'')(G'-G''')}, \\
\dfrac{\alpha''\alpha''}{k} = \dfrac{(a'''-G')(a+G')(a'-G') - c''c''(a+G') - b'b'(a'''-G') - b'''b'''(a'-G') + 2b'''b'c''}{(G'+G)(G'-G'')(G'-G''')}, \\
\dfrac{\alpha'''\alpha'''}{k} = \dfrac{(a+G')(a'-G')(a''+G') - c'''c'''(a+G') - b''b''(a'-G') - b'b'(a''-G') + 2b'b''c'''}{(G'+G)(G'-G'')(G'-G''')}
\end{cases}
$$

From these equations one can derive three other systems, if instead of

$$G,\quad G',\quad G'',\quad G''',\quad \alpha\alpha,\quad \alpha'\alpha',\quad \alpha''\alpha'',\quad \alpha'''\alpha'''$$

one respectively puts

$$
\begin{array}{cccccccc}
G, & G'', & G', & G''', & \beta\beta, & \beta'\beta', & \beta''\beta'', & \beta'''\beta''', \\
G, & G''', & G'', & G', & \gamma\gamma, & \gamma'\gamma', & \gamma''\gamma'', & \gamma'''\gamma''', \\
-G', & -G, & G'', & G', & -\delta\delta, & -\delta'\delta', & -\delta''\delta, & -\delta'''\delta''',
\end{array}
$$

Further, one has to note that the products of two of the quantities α, α', α'', α''', two of the quantities β, β', β'', β''' etc., two of the quantities γ, γ', γ'', γ''' etc., two of the quantities δ, δ', δ'', δ''' etc. can be expressed rationally. Hence having taken the signs of α, β, γ, δ arbitrarily, the signs of the remaining quantities are determined by them. But:

$$\text{(XI.)}\quad \begin{cases} \dfrac{\alpha\alpha'}{k} = \dfrac{b'(a''-G')(a'''-G') - c''b'''(a''-G') - c'''b''(a'''-G') - b'c'c' + b''c'c'' + b'''c'c'''}{(G'+G)(G'-G'')(G'-G''')}, \\ \dfrac{\alpha\alpha''}{k} = \dfrac{b''(a'''-G')(a'-G') - c'''b''(a'''-G') - c'b'''(a'-G') - b''c''c'' + b'''c''c''' + b'c''c'}{(G'+G)(G'-G'')(G'-G''')}, \\ \dfrac{\alpha\alpha'''}{k} = \dfrac{b'''(a'-G')(a''-G') - c'b''(a'-G') - c''b'(a''-G') - b''''c'''c''' + b'c'''c' + b''c'''c''}{(G'+G)(G'-G'')(G'-G''')}, \\ -\dfrac{\alpha''\alpha''}{k} = \dfrac{c'(a+G')(a'-G') - c''c'''(a+G') - b''b'''(a'-G') - c''b'b' + c''b'b'' + c'''b'b'''}{(G'+G)(G'-G'')(G'-G''')}, \\ -\dfrac{\alpha'''\alpha'}{k} = \dfrac{c''(a+G')(a''-G') - c'''c'(a+G') - b'''b'(a''-G') - c''b''b'' + c'''b''b''' + c'b''b''}{(G'+G)(G'-G'')(G'-G''')}, \\ -\dfrac{\alpha'\alpha''}{k} = \dfrac{c'''(a+G')(a'''-G') - c'c''(a+G') - b'b''(a'''-G') - c''b'''b''' + c'b'''b' + c''b'''b''}{(G'+G)(G'-G'')(G'-G''')}. \end{cases}$$

From these formulas one derives the other remaining ones, if instead of

$$k,\quad G,\quad G',\quad G'',\quad G''',\quad \alpha,\quad \alpha',\quad \alpha'',\quad \alpha$$

one writes

$$\begin{array}{ccccccccc} k, & G, & G'', & G', & G''', & \beta, & \beta', & \beta'', & \beta''', \\ k, & G, & G''', & G'', & G', & \gamma, & \gamma', & \gamma'', & \gamma''', \\ -k, & -G', & -G, & G'', & G''', & \delta, & \delta', & \delta'', & \delta'''. \end{array}$$

respectively. These formulas, among many other things, tell us that one has to set different roots of equation (VI.) for the quantities G, $-G'$, $-G''$, $-G'''$ etc., for one of the quantities α, β etc. not to become infinite. Furthermore, for the sake of brevity, we presented the analysis, by means of which the equations (X.) and (XI.) are found, in compact form.

10.

