A Transformation Formula for definite Integrals *

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1.

It is a known theorem and one of highest importance that, having expanded a function U into a series of cosines and sinus of multiples of the angle x, the coefficients of the expansions are determined by the definite integrals

$$\int\limits_0^{2\pi} U \cos ixdx, \quad \int\limits_0^{2\pi} U \sin ixdx.$$

Since the values of these integrals can certainly always be found by quadratures, one has a general method, to do expansions of such a kind.
If the expansion converges well, the value of the integrals, while i increases, decreases rapidly; how this happens, is easily understood. For, for larger numbers i the positive and negative values of the functions under the integral sign alternate more rapidly, and cancel each other for the most part. But hence a certain inconvenience of the method results; for, we see that the value of a very small quantity in question is to be determined by differences of very large quantities. In astronomical calculations the determination of huge inequalities is extremely problematic because of this inconvenience.
In the special case, in which the following expression is propounded to be expanded

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$$\frac{1}{\sqrt{1-2a\cos x+a^2}},$$

Legendre once found an ingenious transformation of the integrals exhibiting the coefficients of the expansion, by which this inconvenience is avoided. This transformation is contained in the formula

$$\int\limits_0^\pi \frac{\cos ixdx}{\sqrt{1-2a\cos x+a^2}} = a^i\int\limits_0^\pi \frac{\sin^{2i} xdx}{\sqrt{1-a^2\sin^2 x}}.$$

The transformed integral is multiplied by a constant small factor a^i; furthermore, even under the integral sign one finds the small factor $\sin^{2i} x$; so that, if you apply quadratures to the transformed integral, you find the value of the integral as the sum of very small positive quantities; this yields quick and appropriate calculations. Legendre believed that that transformation formula is the only one of its kind. But I recently coincidentally discovered a general formula, by which, having propounded the expansion of function into a series of cosines of multiple angles, the integrals exhibiting the coefficients of the expansion are transformed into others, in which under the integral sign instead of $\cos ix$ one finds the factor $\sin^{2i} x$, and instead of the function to be expanded one finds its differential of order i with respect to $\cos x$. If the function to be expanded has several angles, e.g, x, y, having applied the transformation to one variable after the other, double integrals exhibiting the coefficients of the expansion are changed into others, in which instead of the factor $\cos ix\cos i'y$ one finds the factor $\sin^{2i} x\sin^{2i'} y$ and instead of the functions its differential of order i with respect to $\cos x$ and of order i' with respect to $\cos y$. This transformation formula is of the same kind as the one once propounded by Legendre. In the following I will explain this subject and will illustrate it in various examples.

2.

While m, n are positive integers, one has the known formulas

$$(1)\qquad \int\limits_0^{\frac{1}{2}\pi} \sin^{2m} x\cos^{2n} xdx = \frac{(2m-1)(2m-3)\cdot 1\cdot(2n-1)(2n-3)\cdots 1}{(2m+2n)(2m+2n-2)\cdots 2}\cdot\frac{\pi}{2},$$

$$(2.)\quad \int_0^{\frac{1}{2}\pi} \cos^{2m} x \cos 2nx dx = (-1)^n \int_0^{\frac{1}{2}\pi} \sin^{2m} x \cos 2nx dx$$
$$= \frac{1}{2^{2m}} \cdot \frac{2m(2m-1)\cdots(m+n+1)}{1\cdot 2\cdots(m-n)} \cdot \frac{\pi}{2},$$

$$(3.)\quad \int_0^{\frac{1}{2}\pi} \cos^{2m+1} x \cos(2n+1) x dx = (-1)^n \int_0^{\frac{1}{2}\pi} \sin^{2m+1} x \cos(2n+1) x dx$$
$$= \frac{1}{2^{2m+1}} \cdot \frac{(2m+1)2m\cdots(m+n+2)}{1\cdot 2\cdots(m-n)} \cdot \frac{\pi}{2}.$$

The following single formula, which holds, if $p-i$ denotes an even positive number, contains the last two formulas:

$$(4.)\quad \int_0^{\frac{1}{2}\pi} \cos^p x \cos ix dx = \frac{1}{2^p} \cdot \frac{p(p-1)\cdots\left(\frac{p+i}{2}+1\right)}{1\cdot 2\cdots\left(\frac{p-i}{2}\right)} \cdot \frac{\pi}{2},$$

which formula can also be exhibited this way:

$$\int_0^{\frac{1}{2}\pi} \cos^p x \cos ix dx$$
$$= \frac{p(p-1)\cdots(p-i+1)}{1\cdot 3\cdots(2i-1)} \cdot \frac{(p-i-1)(p-i-3)\cdots(2i-1)(2i-3)\cdots 1}{2\cdot 4\cdot 6\cdots(p+i)} \cdot \frac{\pi}{2},$$

whence from (1) the following formula results

$$(5.)\quad \int_0^{\frac{1}{2}\pi} \cos^p x \cos ix dx = \frac{p(p-1)\cdots(p-i+1)}{1\cdot 3\cdots(2i-1)} \int_0^{\frac{1}{2}\pi} \sin^{2i} x \cos^{p-i} x dx.$$

For this formula also to hold for and odd number $p-i$, let us extend both integrals from 0 to π; having done this both vanish for odd $p-i$. *Therefore, while p, i denote positive integer numbers, it will be*

$$(6.)\quad \int_0^{\pi} \cos^p x \cos ix dx = \frac{p(p-1)\cdots(p-i+1)}{1\cdot 3\cdots(2i-1)} \int_0^{\pi} \sin^{2i} x \cos^{p-1} x dx.$$

3.

Let us suppose that a function of z, $f(z)$, can be expanded into a power series in z, and its expansion is

$$f(z) = \sum A_p z^p;$$

further, following Lagrange, let us put

$$\frac{d^i f(z)}{dz^i} = f^{(i)}(z),$$

whence

$$f^{(i)}(z) = \sum p(p-1)\cdots(p-i+1)A_p z^{p-i};$$

from (6) it will be

$$\int_0^\pi f(\cos x)\cos ix dx = \sum A_p \int_0^\pi \cos^p x \cos ix dx$$

$$= \frac{1}{1\cdot 3\cdots(2i-1)} \int_0^\pi dx \sin^{2i} x \left\{\sum p(p-1)\cdots(p-i+1)A_p \cos^{p-i} x\right\}$$

or

$$(7) \qquad \int_0^\pi f(\cos x)\cos ix dx = \frac{1}{1\cdot 3\cdots(2i-1)} \int_0^\pi f^{(i)}(\cos x)\sin^{2i} x dx.$$

This formula for the definite integral suggests the propounded transformation.

4.

Formula (7), found in the preceding, can also be demonstrated using the following memorable lemma:

"The differential of order $(i-1)$ of $\sin^{2i-1} x$ with respect to $\cos x$ is

$$(-1)^{i-1} 1\cdot 3\cdot 5\cdots(2i-1)\frac{\sin ix}{i},$$

or, having put $\cos x = z$, one has

$$\frac{d^{i-1}(1-z^2)^{\frac{2i-1}{2}}}{dz^{i-1}} = (-1)^{i-1} 1 \cdot 3 \cdot 5 \cdots (2i-1) \frac{\sin ix}{i}."$$

To demonstrate it I observe that, having put

$$p = a + bz + cz^2, \quad q = b + 2cz,$$

one has in general

$$\frac{d^n p^r}{dz^n} = r(r-1)\cdots(r-n+1)p^{r-n}q^n \begin{cases} 1 + \dfrac{n(n-1)}{r-n+1}\dfrac{cp}{q^2} + \dfrac{n(n-1)(n-2)(n-3)}{(r-n+1)(r-n+2)\cdot 2}\dfrac{c^2p^2}{q^4} \\ + \dfrac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{(r-n+1)(r-n+2)(r-n+3)\cdot 2\cdot 3}\dfrac{c^3p^3}{q^6} + \cdots, \end{cases}$$

cf. Lacroix, *Traite du calculu differentiel et du calcul integral, Seconde edition, T. I., pag. 183*. Hence, having substituted the values

$$p = a + bz + cz^2 = 1 - z^2 = \sin^2 x, \quad q = -2z = -2\cos x,$$
$$c = -1, \quad r = \frac{2i-1}{2}, \quad n = i - 1,$$

it results

$$\frac{d^{i-1}(1-z^2)^{\frac{2i-1}{2}}}{dz^{i-1}}$$
$$(-1)^{i-1} 3 \cdot 5 \cdots (2i-1) \left[\cos^{i-1} x \sin x - \frac{(i-1)(i-2)}{2\cdot 3}\cos^{i-3} x \sin^2 x \right.$$
$$\left. + \frac{(i-1)(i-2)(i-3)(i-4)}{2\cdot 3\cdot 4\cdot 5}\cos^{i-5} x \sin^5 x - \cdots \right]$$

or by known trigonometric formulas

$$\frac{d^{i-1}(1-z^2)^{\frac{2i-1}{2}}}{dz^{i-1}} = (-1)^{i-1} 3 \cdot 5 \cdots (2i-1) \frac{\sin ix}{i}, \tag{8}$$

what was to be demonstrated.

Having demonstrated the lemma, formula (7) is easily proven by partial integration, repeated i times. For, if a certain function w and its differentials up to the order $i-1$ vanish in the limits of the integration, it is known that by partial integration one has

$$\int w\frac{d^i v}{dz^i}dz = (-1)^i \int v\frac{d^i w}{dz^i}dz.$$

Hence, having put

$$v = f(z), \quad w = (1-z^2)^{\frac{2i-1}{2}},$$

and having extended the integration from -1 to $+1$, it results

$$\int\limits_{-1}^{+1} f^{(i)}(z)(1-z^2)^{\frac{2i-1}{2}}dz = (-1)^i \int\limits_{-1}^{+1} f(z)\frac{d^i(1-z^2)^{\frac{2i-1}{2}}}{dz^i}dz.$$

For, $(1-z^2)^{\frac{2i-1}{2}}$ and its differentials up to order $i-1$ vanish for the limits $z=-1$, $z=+1$. But having differentiated (8) with respect to z, we have

$$\frac{d^i(1-z^2)^{\frac{2i-1}{2}}}{dz^i} = (-1)^{i-1}3\cdot 5\cdots(2i-1)\cos ixdx.$$

Hence, having put $z = \cos x$, the preceding formula goes over into the following:

$$\int\limits_0^{\pi} f^{(i)}\sin^{2i}xdx = 3\cdot 5\cdots(2i-1)\int\limits_0^{\pi} f(\cos x)\cos ixdx,$$

which is the propounded formula (7).
The preceding proof assumes nothing but that the function $f(\cos x)$ and its differentials up to order i for the assigned limits of integration do not become infinite; and it does not assume, as the first proof, that the function $f(\cos x)$ can be expanded into a series of *integer* powers of $\cos x$. Therefore, formula (7) is not restricted to this case. Hence, having put $f(\cos x) = \cos^p x$, it is plain that *formula (6) also holds if p is not an integer number, if just $p > i$.*

5.

To these results we want to add the following considerations. For the sake of brevity let us set

$$B_i = \frac{1 \cdot 3 \cdots (2i-1)}{2 \cdot 4 \cdots 2i},$$

from (7) it will be

$$B_i \int_0^\pi f(\cos x) \cos ix dx = \int_0^\pi \frac{f^{(i)}(\cos x) \sin^{2i} x dx}{2^i \cdot 1 \cdot 2 \cdot 3 \cdots i}.$$

Hence, since, while h denotes a constant smaller than 1,

$$\frac{1}{2} \left\{ \frac{1}{\sqrt{1 - he^{x\sqrt{-1}}}} + \frac{1}{\sqrt{1 - he^{-x\sqrt{-1}}}} \right\} = 1 + B_1 h \cos x + B_2 h^2 \cos 2x + B_3 h^3 \cos 3x + \cdots,$$

by means of Taylor's theorem one finds

$$(9) \quad \frac{1}{2} \int_0^\pi dx f(x) \left\{ \frac{1}{\sqrt{1 - he^{x\sqrt{-1}}}} + \frac{1}{\sqrt{1 - he^{-x\sqrt{-1}}}} \right\} = \int_0^\pi f\left(\cos x + \frac{h \sin^2 x}{2}\right) dx.$$

This formula can also be exhibited this way:

$$(10) \qquad \frac{1}{2} \int_0^{2\pi} \frac{f(\cos x) dx}{\sqrt{1 - he^{x\sqrt{-1}}}} = \int_0^\pi f\left(\cos x + \frac{h \sin^2 x}{2}\right) dx.$$

Formula (9) can also be deduced from an indefinite transformation. For, having put

$$\cos \eta = \cos x + \frac{h \sin^2 x}{2},$$

it follows

$$\sqrt{1 - 2h \cos \eta + h^2} = 1 - h \cos x,$$

whence

$$\sqrt{1 - 2h \cos \eta + h^2} - (1 - h \cos \eta) = \frac{h^2 \sin^2 x}{2}.$$

From this equation, having extracted the roots, this one results:

$$\sqrt{1-he^{\eta\sqrt{-1}}}-\sqrt{1-he^{-\eta\sqrt{-1}}}=-h\sin x\sqrt{-1}.$$

Having multiplied this by

$$\sqrt{1-he^{\eta\sqrt{-1}}}+\sqrt{1-he^{-\eta\sqrt{-1}}},$$

and after a division by h, it results

$$2\sin\eta=\sin\left\{\sqrt{1-he^{\eta\sqrt{-1}}}+\sqrt{1-he^{-\eta\sqrt{-1}}}\right\}.$$

Now, having differentiated the propounded equation, we obtain

$$\sin\eta d\eta=\sin\left[1-h\cos x\right]dx.$$

But from the preceding

$$\frac{2\sin\eta}{\sin x\left[1-h\cos x\right]}=\frac{1}{\sqrt{1-he^{\eta\sqrt{-1}}}}+\frac{1}{\sqrt{1-he^{-\eta\sqrt{-1}}}};$$

hence we see *that having put*

$$(11)\qquad \cos\eta=\cos x+\frac{h\sin^2 x}{2},$$

we have

$$(12)\qquad dx=\frac{1}{2}\left\{\frac{1}{\sqrt{1-he^{\eta\sqrt{-1}}}}+\frac{1}{\sqrt{1-he^{-\eta\sqrt{-1}}}}\right\}d\eta,$$

and hence also

$$(13)\quad \int f\left(\cos x+\frac{h\sin^2 x}{2}\right)dx=\frac{1}{2}\int f(\cos\eta)\left\{\frac{1}{\sqrt{1-he^{\eta\sqrt{-1}}}}+\frac{1}{\sqrt{1-he^{-\eta\sqrt{-1}}}}\right\}d\eta.$$

If h is smaller than 1, while x grows from 0 to π, the expression $\cos x+\frac{h\sin^2 x}{2}$ continuously decreases from 1 to -1, whose differential $-\sin x\left[1-h\cos x\right]$ always gives negative values; hence at the same time the angle η continuously increases from 0 to π. Hence it is plain, that in formula (13), having extended the one integral from 0 to π, also the other is extended from 0 to π. Formula

(9) yields this.
Additionally, I observe that from the given formulas

$$\sqrt{1-2h\cos\eta+h^2}=1-h\cos x,$$
$$\sqrt{1-he^{\eta\sqrt{-1}}}+\sqrt{1-he^{-\eta\sqrt{-1}}}=-h\sin x\sqrt{-1},$$

it follows

$$(14)\qquad \begin{aligned}&\left\{1-\sqrt{1-he^{\eta\sqrt{-1}}}\right\}\left\{1+\sqrt{1-he^{-\eta\sqrt{-1}}}\right\}=he^{x\sqrt{-1}}\\ &\left\{1+\sqrt{1-he^{\eta\sqrt{-1}}}\right\}\left\{1-\sqrt{1-he^{-\eta\sqrt{-1}}}\right\}=he^{-x\sqrt{-1}}.\end{aligned}$$

From formula (12) by integration one has the expression for the angle x

$$(15)\quad x=\eta+\frac{1}{2}\eta\sin\eta+\frac{1\cdot3}{2\cdot4}\frac{h^2\sin2\eta}{2}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\frac{h^3\sin3\eta}{3}+\cdots$$

The same is also deduced from the Lagrangian theorem, that, given

$$\alpha-z+\varphi(z)=0,$$

one has

$$\psi(z)=\psi(a)+\varphi(a)\psi'(a)+\frac{1}{2}\frac{d\left[\varphi(\alpha)^2\psi'(\alpha)\right]}{d\alpha}+\frac{1}{2\cdot3}\frac{d^2\left[\varphi(\alpha)^3\psi'(\alpha)\right]}{d\alpha^2}+\frac{1}{2\cdot3\cdot4}\frac{d\left[\varphi(\alpha)^4\psi'(\alpha)\right]}{d\alpha^3}+\cdots$$

From this series, having put

$$\psi(z)=\arccos z,\quad \alpha=\cos\eta,\quad \varphi(z)=-\frac{h(1-z^2)}{2},$$

and recalling (8), formula (15) results. Vice versa from formula (15) applying the Lagrangian theorem one can deduce (8).

6.

To deduce the formula mentioned above, found by Legendre, from the general formula (7),

$$\int_0^\pi \frac{\cos ixdx}{\sqrt{1-2a\cos x+a^2}} = a^i \int_0^\pi \frac{\sin^{2i} xdx}{\sqrt{1-a^2\sin^2 x}},$$

one can argue as this:
Having put

$$f(\cos x) = [1-2a\cos x+a^2]^{-\frac{1}{2}},$$

one has

$$\frac{f^{(i)}(\cos x)}{1\cdot 3\cdots(2i-1)} = a^i[1-2a\cos x+a^2]^{-\frac{2i+1}{2}},$$

whence from (7)

$$(16)\qquad \int_0^\pi \frac{\cos ixdx}{\sqrt{1-2a\cos x+a^2}} = a^i \int_0^\pi \frac{\sin^{2i} xdx}{(1-2a\cos x+a^2)^{\frac{1}{2}(2i+1)}}.$$

Now having put

$$(17)\qquad \sin y = \frac{\sin x}{\sqrt{1-2a\cos x+a^2}},$$

one obtains, what is a known transformation of elliptic integrals due to Landen

$$(18)\qquad \frac{dy}{\sqrt{1-a^2\sin^2 y}} = \frac{dx}{\sqrt{1-2a\cos x+a^2}}.$$

Where the limits of x are 0 and π, the limits of y are also 0 and π, whence from (17), (18)

$$(19)\qquad \int_0^\pi \frac{\sin^{2i} xdx}{(1-2a\cos x+a^2)^{\frac{1}{2}(2i+1)}} = \int_0^\pi \frac{\sin^{2i} ydy}{\sqrt{1-a^2\sin^2 y}},$$

which substituted in (16) gives the propounded formula.
On the given occasion I want to mention an indefinite transformation, which reveals the true nature of Legendre's formula. In its proof I will use signs and notations introduced in the my book Fundamenta nova etc.
If $f(u)$ is a periodic function, this means, a function which does not change

its value, having increased the argument u by a certain constant, which we call the index of the period: The integral

$$\int f(u)du,$$

taken for two arbitrary limits, whose difference is equal to the index of the period, gives the same value, having increased the argument u by an arbitrary real or imaginary quantity, as long as the function to be integrated does not become infinite within the boundaries of integration.
Hence, if we set

$$f(u) = \sin^{2n} \operatorname{am} u,$$

while i denotes the imaginary quantity $\sqrt{-1}$, it will be

$$\int\limits_0^{2\pi} \frac{\sin^{2n} \varphi d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \int\limits_0^{4K} \sin^{2n} \operatorname{am} u du = \int\limits_0^{4K} \sin^{2n} \operatorname{am}\left(u + \frac{iK'}{2}\right) du.$$