Now, if one wants to transform the double integral

$$\int\int UdPd\vartheta,$$

while U denotes a certain function of the quantities P, ϑ, by means of the equations

$$f(P,\vartheta) = \Pi(\psi,\vartheta),$$
$$F(P,\vartheta) = \chi(\psi,\vartheta),$$

it is known that

$$\int\int UdPd\vartheta = \int\int Ud\psi d\varphi \frac{\frac{\partial\Pi}{\partial\psi}\cdot\frac{\partial\chi}{\partial\varphi} - \frac{\partial\Pi}{\partial\varphi}\cdot\frac{\partial\chi}{\partial\psi}}{\frac{\partial f}{\partial P}\cdot\frac{\partial F}{\partial\vartheta} - \frac{\partial f}{\partial\vartheta}\cdot\frac{\partial F}{\partial P}}.$$

In our case, since

$$\delta + +\delta'\cos\psi + \delta''\sin\psi\cos\varphi + \delta'''\sin\psi\sin\varphi$$
$$= \frac{k}{\delta - \alpha\cos P - \beta\sin P\cos\vartheta - \gamma\sin P\sin\vartheta},$$

abbreviating

$$t = \delta - \alpha\cos P - \beta\sin P\cos\vartheta - \gamma\sin P\sin\vartheta,$$

one can put

$$f(P,\vartheta) = \frac{k}{t}\sin P\cos\vartheta,$$

$$F(P,\vartheta) = \frac{k}{t}\sin P\sin\vartheta,$$

$$\Pi(\psi,\vartheta) = \beta + \beta'\cos\psi + \beta''\sin\psi\cos\varphi + \beta'''\sin\psi\sin\varphi,$$

$$\chi(\psi,\varphi) = \gamma + \gamma'\cos\psi + \gamma''\sin\psi\cos\varphi + \gamma'''\sin\psi\sin\varphi.$$

Hence it will be

$$t^4\left(\frac{\partial f}{\partial P}\cdot\frac{\partial F}{\partial\vartheta} - \frac{\partial f}{\partial\vartheta}\cdot\frac{\partial F}{\partial P}\right)$$

$$= kk \sin P \cdot t \left(t \cos P - \frac{\partial t}{\partial P} \sin P \right) = kk \sin P \cdot P \cdot t(\delta \cos P - \alpha)$$

$$= kk \sin P \cdot tt \left\{ (\delta\alpha' - \delta'\alpha) \cos\psi + (\delta\alpha'' - \delta''\alpha) \sin\psi \cos\varphi + (\delta\alpha''' - \alpha\delta''') \sin\psi \sin\varphi \right\},$$

which expression because of the equation (VIII.) we see to go over into this one:

$$-k\varepsilon \sin P \cdot tt \left\{ (\beta''\gamma''' - \beta'''\gamma'') \cos\psi + (\beta'''\gamma' - \beta'\gamma''') \sin\psi \cos\varphi + (\beta'\gamma'' - \beta''\gamma') \sin\psi \sin\varphi \right\}.$$

Further, it will be

$$\frac{\partial \Pi}{\partial \psi} \cdot \frac{\partial \chi}{\partial \varphi} - \frac{\partial \Pi}{\partial \varphi} \cdot \frac{\partial \chi}{\partial \psi}$$

$$= \sin\psi \left\{ -\beta' \sin\psi + \beta'' \cos\psi \cos\varphi + \beta''' \cos\psi \sin\varphi \right\} \left\{ -\gamma'' \sin\varphi + \gamma''' \cos\varphi \right\}$$

$$- \sin\psi \left\{ -\gamma' \sin\psi + \gamma'' \cos\psi \cos\varphi + \gamma''' \cos\psi \sin\varphi \right\} \left\{ -\beta'' \sin\varphi + \beta''' \cos\varphi \right\}$$

$$= \sin\psi \left\{ (\beta''\gamma''' - \beta'''\gamma'') \cos\psi + (\beta'''\gamma' - \beta'\gamma''') \sin\psi \cos\varphi + (\beta'\gamma'' - \beta''\gamma') \sin\psi \sin\varphi \right\}.$$

Hence

$$\frac{\frac{\partial \Pi}{\partial \psi} \cdot \frac{\partial \chi}{\partial \varphi} - \frac{\partial \Pi}{\partial \varphi} \cdot \frac{\partial \chi}{\partial \psi}}{\frac{\partial f}{\partial P} \cdot \frac{\partial F}{\partial \vartheta} - \frac{\partial f}{\partial \vartheta} \cdot \frac{\partial F}{\partial P}} = -\frac{tt \sin\psi}{k\varepsilon \sin P}.$$

Now, since

$$G + G' \cos^2 P + G'' \sin^2 P \cos^2 \vartheta + G''' \sin^2 P \sin^2 \vartheta = \frac{tte}{k},$$

finally:

$$\pm \int\int \frac{\sin P dP d\vartheta}{G + G' \cos^2 P + G'' \sin^2 P \cos^2 \vartheta + G''' \sin^2 P \sin^2 \vartheta} = \int\int \frac{\sin\psi d\psi d\varphi}{e}.$$

Since these disquisitions are to be continued, I wanted to publish this comment for the analysts.
Written in the month of June 1827, at the University of Koenigsberg