Having put $\operatorname{am} u = \varphi$, $\operatorname{am} a = \alpha$, from the Eulerian theorem one has

$$\sin \operatorname{am}(u+a) = \frac{\cos\alpha\Delta\alpha\sin\varphi + \sin\alpha\cos\varphi\Delta\varphi}{1-k^2\sin^2\alpha\sin^2\varphi},$$

having put $a = \frac{iK'}{2}$ in which formula, whence

$$\sin\alpha = \frac{i}{\sqrt{k}}, \quad \cos\alpha = \sqrt{\frac{1+k}{k}}, \quad \Delta\alpha = \sqrt{1+k},$$

one finds

$$\sqrt{k}\sin\operatorname{am}\left(u+\frac{iK'}{2}\right) = \frac{(1+k)\sin\varphi + i\cos\varphi\Delta\varphi}{1+k\sin^2\varphi}.$$

Now let us set

$$\frac{(1+k)\sin\varphi}{1+\sin^2\varphi} = \sin\psi, \quad \frac{2\sqrt{k}}{1+k} = \lambda,$$

whence also

$$\frac{\cos\varphi\Delta\varphi}{1+k\sin^2\varphi} = \cos\psi, \qquad \frac{1-k\sin^2\varphi}{1+k\sin^2\varphi} = \Delta(\psi,\lambda),$$

$$du = \frac{d\varphi}{\Delta\varphi} = \frac{d\psi}{(1+k)\Delta(\psi,\lambda)} = \frac{d\psi}{\sqrt{1+2k\cos 2\psi + k^2}}.$$

This is the substitution, by which Gauss exhibited Landen's transformation connected to the *bisection*. Having substituted the preceding formulas, it results

$$\sqrt{k}\sin\operatorname{am}\left(u + \frac{iK'}{2}\right) = ie^{-i\psi},$$

and hence

$$(20) \qquad \int \frac{\cos 2n\psi - i\sin 2n\psi}{\sqrt{1+2k\cos 2\psi + k^2}} d\psi = (-k)^n \int \sin^{2n}\operatorname{am}\left(u + \frac{iK'}{2}\right) du.$$

This is the indefinite transformation, from which for definite limits Legendre's formula follows.

For, while u grows from 0 to $4K$ or φ from 0 to 2π, also ψ grows from 0 to 2π, for which limits the imaginary part multiplied by $\sin 2n\psi$ vanishes; hence it results

$$\begin{aligned}(-k)^n \int\limits_0^{2\pi} \frac{\sin^{2n}\varphi d\varphi}{\sqrt{1-k^2\sin^2\varphi}} &= (-k)^n \int\limits_0^{4K} \sin^{2n}\operatorname{am}\left(u+\frac{iK'}{2}\right) du \\ &= \int\limits_0^{2\pi} \frac{\cos 2n\psi d\psi}{\sqrt{1+2k\cos 2\psi + k^2}},\end{aligned}$$

which formula, having put $k = -a$, $n = i$ goes over into the propounded one.

7.

The formula

$$\int\limits_0^{\pi} \frac{\cos ix dx}{\sqrt{1-2a\cos x + a^2}} = a^i \int\limits_0^{\pi} \frac{\sin^{2i} x dx}{\sqrt{1-a^2\sin^2 x}}$$

is conveniently applied, if one discusses the expansion of the integral

$$\int\limits_0^{\pi} \frac{\cos ix dx}{\sqrt{1-2a\cos x + a^2}}$$

into a series, which we want to converge rapidly for large values of i. For, since from (1)

$$\int_0^\pi \sin^{2i} x \cos^{2n} x dx = \frac{1\cdot 3\cdot 5\cdots(2i-1)\cdot 1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2i+2n)}\pi$$
$$= \frac{1\cdot 3\cdots(2i-1)}{2\cdot 4\cdots 2i}\cdot\frac{1\cdot 3\cdots(2n-1)}{(2i+2)(2i+4)\cdots(2i+2n)}\pi$$

and one has

$$\frac{1}{\sqrt{1-a^2\sin^2 x}} = \frac{1}{\sqrt{1-a^2+a^2\cos^2 x}}$$
$$= \frac{1}{\sqrt{1-a^2}}\left[1-\frac{1}{2}\frac{a^2\cos^2 x}{1-a^2}+\frac{1\cdot 3}{2\cdot 4}\frac{a^4\cos^4 x}{(1-a^2)^2}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{a^6\cos^6 x}{(1-a^2)^8}+\cdots\right],$$

one finds

$$(21.)\qquad \int_0^\pi \frac{\cos ixdx}{\sqrt{1-2a\cos x+a^2}}$$
$$= \frac{1\cdot 3\cdot 5\cdots(2i-1)}{2\cdot 4\cdot 6\cdots 2i}\frac{\pi a^i}{\sqrt{1-a^2}}\left\{1-\frac{1}{2}\frac{1}{2i+2}\frac{a^2}{1-a^2}+\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 3}{(2i+2)(2i+4)}\frac{a^4}{(1-a^2)^2}\right.$$
$$\left.-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1\cdot 3\cdot 5}{(2i+2)(2i+4)(2i+6)}\frac{a^6}{(1-a^2)}+\cdots\right\}$$

which series is seen to converge very fast for larger values of i.
Legendre found a memorable more general expansion

$$(22)\qquad \int_0^\pi \frac{\cos ixdx}{(1-2a\cos x+a^2)^n}$$
$$= \frac{n(n+1)\cdots(n+i-1)}{1\cdot 2\cdots i}\frac{\pi a^i}{(1-a^2)^n}\left\{1+\frac{n(n-1)}{1\cdot(i+1)}\frac{a^2}{1-a^2}+\frac{(n+1)n(n-1)(n-2)}{1\cdot 2\cdot(i+1)(i+2)}\frac{a^4}{(1-a^2)^2}\right.$$
$$\left.\frac{(n+2)(n+1)n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot(i+1)(i+2)(i+3)}\cdot\frac{a^6}{(1-a^2)^3}+\cdots\right\},$$

which is also plain to converge very fast for large values of i. In order to find this expansion, Legendre, having explored the first term of the series by a particular artifice, he assumed the following form of the series:

$$\int_0^\pi \frac{\cos ixdx}{(1-2a\cos x+a^2)^n} = \pi P,$$

$$= \frac{n(n+1)\cdots(n+i-1)}{1\cdot 2\cdots i}\frac{\pi a^i}{(1-a^2)^n}\left[1+\frac{c'}{i+1}+\frac{c''}{(i+1)(i+2)}+\frac{c'''}{(i+1)(i+2)(i+3)}+\cdots\right],$$

while c', c'', c''', $\cdots$ do not depend on the number i. Having done this, by means of linear relation, which holds among three terms P_{i-1}, P_i, P_{i+1},

$$(i+1-n)P_{i+1} - \frac{1+a^2}{a}iP_i + (i-1+n)P_{i-1} = 0$$

he determined the terms c', c'', c''', $\cdots$ one after the another.
You might obtain a more direct proof of formula (22) by means of our theorem (7) in the following way:
Having put

$$f(\cos x) = (1-2a\cos x+a^2)^{-n},$$

one has

$$f^{(i)}(\cos x) = (2a)^i\cdot n(n+1)\cdots(n+i-1)\cdot(1-2a\cos x+a^2)^{-(n+i)},$$

whence from (7):

$$\int_0^\pi \frac{\cos ixdx}{(1-2a\cos x+a^2)^n} = \frac{n(n+1)\cdots(n+i-1)}{1\cdot 3\cdots(2i-1)}(2a)^i\int_0^\pi \frac{\sin^{2i}xdx}{(1-2a\cos x+a^2)^{n+1}}. \tag{23}$$

Let us put $\sqrt{1-2a\cos x+a^2} = R$, $\frac{\sin x}{R} = \sin y$, it will be

$$\frac{\sin^{2i}xdx}{(1-2a\cos x+a^2)^{n+1}} = -\frac{1}{(2n-1)a}\sin^{2i-1}ydR^{-(2n-1)},$$

which, integrated from the limits 0 to π with respect to each of both variables, gives

$$\int_0^\pi \frac{\sin^{2i} x dx}{(1-2a\cos x+a^2)^{n+i}} = -\frac{1}{(2n-1)a}\int_0^\pi \sin^{2i-1} y \frac{dR^{-(2n-1)}}{dy} dy.$$

Let us expand the expression $R^{-(2n-1)}$ into a power series in $\cos y$. For this purpose I observe that one has

$$R^2 - \sin^2 x = (\cos x - a)^2 = \cos^2 y \cdot R^2,$$

and hence, since $2a(\cos x - a) = 1 - a^2 - R^2$, we have

$$R^2 + 2aR\cos y = 1 - a^2.$$

Therefore, one has the expansion in question[1]

$$(24) \qquad R^{-(2n-1)} = \sqrt{(1-2a\cos x+a^2)^{-(2n-1)}}$$

$$= \frac{2n-1}{\sqrt{(1-a^2)^{2n-1}}}\left[\frac{1}{2n-1} + \frac{a\cos y}{\sqrt{1-a^2}} + \frac{2n-1}{2}\frac{a^2\cos^2 y}{(1-a^2)^2} + \frac{(2n-2)\cdot 2n}{2\cdot 3}\frac{a^3\cos^3 y}{\sqrt{(1-a^2)^3}} + \frac{(2n-3)(2n-1)(2n+1)}{2\cdot 3\cdot 4}\frac{a^4\cos^4 y}{\sqrt{(1-a^2)^4+\cdots}}\right],$$

whence

$$-\frac{\sin^{2i-1} y}{(2n-1)a}\frac{dR^{-(2n-1)}}{dy}$$

$$= (1-a^2)^{-n}\sin^{2i} y\left[1 + (2n-1)\frac{a\cos y}{\sqrt{1-a^2}} + \frac{(2n-2)\cdot 2n}{1\cdot 2}\frac{a^2\cos^2 y}{\sqrt{(1-a^2)^2}} + \frac{(2n-3)(2n-1)(2n+1)}{1\cdot 2\cdot 3}\frac{a^3\cos^3 y}{\sqrt{(1-a^2)^3+\cdots}}\right].$$

Having integrated this expression with respect to y from 0 to π, the terms multiplied by the odd powers of $\cos y$ vanish; for the remaining ones from (1)

$$\frac{(2n-2m)(2n-2m+2)\cdots(2n+2m-2)}{1\cdot 2\cdot 3\cdots 2m}\int_0^\pi \sin^{2i} y\cos^{2m} y dy$$

[1] V. Lacroix, Traite du calcul differentiel er du calcul integral, Seconde edition, T. I p. 286, where instead of α, β, γ, y, m, n you have to write $1-a^2$, 1, $-2a\cos y$, R^3, $\frac{1}{2}$, $-\frac{2n-1}{2}$.

$$= \frac{1\cdot 3\cdots(2i-1)}{2\cdot 4\cdots 2i}\cdot\frac{(n-m)(n-m-1)\cdots(n+m-1)}{1\cdot 2\cdots m\cdot(i+1)(i+2)\cdots(i+m)}\pi,$$

whence

$$\int_0^\pi \frac{\sin^{2i} x dx}{(1-2a\cos x+a^2)^{n+i}} = -\frac{1}{(2n-1)a}\int_0^\pi \sin^{2i-1} y\frac{dR^{-(2n-1)}}{dy}dy$$

$$= \frac{1\cdot 3\cdots(2i-1)}{2\cdot 4\cdots 2i}(1-a^2)^{-n}\pi\left[1+\frac{(n-1)n}{1\cdot(i+1)}\frac{a^2}{1-a^2}+\frac{(n-2)(n-1)n(n+1)}{1\cdot 2\cdot(i+1)(i+2)}\cdot\frac{a^4}{(1-a^2)^2}+\cdots\right],$$

which, substituted in (23), gives formula (22) propounded by Legendre.

8.

From formula (23) you even easily deduce a memorable formula due to Euler. For, having put $2x$ instead of x, $-a$ instead of a in (23), we have

$$\int_0^\pi \frac{\cos 2ixdx}{(1+2a\cos 2x+a^2)^n} = \frac{n(n+1)\cdots(n+i-1)}{1\cdot 3\cdots(2i-1)}(-2a)^i\int_0^\pi \frac{\sin^{2i} 2xdx}{(1+2a\cos 2x+a^2)^{n+i}};$$

in the one integral let us put

$$\frac{1-a}{1+a}\tan x = \tan y,$$

whence

$$\frac{(1-a^2)\sin 2x}{1+2a\cos 2x+a^2} = \sin 2y, \quad 1+2a\cos 2x+a^2 = \frac{(1-a^2)^2}{1-2a\cos 2y+a^2},$$

$$\frac{(1-a^2)dx}{1+2a\cos 2x+a^2} = dy,$$

and hence

$$\int_0^\pi \frac{\cos 2ixdx}{(1+2a\cos 2x+a^2)^n}$$

$$= \frac{n(n+1)\cdots(n+i-1)}{1\cdot 3\cdots(2i-1)}\cdot\frac{(-2a)^i}{(1-a^2)^{2n-1}}\int_0^\pi (1-2a\cos 2y+a^2)^{n-i-1}\sin^{2i} 2ydy.$$

But having written $1 - n$ instead of n in (23), it results

$$\int_0^\pi (1 - 2a\cos 2x + a^2)^{n-1} \cos 2ixdx$$

$$= \frac{(n-1)(n-2)\cdots(n-i)}{1\cdot 3\cdots(2i-1)}(-2a)^i \int_0^\pi (1 - 2a\cos 2y + a^2)^{n-i-1} \sin^{2i} 2ydy,$$

having substituted which formula in the preceding one, one obtains

$$\int_0^\pi \frac{\cos 2ixdx}{(1 + 2a\cos 2x + a^2)^n}$$

$$= \frac{n(n+1)\cdots(n+i-1)}{(n-1)(n-2)\cdots(n-i)} \cdot \frac{1}{(1-a^2)^{2n-1}} \int_0^\pi (1 - 2a\cos 2x + a^2)^{n-1} \cos 2ixdx,$$

which is the extraordinary formula, which Euler once studied very extensively.

9.

Let ε, μ, e be the eccentric anomaly, mean anomaly, eccentricity, whence

$$\mu = \varepsilon - e\sin\varepsilon.$$

Let the cosines and sines of multiplies of the eccentric anomaly be expanded into infinite series of sines and cosines of multiplies of the mean anomaly,

$$\begin{aligned} \cos n\varepsilon &= p_n + 2p'_n \cos\mu + 2p''_n \cos 2\mu + 2p'''_n \cos 3\mu + \cdots, \\ \sin n\varepsilon &= \phantom{p_n +{}} q'_n \sin\mu + q''_n \sin 2\mu + q'''_n \sin 3\mu + \cdots, \end{aligned}$$

it will be

$$
\begin{aligned}
p_n^i &= \frac{1}{\pi}\int\limits_0^\pi \cos i\mu \cos n\varepsilon d\mu = \frac{n}{i\pi}\int\limits_0^\pi \sin i\mu \sin n\varepsilon d\varepsilon \\
&= \frac{n}{2i\pi}\int\limits_0^\pi d\varepsilon[\cos((i-n)\varepsilon - ie\sin\varepsilon) - \cos((i+n)\varepsilon - ie\sin\varepsilon)], \\
q_n^i &= \frac{2}{\pi}\int\limits_0^\pi \sin i\mu \sin n\varepsilon d\mu = \frac{2n}{i\pi}\int\limits_0^\pi \cos i\mu \cos n\varepsilon d\varepsilon \\
&= \frac{n}{i\pi}\int\limits_0^\pi d\varepsilon[\cos((i-n)\varepsilon - ie\sin\varepsilon) + \cos((i+n)\varepsilon - ie\sin\varepsilon)],
\end{aligned}
$$

which integral transformations are obtained by partial integration. Since if we, following Bessel, put

$$
\frac{1}{\pi}\int\limits_0^\pi \cos(i\varepsilon - k\sin\varepsilon)d\varepsilon = I_k^{(i)},
$$

it will be

$$
\begin{aligned}
p_n^{(i)} &= \frac{n}{2i}\left(I_{ie}^{(i-n)} - I_{ie}^{(i+n)}\right), \\
q_n^{(i)} &= \frac{n}{i}\left(I_{ie}^{(i-n)} - I_{ie}^{(i+n)}\right).
\end{aligned}
$$

Depending on whether i is an even or odd number, one also has

$$
\begin{aligned}
I_k^{(2i)} &= \frac{1}{\pi}\int\limits_0^\pi \cos(k\sin\varepsilon)\cos 2i\varepsilon d\varepsilon &&= \frac{(-1)^i}{\pi}\int\limits_0^\pi \cos(k\cos\varepsilon)\cos 2i\varepsilon d\varepsilon, \\
I_k^{(2i+1)} &= \frac{1}{\pi}\int\limits_0^\pi \sin(k\sin\varepsilon)\sin(2i+1)\varepsilon d\varepsilon &&= \frac{(-1)^i}{\pi}\int\limits_0^\pi \sin(k\cos\varepsilon)\cos(2i+1)\varepsilon d\varepsilon,
\end{aligned}
$$

whence the transcendents $I_k^{(2i)}$, $I_k^{(2i+1)}$ are the coefficients of the expansion of $\cos(k\cos\varepsilon)$, $\sin(k\cos\varepsilon)$ into a series of cosines of multiples of ε

$$
\begin{aligned}
\cos(k\cos\varepsilon) &= I_k^{(0)} - 2I_k^{(2)}\cos 2\varepsilon + 2I_k^{(4)}\cos 4\varepsilon - 2I_k^{(6)}\cos 6\varepsilon + \cdots, \\
\sin(k\cos\varepsilon) &= 2I_k^{(1)}\cos\varepsilon - 2I_k^{(3)}\cos 3\varepsilon + 2I_k^{(5)}\cos 5\varepsilon - \cdots
\end{aligned}
$$

If the cosines and sines of a multiple of the mean anomaly are to be expanded into a series of cosines and sines of multiples of the eccentricity, put

$$\begin{aligned}
\cos i\mu &= k^{(i)} + 2k_1^{(i)}\cos\varepsilon + 2k_2^{(i)}\cos 2\varepsilon + 2k_3^{(i)}\cos 3\varepsilon + \cdots,\\
\sin i\mu &= \qquad\quad l_1^{(i)}\sin\varepsilon + \ \ l_2^{(i)}\sin 2\varepsilon + \ \ l_3^{(i)}\sin 3\varepsilon + \cdots,
\end{aligned}$$

it will be

$$\begin{aligned}
k_n^{(i)} &= \frac{1}{\pi}\int\limits_0^\pi \cos i\mu\cos n\varepsilon d\varepsilon = \frac{1}{2\pi}\int\limits_0^\pi d\varepsilon[\cos((i-n)\varepsilon - ie\sin\varepsilon) + \cos((i+n)\varepsilon - ie\sin\varepsilon)],\\
l_n^{(i)} &= \frac{2}{\pi}\int\limits_0^\pi \sin i\mu\sin n\varepsilon d\varepsilon \ = \ \frac{1}{\pi}\int\limits_0^\pi d\varepsilon[\cos((i-n)\varepsilon - ie\sin\varepsilon) - \cos((i+n)\varepsilon - ie\sin\varepsilon)]
\end{aligned}$$

or

$$\begin{aligned}
k_n^{(i)} &= \frac{1}{2}\left(I_{i\varepsilon}^{(i-n)} + I_{ie}^{(i+n)}\right) = \frac{i}{2n}q_n^{(i)},\\
l_n^{(i)} &= \quad I_{i\varepsilon}^{(i-n)} - I_{ie}^{(i+n)} \quad = \frac{2i}{n}p_n^{(i)}.
\end{aligned}$$

Bessel explained the nature and the various applications of the transcendents $I_k^{(i)}$ for the determination of definte integrals in his celebrated paper *De Perturbationibus, quae a motu solis pendent (Acad. Berol. ad annum* 1824*)*. In this paper he proved that the functions $I_k^{(0)}$, $I_k^{(1)}$, $I_k^{(2)}$, $I_k^{(3)}$, $\cdots$ are all expressed by two of them linearly. Hence it is plain, having found the coefficients of the expansion of $\cos\varepsilon$, $\sin\varepsilon$ into a series of multiples of the mean anomaly, that the coefficients of the expansion of $\cos n\varepsilon$, $\sin n\varepsilon$ are determined linearly from them. Since the same transcendents also occur in the theory of heat, many famous men, which treated the subject of heat, noted various of their properties.

But having mentioned all this, let us transform the integral $I_k^{(i)}$ by means of formula (7). By means of it, having respectively put $f(z) = \cos(kz)$, $f(z) = \sin(kz)$, one finds

$$\begin{aligned}\pi I_k^{(2i)} &= (-1)^i \int_0^\pi \cos(k\cos\varepsilon)\cos 2i\varepsilon d\varepsilon \\ &= \frac{k^{2i}}{1\cdot 3\cdot 5\cdots(4i-1)}\int_0^\pi \cos(k\cos\varepsilon)\sin^{4i}\varepsilon d\varepsilon, \\ \pi I_k^{(2i+1)} &= (-1)^i \int_0^\pi \sin(k\cos\varepsilon)\cos(2i+1)\varepsilon d\varepsilon \\ &= \frac{k^{2i+1}}{1\cdot 3\cdot 5\cdots(4i+1)}\int_0^\pi \cos(k\cos\varepsilon)\sin^{4i+2}\varepsilon\varepsilon d\varepsilon,\end{aligned}$$

whence, depending on whether i is even or odd,

$$\pi I_k^{(i)} = \frac{k^i}{1\cdot 3\cdots(2i-1)}\int_0^\pi \cos(k\cos\varepsilon)\sin^{2i}\varepsilon d\varepsilon.$$

And Bessel himself (formula 53 in the mentioned paper) demonstrated this expression for the transcendent $I_k^{(i)}$ by particular artifices.

10.

I want to add an example of a transformation of a double integral, which can also be useful in astronomical calculations. Let

$$f^{(i,i')}(\cos x, \cos x') = \frac{\partial^{i+i'} f(y,z)}{\partial y^i \partial z^{i'}},$$

if after the differentiations one puts $y = \cos x$, $z = \cos x'$: From formula (7), applying it to the variables x, x' one after the other, one obtains:

$$(25) \qquad \int_0^\pi \int_0^\pi f(\cos x, \cos x')\cos ix \cos i'x' dx dx'$$

$$= \frac{1}{1\cdot 3\cdots(2i-1)\cdot 1\cdot 3\cdots(2i'-1)}\int_0^\pi\int_0^\pi f^{(i,i')}(\cos x,\cos x')\sin^{2i} x \sin^{2i'} x' dx dx'.$$

Let

$$f(\cos x, \cos x') = (l + 2l'\cos x + 2l''\cos x')^{-n},$$

from (25) it will be

$$(26) \qquad \int_0^\pi\int_0^\pi \frac{\cos ix \cos i'x' dxdx'}{(l+2l'\cos x + 2l''\cos x')^n}$$

$$= (-2)^{i+i'}(l')^i(ll'')^{i'}\frac{n(n+1)(n+2)\cdots(n+i+i'-1)}{1\cdot 3\cdots(2i-1)\cdot 1\cdot 3\cdots(2i'-1)}\int_0^\pi\int_0^\pi \frac{\sin^{2i}x\sin^{2i'}x dxdx'}{(l+2l'\cos x+2l''\cos x')^{n+i+i'}}.$$

Let the radii of the circular orbit of two planets be a, a', the inclination I, the anomalies φ, φ'. Let it be propounded to expand the n-th power of the reciprocal distance of these planets into a series of multiples of $\varphi+\varphi'$, $\varphi-\varphi'$; let this expansion be

$$\frac{1}{[a^2-2aa'(\cos\varphi\cos\varphi'+\cos I\sin\varphi\sin\varphi')+a'^2]^{\frac{1}{2}n}} = \sum p_{i,i'}\cos i(\varphi-\varphi')\cos i'(\varphi+\varphi'),$$

having extended the sum to the numbers i, i', from $-\infty$ to $+\infty$ for each of them. Having put $\frac{1}{2}n$ instead of n, further,

$$l=a^2+a'^2,\quad l'=-aa'\cos^2\left(\frac{1}{2}I\right),\quad l''=-aa'\sin^2\left(\frac{1}{2}I\right),$$
$$\varphi-\varphi'=x,\quad \varphi+\varphi'=x',$$

from (26) it will be

$$p_{i,i'} = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \frac{\cos ix\cos i'x'dxdx'}{[a^2-2aa'(\cos^2(\frac{1}{2}I)\cos x+\sin^2(\frac{1}{2}I)\cos x')]^{\frac{1}{2}}}$$

$$(27) \qquad = \frac{n(n+2)(n+4)\cdots(n+2i+2i'-2)}{1\cdot 3\cdots(2i-1)\cdot 1\cdot 3\cdots(2i'-1)}a^{i+i'}a'^{i+i'}\cos^{2i}\left(\frac{1}{2}I\right)\sin^{2i'}\left(\frac{1}{2}I\right)$$

$$\cdot\frac{1}{\pi^2}\int_0^\pi\int_0^\pi\frac{\sin^{2i}x\sin^{2i'}x dxdx'}{[a^2-2aa'(\cos^2(\frac{1}{2}I)\cos x+\sin^2(\frac{1}{2}I)\cos x')+a'^2]^{\frac{1}{2}n+i+i'}}.$$

This last expression, since both it shows the structure of the coefficient $p_{i,i'}$ very well, and, if one likes the computation by quadratures better, is convenient, can be useful in perturbations, if the inclination, as it is the case for the newer planets, is small.

11.

Formula (7) can also be applied for the determination of the value of the integral $\int\limits_0^\pi U \cos i\varphi d\varphi$, if i grows to infinity. This determinations demands, that, having expanded U into a series of cosines of multiples of φ, the series converges. For, having transformed the propounded integral $\int\limits_0^\pi U \cos i\varphi d\varphi$ into $\int\limits_0^\pi V \sin^{2i} \varphi d\varphi$ by means of (7), for the determination of this for an infinite i one can apply Laplace's method for the approximate evaluation of integrals, which contain large exponents under the integral sign.
For the sake of an example let

$$A = \int\limits_0^\pi \frac{\cos ixdx}{(l+2l'\cos x)^n},$$

from (7) it will be

$$A = \frac{n(n+1)\cdots(n+i-1)}{1\cdot 3\cdots(2i-1)}(-2l')^i \int\limits_0^\pi \left(\frac{\sin^2 x}{l+2l'\cos x}\right)^i \frac{dx}{(l+2l'\cos x)^n}.$$

Let us find the maximal value of the expression raised to the i-th power under the integral sign, which, having put $\cos x = y$, becomes

$$\frac{\sin^2 x}{l+2l'\cos x} = \frac{1-y^2}{l+2l'y}.$$

Having put its differential $= 0$, we have

$$0 = y(l+2l'y) + l'(1-y^2) = l' + ly + l'y^2,$$

whence two values for y result

$$y = \frac{-l \pm \sqrt{l^2 - 4l'^2}}{2l'},$$

since the product of which is $= 1$, the one will be absolutely greater than 1, the other absolutely smaller tham 1. One has to chose the second, since

$y = \cos x$ and hence absolutely smaller than 1; this value, if, what we assume, l is positive, corresponds to the positive root. But for that value one has

$$\frac{1-y^2}{l+2l'y} = \frac{y(1-y^2)}{ly+2l'y^2} = -\frac{y}{l'} = \frac{l-\sqrt{l^2-4l'^2}}{2l'^2} = \frac{2}{l+\sqrt{l^2-4l'^2}},$$

which is the maximum value in question. The second differential of the expression

$$\frac{1-y^2}{l+2l'y} = \frac{4l'^2-l^2}{4l'^2(l+2l'^2y)} + \frac{l}{2l'^2} - \frac{l+2l'y}{4l'^2},$$

taken with respect to y, for the assigned value of y becomes

$$-\frac{2(l^2-4l'^2)}{(l+2l'y)^3} = -\frac{2}{\sqrt{l^2-4l'^2}}.$$

Hence, having put

$$y = \frac{-l+\sqrt{l^2-4l'^2}}{2l'} - \frac{t}{\sqrt{i}},$$

it results:

$$\frac{1-y^2}{l+2l'y} = \frac{2}{l+\sqrt{l^2-4l'^2}} - \frac{t^2}{i\sqrt{l^2-4l'^2}} + \frac{\alpha t^3}{\sqrt{i^3}} + \cdots$$

and hence for the infinite i

$$\left(\frac{1-y^2}{l+2l'y}\right)^i = \left(\frac{2}{l+\sqrt{l^2-4l'^2}}\right)^i e^{-\frac{l+\sqrt{l^2+4l'^2}}{2\sqrt{l^2-4l'^2}}t^2}.$$

Further, for infinite i

$$\frac{dx}{(l+2l'\cos x)^n} = -\frac{dy}{\sqrt{1-y^2}(l+2l'y)^n} = \left(\frac{l+\sqrt{l^2-4l'^2}}{2}\right)^{\frac{1}{2}} (l^2-4l'^2)^{-\frac{2n+1}{4}} \frac{dt}{\sqrt{i}}.$$

The limits of the integral for infinite i can be taken from $-\infty$ to $+\infty$; for these limits one has

$$\left(\frac{l+\sqrt{l^2-4l'^2}}{2}\right)^{\frac{1}{2}}\int\limits_{-\infty}^{+\infty} e^{-\frac{l+\sqrt{l^2-4l'^2}}{2\sqrt{l^2-4l'^2}}t^2}dt = \sqrt[4]{l^2-4l'^2}\sqrt{\pi}.$$

Having substituted everything, for infinite i it results

$$(28)\quad A = \frac{n(n+1)\cdots(n+i-1)}{1\cdot 3\cdots(2i-1)}(l^2-4l'^2)^{-\frac{1}{2}n}\left(\frac{-4l'}{l+\sqrt{l^2-4l'^2}}\right)^i\sqrt{\frac{\pi}{i}}.$$

If one sets $l = 1+a^2$, $l' = a$, from (28)

$$A = \frac{n(n+1)\cdots(n+i-1)}{1\cdot 3\cdots(2i-1)}(1-a^2)^{-n}(-2a)^i\sqrt{\frac{\pi}{i}}.$$

One has the same expression from Legendre's formula (22)

$$(30)\qquad A = \frac{n(n+1)\cdots(n+i-1)}{1\cdot 2\cdots i}(1-a^2)^{-n}(-a)^i\pi.$$

Hence having compared (29) and (30) to each other, for infinite i it results

$$(31)\qquad \frac{1\cdot 3\cdots(2i-1)}{2\cdot 4\cdots 2i} = \frac{1}{\sqrt{i\pi}},$$

which is the known formula due to Wallis.

12.

Now let us find the value of the double integral

$$B = \int\limits_0^\pi\int\limits_0^\pi \frac{\cos ix\cos ix' dxdx'}{(l+2l'\cos x+2l''\cos x')^n}$$

first, if one of the numbers i, i' is infinite; *then*, if both become infinite. Therefore, let i be infinite, i' finite; by putting $l+2l''\cos x'$ instead of l in (28), we obtain

$$B = \frac{n(n+1)\cdots(n+i-1)}{1\cdot 3\cdots(2i-1)}\sqrt{\frac{\pi}{i}}(-4l')^i\int\limits_0^\pi \frac{[(l+2l''\cos x')^2-4l'^2]^{-\frac{1}{2}n}\cos i'xdx'}{[l+2l''\cos x'+\sqrt{(l+2l''\cos x')^2-4l'^2}]^i}.$$

The maximal value of the expression under the integral raised to the i-th power, if l'' is positive, corresponds to the value $x' = \pi$. Therefore, having put $x' = \pi - \frac{t}{\sqrt{i}}$,

$$\begin{aligned} &l + 2l'' \cos x' + \sqrt{(l + 2l'' \cos x')^2 - 4l'^2} \\ &= l - 2l'' + \sqrt{(l - 2l'')^2 - 4l'^2} + \frac{l - 2l'' + \sqrt{(l - 2l'')^2 - 4l'^2}}{\sqrt{(l - 2l'')^2 - 4l'^2}} \frac{l''t^2}{i} + \frac{\alpha t^4}{i^2} + \cdots, \end{aligned}$$

whence for infinite i

$$\begin{aligned} &[l + 2l'' \cos x' + \sqrt{(l + 2l'' \cos x')^2 - 4l'^2}]^{-i} \\ &= [l - 2l'' + \sqrt{(l - 2l'')^2 - 4l'^2}]^{-i} e^{-\frac{l''t^2}{\sqrt{(l-2l'')^2 - 4l'^2}}}. \end{aligned}$$

Since π is the one limit of the propounded integration and x' is not extended further, the limits for t will be 0 and ∞. Having done the integration, for infinite i, finite i', it results

$$\begin{aligned} (32) \qquad &\int_0^\pi \int_0^\pi \frac{\cos ix \cos i'x' dx dx'}{(l + 2l' \cos x + 2l'' \cos x')^n} \\ &= (-1)^{i+i'} \frac{n(n+1) \cdots (n+i-1)}{1 \cdot 3 \cdots (2i-1)} \frac{\pi}{2i} \frac{[(l - 2l'')^2 - 4l'^2]^{-\frac{2n-1}{4}}}{\sqrt{l''}} \left(\frac{4l'}{l - 2l'' + \sqrt{(l - 2l'')^2 - 4l'^2}} \right)^i, \end{aligned}$$

if l'' is assumed to be positive. But we see the number i' to affect only the sign of the mentioned values. The same formula, using (31), can also be represented this way:

$$\begin{aligned} (33) \qquad &\frac{1 \cdot 2 \cdot 3 \cdots i \cdot 1 \cdot 2 \cdot 3 \cdots i}{1 \cdot 3 \cdots (2i-1) \cdot n(n+1) \cdots (n+i-1)} \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \frac{\cos ix \cos i'x' dx dx'}{(l + 2l' \cos x + 2l'' \cos x')^n} \\ &= (-1)^{i+i'} \frac{[(l - 2l'')^2 - 4l'^2]^{-\frac{2n-1}{4}}}{2\sqrt{l''}} \left(\frac{l'}{l - 2l'' + \sqrt{(l - 2l'')^2 - 4l'^2}} \right)^i. \end{aligned}$$

If i' is of the same order as $\sqrt{i}$, one puts

$$\frac{i'}{\sqrt{i}} = r,$$

which will be a finite quantity; we have

$$\cos i'x' = \cos i' \left(\pi - \frac{t}{\sqrt{i}} \right) = (-1)^{i'} \cos rt.$$

Hence, since one has the known formula

$$\int\limits_0^\infty dt \cos rt e^{-a^2t^2} = \frac{\sqrt{\pi}}{2a} e^{-\frac{r^2}{4a^2}} = e^{-\frac{r^2}{4a^2}} \int\limits_0^\infty dt e^{-a^2t^2},$$

the one side of the equation (32) or (33) is still to be multiplied by

$$e^{-\frac{r^2\sqrt{(l-2l'')^2-4l'^2}}{4l''}}.$$

Now let us go over to the case, in which $\frac{i'}{i}$ is a finite quantity.

13.

Therefore, let $\frac{i'}{i} = r$ be a finite quantity: By means of formula (25) we find (26)

$$B = \int\limits_0^\pi \int\limits_0^\pi \frac{\cos ix \cos i'x' dx dx'}{(l + 2l' \cos x + 2l'' \cos x')^n}$$

$$= (-2)^{i+i'} \frac{n(n+1) \cdots (n+i+i'-1)}{1 \cdot 3 \cdots (2i-1) \cdot 1 \cdot 3 \cdots (2i'-1)} (l')^i (l'')^{i'}$$

$$\times \int\limits_0^\pi \int\limits_0^\pi \left(\frac{\sin^2 x \sin^{2r} x'}{(l + 2l' \cos x + 2l'' \cos x')^{1+r}} \right)^i \frac{dx dx'}{(l + 2l' \cos x + 2l'' \cos x')^n}.$$

Let $\cos x = y$, $\cos x' = z$, and let us find the maximum value of the expression

$$\frac{\sin^2 x \sin^{2r} x'}{(l + 2l' \cos x + 2l'' \cos x')^{1+r}} = \frac{(1-y^2)(1-z^2)^r}{(l + 2l'y + 2l''z)^{1+r}}.$$

Having differentiated this expression with respect to y and z, and having put the differential equal to zero, these equations result

$$(34) \qquad \begin{aligned} (1+r)l' \ + \ ly + (1-r)l'y^2 &= -2l''yz, \\ (1+r)l'' + rlz - (1-r)l''z^2 &= -2rl'yz; \end{aligned}$$

the values of y, z, which render the propounded expression a maximum, are to be derived from these equations. Having found these, if we eliminate z from the first equation or y from the second, one has

$$l + 2l'y + 2l''z = -(1+r)l'\frac{1-y^2}{y} = -\frac{(1+r)l''}{r}\frac{1-z^2}{z},$$

and hence the maximum value in question

$$(35) \qquad \frac{(1-y^2)(1-z^2)^r}{(l+2l'y+2l''z)^{1+r}} = \left(-\frac{1}{1+r}\right)^{1+r}\frac{r^r}{l'(l'')^r}yz^r.$$

I observe, what the nature of the problem demands, that the one of the equations (34) goes over into the other, having permuted l' and l'', y and z, and having simultaneously put $\frac{1}{r}$ instead of r.
Let $y = a$, $z = b$ be the value in question, from (34) it will be

$$\begin{aligned} al + [1+r+(1-r)a^2]l' + 2abl'' &= 0, \\ rbl + 2rabl' + [1+r-(1-r)b^2]l'' &= 0, \end{aligned}$$

whence

$$(36) \qquad l : l' : l'' = \frac{1+a^2}{1-a^2} + r\frac{1+b^2}{1-b^2} : -\frac{a}{1-a^2} : -\frac{br}{1-b^2}.$$

If these equations are satisfied for given values of a, b, it is plain that the equations same are also satisfied by their reciprocals.
Having introduced the multiplicator p, let us substitute the following equations for formula (36)

$$\frac{1+a^2}{1-a^2} + r\frac{1+b^2}{1-b^2} = pl, \quad -\frac{a}{1-a^2} = pl', \quad -\frac{br}{1-b^2} = pl'',$$

whence, having put

$$\frac{1+a^2}{1-a^2} = \sqrt{1+4p^2l'^2} = A, \quad r\frac{1+b^2}{1-b^2} = \sqrt{r^2+4p^2l''^2} = B,$$

one finds

$$A + B = pl;$$

having multiplied this equation by $A - B$, we have

$$pl(A - B) = 1 - r^2 + 4p^2(l'l' - l''l''),$$

whence

$$\begin{aligned} 2plA &= 1 - r^2 + p^2(ll + 4l'l' - 4l''l'') \\ 2plB &= -(1 - r^2) + p^2(ll - 4l'l' + 4l''l''). \end{aligned}$$

Having squared the one of these equations, it results

$$0 = (1 - r^2)^2 - 2[(1 + r^2)ll - 4(1 - r^2)(l'l' - l''l'')]p^2 + Ep^4,$$

if, for the sake of brevity, one puts

$$E = (l + 2l' + 2l'')(l + 2l' - 2l'')(l - 2l' + 2l'')(l - 2l' - 2l'').$$

For the propounded integral for the assigned limits of integration not to become infinite, one has to set that the sum of $2l'$, $2l''$, assumed to be positive, is smaller than l; hence E will always be positive. In this case one has two positive values of p^2, given by the equation

$$Ep^2 = M + 2l\sqrt{R}$$

or

$$p^2 = \frac{(1 - r^2)^2}{M - 2l\sqrt{R}} = \frac{M + 2l\sqrt{R}}{E},$$

if for the sake of brevity one sets

$$\begin{aligned} M &= (1 + r^2)ll - 4(1 - r^2)(l'l' - l''l''), \\ R &= r^2ll - 4(1 - r^2)(r^2l'l' - l''l''). \end{aligned}$$

From the formulas, by which we exhibited pA, pB rationally in terms of p^2, we find

$$\frac{pA}{1 - r^2} = \frac{l\sqrt{R}}{M - 2k\sqrt{R}}, \quad \frac{pB}{1 - r^2} = -\frac{r^2l - \sqrt{R}}{M - 2l\sqrt{R}},$$

or, since

$$p = \frac{1 - r^2}{\sqrt{M - 2l\sqrt{R}}},$$

it results

$$A = \frac{l - \sqrt{R}}{\sqrt{M - 2l\sqrt{R}}}, \quad B = -\frac{r^2 l - \sqrt{R}}{\sqrt{M - 2l\sqrt{R}}},$$

whence

$$a = -\frac{A - 1}{2l' p} = -\frac{l - \sqrt{R} - \sqrt{M - 2l\sqrt{R}}}{2(1 - r^2)l'},$$

$$b = -\frac{B - r}{2l'' p} = \frac{r^2 l - \sqrt{R} + r\sqrt{M - 2l\sqrt{R}}}{2(1 - r^2)l''}$$

or even

$$a = -\frac{2l' p}{A + 1} = -\frac{2(1 - r^2)l'}{l - \sqrt{R} + \sqrt{M - 2l\sqrt{R}}},$$

$$b = -\frac{2l'' p}{B + r} = -\frac{2(1 - r^2)l''}{\sqrt{R} - r^2 l + r\sqrt{M - 2l\sqrt{R}}}.$$

In the preceding expressions one finds two radicals, $\sqrt{R}$ and $\sqrt{M - 2l\sqrt{R}}$, from whose two signs four systems of values for a, b result. But in the expressions A, B, p, a, b those radicals are to be taken with the same sign; having done this, their corresponding values are determined without any ambiguity.

If you change the sign of the radical $\sqrt{M - 2l\sqrt{R}}$ into its opposite, while $\sqrt{R}$ remains the same, p goes over into $-p$, and at the same time a, b into $\frac{1}{a}$, $\frac{1}{b}$. This is plain from the formula

$$A = \frac{1 + a^2}{1 - a^2} = \frac{l - \sqrt{R}}{\sqrt{M - 2l\sqrt{R}}}, \quad B = r\frac{1 + b^2}{1 - b^2} = -\frac{r^2 l - \sqrt{R}}{\sqrt{M - 2l\sqrt{R}}}$$

or even from the values of a, b, since

$$M - R = ll - 4(1 - r^2)^2 l' l',$$

$$r^2 M - R = r^4 ll - 4(1 - r^2)^2 l'' ll''.$$

The quantities E, M, R are always positive, further, since

$$M^2 - 4llR = (1-r^2)^2 E,$$

each of the expressions $M \pm 2l\sqrt{R}$ will be positive. Let us suppose, what is possible, that $r = \frac{i'}{i} < 1$, from the equations

$$\begin{aligned} R &= ll - (1-r^2)[ll + 4r^2 l'l' - 4l''l''], \\ R &= r^4 ll + (1-r^2)[r^2(ll - 4l'l') + 4l''l''], \end{aligned}$$

it follows that

$$l > \pm\sqrt{R} > r^2 l.$$

Hence it is plain, if $\sqrt{R}$ is assumed to be positive, that the expressions

$$A = \frac{1+a^2}{1-a^2} = \frac{l-\sqrt{R}}{\sqrt{M-2l\sqrt{R}}}, \quad B = r\frac{1+b^2}{1-b^2} = \frac{\sqrt{R}}{\sqrt{M-2l\sqrt{R}}}$$

have the same sign, and, if $\sqrt{M-2l\sqrt{R}}$ is positive, that both will be positive, and hence both a, b will be absolutely smaller than 1; if $\sqrt{M-2l\sqrt{R}}$ is negative, both A, B will be negative, and hence both a, b will be absolutely greater than 1. Further, if $\sqrt{R}$ is negative, depending on whether $\sqrt{M-2l\sqrt{R}}$ is positive or negative, either A will be positive, B negative, and hence a will be absolutely smaller than 1, b absolutely greater than 1; or A will be negative, B positive, and hence a absolutely greater than 1, b absolutely smaller than 1. From the preceding it follows, if the sum of $2l'$, $2l''$, here both assumed to be positive, is smaller than l, what must be assumed in the propounded integral, that a system always exists and one of the values $y = a$, $z = b$ is absolutely smaller than 1; these values, if $r < 1$, what can be assumed, correspond to the positive radicals $\sqrt{R}$, $\sqrt{M-2l\sqrt{R}}$. In the same way it is demonstrated, if $r > 1$, that those values correspond to a positive $\sqrt{R}$, a negative $\sqrt{M-2l\sqrt{R}}$. The values of y, z, which were used in the propounded question, must be absolutely smaller than 1, since $y = \cos x$, $z = \cos x'$. Hence the propounded expression

$$\frac{\sin^2 x \sin^{2r} x'}{(l + 2l'\cos x + 2l''\cos x')^{1+r}}$$

has only *one* maximum. This is found from (35), if $r < 1$,

$$\frac{r^r[2(1-r)]^{1+r}}{(l-\sqrt{R}+\sqrt{M-2l\sqrt{R}})(\sqrt{R}-r^2l+r\sqrt{M-2l\sqrt{R}})^r}=\mu,$$

having assumed both radicals to be positive.
If $r=1$, we have

$$p=\frac{2l}{\sqrt{E}},\quad A=\frac{ll+4l'l'-4l''l''}{\sqrt{E}},\quad B=\frac{ll-4l'l'+4l''l''}{\sqrt{E}},$$

$$a=-\frac{4l'l'}{ll+4l'l'-4l''l''+\sqrt{E}},\quad b=-\frac{4ll''}{ll-4l'l'+4l'l'+4l''l''+\sqrt{E}};$$

the maximum in question is

$$\mu=\frac{2}{ll-4l'l'-4l''l''+\sqrt{E}}.$$

Now let is find the values, which the second differential of the following expression has

$$u=\frac{(1-y^2)(1-z^2)^r}{(l+2l'y+2l''z)^{1+r}},$$

if after the differentiations one puts $y=a$, $z=b$. We have the following first differentials of u

$$\frac{\partial u}{\partial y}=-\frac{2u}{(1-y^2)(l+2l'y+2l''z)}[(1+r)l'\ +\ ly+(1-r)l'y^2\ +2l''yz],$$

$$\frac{\partial u}{\partial z}=-\frac{2u}{(1-z^2)(l+2l'y+2l''z)}[(1+r)l''+rlz-(1-r)l''z^2+2rl'yz],$$

it results

$$\frac{\partial^2 u}{\partial y^2}=-\frac{2\mu[l+2(1-r)l'a+2l''b]}{(1-a^2)(l+2l'a+2l''b)},$$

$$\frac{\partial^2 u}{\partial z^2}=-\frac{2\mu[rl-2(1-r)l''b+2rl'a]}{(1-b^2)(l+2l'a+2l''b)},$$

$$\frac{\partial^2 u}{\partial y\partial z}=-\frac{4\mu l''a}{(1-a^2)(l+2l'a+2l''b)}=-\frac{4\mu rl'b}{(1-b^2)(l+2l'a+2l''b)}.$$

Using the found formulas these expressions go over into the following:

$$
\begin{aligned}
\tfrac{1}{2}\frac{\partial^2 u}{\partial y^2} &= -\frac{\mu[1+r-(1-r)a^2]}{(1+r)(1-a^2)^2} = -\mu\alpha,\\
\tfrac{1}{2}\frac{\partial^2 u}{\partial z^2} &= -\frac{\mu r[1+r+(1-r)b^2]}{(1+r)(1-b^2)^2} = -\mu\gamma,\\
\frac{\partial^2 u}{\partial y \partial z} &= \frac{4\mu rab}{(1+r)(1-a^2)(1-b^2)} = 2\mu\beta,
\end{aligned}
$$

whence

$$
\begin{aligned}
\alpha\gamma - \beta\beta &= \frac{r[1+r-(1-r)(a^2-b^2)-(1+r)a^2b^2]}{(1+r)(1-a^2)^2(1-b^2)^2}\\
&= \frac{(B+r^2A)}{(1+r)(1-a^2)(1-b^2)} = \frac{p\sqrt{R}}{(1+r)(1-a^2)(1-b^2)}.
\end{aligned}
$$

Now let us put

$$\cos x = y = a - \frac{t}{\sqrt{i}}, \quad \cos x' = z = b - \frac{t'}{\sqrt{i}},$$

it will be

$$\frac{\sin^2 x \sin^{2r} x'}{(l+2l'\cos x + 2l''\cos x')^{1+r}} = \mu\left(1 - \frac{\alpha tt - 2\beta tt' + \gamma t't'}{i} + \frac{\delta}{\sqrt{i^3}} + \cdots\right),$$

whence for infinite i

$$\left[\frac{\sin^2 x \sin^{2r} x'}{(l+2l'\cos x + 2l''\cos x')^{1+r}}\right]^i = \frac{\sin^{2i} x \sin^{2i'} x}{(l+2l'\cos x + 2l''\cos x')^{i+i'}} = \mu^i e^{-(\alpha tt - 2\beta tt' + \gamma t't')}$$

Further, for infinite i

$$
\begin{aligned}
&\frac{dx dx'}{(l+2l'\cos x + 2l''\cos x')^n}\\
&= \frac{1}{\sqrt{(1-a^2)(1-b^2)}}\frac{1}{(l+2l'a+2l''b)^n}\frac{dtdt'}{i} = \frac{p^n dtdt'}{i(1+r)^n\sqrt{(1-a^2)(1-b^2)}}.
\end{aligned}
$$

The limits of integration for t, t' become $-\infty$, ∞; for these limits one has

$$\int\int dtdt'e^{-(\alpha tt-2\beta tt'+\gamma t't')} = \frac{\pi}{\sqrt{\alpha\gamma-\beta\beta}} = \frac{\pi\sqrt{1+r}\sqrt{(1-a^2)(1-b^2)}}{\sqrt{p}\sqrt[4]{R}}.$$

Hence finally *for infinite i, i' the value of the integral*

$$\int_0^\pi \int 0^\pi \frac{\cos ix \cos i'x'dxdx'}{(l+2l'\cos x+2l''\cos x')^n},$$

if i, i' remains in the finite ratio $\frac{i'}{i}=r$, becomes

$$\frac{(-4)^{i+i'}\pi}{i}\frac{n(n+1)\cdots(n+i+i'-1)}{1\cdot 3\cdots(2i-1)\cdot 1\cdot 3\cdots(2i'-1)}\frac{1}{\sqrt[4]{R}(M-2l\sqrt{R})^{\frac{1}{4}(2n-1)}}$$

$$\frac{r^{i'}(1-r)^{i+i'+\frac{1}{2}(2n-1)}(l')^i(l'')^{i'}}{(l-\sqrt{R}+\sqrt{M}-2l\sqrt{R})^i(\sqrt{R}-r^2l+r\sqrt{M}-2l\sqrt{R})^{i'}},$$

where

$$\begin{aligned} R &= r^2ll-4(1-r^2)(r^2l'l'-l''l''),\\ M &= (1+r^2)ll-4(1-r^2)(l'l'-l''l''), \end{aligned}$$

having assumed the radicals to be positive.
The numerical factor can also be exhibited this way from (31):

$$\frac{(-4)^{i+i'}\pi}{i}\frac{n(n+1)\cdots(n+i+i'-1)}{1\cdot 3\cdots(2i-1)\cdot 1\cdot 3\cdots(2i'-1)} = \pi^2(-2)^{i+i'}\frac{n(n+1)\cdots(n+i+i'-1)}{1\cdot 2\cdots i\cdot 1\cdot 2\cdots i'}\sqrt{r}.$$

In the special case, in which $i=i'$, $r=1$, *for infinite i*

$$\int_0^\pi\int_0^\pi \frac{\cos ix\cos ix'dxdx'}{(l+2l'\cos x+2l''\cos x')^n}$$

$$= 2^i\frac{n(n+1)\cdots(n+2i-1)}{1\cdot 2\cdots i\cdot 1\cdot 2\cdots i}\frac{l^{n-1}}{E^{\frac{1}{4}(2n-1)}}\frac{(l')^i(l'')^i\pi^2}{(ll-4l'l'-4l''l''+\sqrt{E})^i},$$

having put

$$E = (l + 2l' + 2l'')(l + 2l' - 2l'')(l - 2l'2l'')(l - 2l'2l'').$$

If one sets

$$l = \frac{1+a^2}{1-a^2} + \frac{1+b^2}{1-b^2}, \quad l' = -\frac{a}{1-a^2}, \quad l'' = -\frac{b}{1-b^2},$$

we find

$$ll - 4l'l' - 4l''l'' = \frac{4(1+a^2b^2)}{(1-a^2)(1-b^2)}, \quad \sqrt{E} = 2l = \frac{4(1-a^2b^2)}{(1-a^2)(1-b^2)},$$

whence, *while a, b are real quantities smaller than* 1, *for infinite i one has*

$$\int\limits_0^\pi \int\limits_0^\pi \frac{\cos ix \cos ix' dx dx'}{\left(\dfrac{1-2a\cos x + a^2}{2(1-a^2)} + \dfrac{1-2b\cos x' + b^2}{2(1-b^2)}\right)^n}$$

$$= \frac{n(n+1)(n+2)\cdots(n+2i-1)}{2\cdot 4\cdots 2i \cdot 2\cdot 4 \cdots 2i}\sqrt{\frac{(1-a^2)(1-b^2)}{1-a^2b^2}}a^i b^i \pi^2.$$

These are sufficiently simple formulas.

14.

On the given occasion I want to add some things about the integrals

$$\int\limits_0^\pi \int\limits_0^\pi \frac{\cos ix \cos i'x' dx dx'}{(l + 2l'\cos x + 2l''\cos x')^n}$$

and similar ones; I will demonstrate them in another paper. First, I observe that generally, what is a theorem of highest importance, *while Δ denotes an arbitrary always positive rational function of* $\cos x$, $\sin x$, $\cos x'$, $\sin x'$ *that the integrals*

$$\int\limits_0^\pi \int\limits_0^\pi \frac{\cos ix \cos i'x' dx dx'}{\Delta^n}, \quad \int\limits_0^\pi \int\limits_0^\pi \frac{\cos ix \sin i'x' dx dx'}{\Delta^n}$$

$$\int\limits_0^\pi \int\limits_0^\pi \frac{\sin ix \cos i'x' dx dx'}{\Delta^n}, \quad \int\limits_0^\pi \int\limits_0^\pi \frac{\sin ix \sin i'x' dx dx'}{\Delta^n}$$

for the different integer values of i, i' can all be expressed linearly by a finite number of them. And those same integrals are expressed linearly by the same for the exponents of Δ differing from the propounded number n by an arbitrary integer.
The integrals

$$\int\limits_0^\pi \int\limits_0^\pi \frac{\cos ix \cos i'x' dx dx'}{(l + 2l' \cos x + 2l'' \cos x')^n}$$

can all be expressed linearly by *four* of them. If

$$\Delta = a + b \cos x + c \sin x + \cos x' (a' + b' \cos x + c' \sin x) + \sin x' (a'' + b'' \cos x + c'' \sin x),$$

the integrals

$$\int\limits_0^\pi \int\limits_0^\pi \frac{\cos ix \cos i'x' dx dx'}{\Delta^n}, \quad \int\limits_0^\pi \int\limits_0^\pi \frac{\cos ix \sin i'x' dx dx'}{\Delta^n}$$

$$\int\limits_0^\pi \int\limits_0^\pi \frac{\sin ix \cos i'x' dx dx'}{\Delta^n}, \quad \int\limits_0^\pi \int\limits_0^\pi \frac{\sin ix \sin i'x' dx dx'}{\Delta^n}$$

can all be expressed linearly by *seven* of them. Let us set that the preceding expression of Δ additionally contains the two terms $d \cos 2x + d' \cos 2x'$, then the form of Δ corresponds to the square of the distance of two planets, expressed by the eccentric anomalies. In this case one has the theorem:
"If it is propounded to expand the distance of two planets, which are moved in elliptic orbits, raised to an arbitrary power, into an infinite series of cosines and sines of multiples of their eccentric anomalies: then the infinitely many coefficients of the expansion can all be expressed linearly by fifteen of them."
In the case, in which the sum of $2l'$, $2l''$, both assumed to be positive, is equal to l, the integral

$$\int\limits_0^\pi \int\limits_0^\pi \frac{\cos ix \cos i'x' dx dx'}{(l + 2l' \cos x + 2l'' \cos x')^{\frac{1}{2}}}$$

can be reduced to a product of elliptic integrals, of which the modulus of the one is the modulus of the other. For, let l', l'' be positive, $l = 2(l' + l'')$, and set

$$\varkappa^2 = \frac{\sqrt{l'+l''}-\sqrt{l''}}{\sqrt{l'+l''}+\sqrt{l''}}, \quad \varkappa'^2 = \frac{2\sqrt{l''}}{\sqrt{l'+l''}+\sqrt{l''}},$$
$$\lambda^2 = \frac{\sqrt{l'+l''}-\sqrt{l'}}{\sqrt{l'+l''}+\sqrt{l'}}, \quad \lambda'^2 = \frac{2\sqrt{l'}}{\sqrt{l'+l''}+\sqrt{l'}},$$

I found, if $i \geq i'$,

$$\frac{4}{\pi^2}\frac{1^2 3^2 \cdots (2i-1)^2}{(2i'+2i-1)(2i'+2i-3)\cdots(2i'-2i+1)}\int_0^\pi\int_0^\pi \frac{\cos ix \cos i'x dx dx'}{(l+2l'\cos x+2l''\cos x')^{\frac{1}{2}}}$$

$$= \frac{(-1)^{i+i'}}{\sqrt{l'+l''}+\sqrt{l''}}\int_0^{\frac{1}{2}\pi}\sin^{2i'-2i}\varphi\cos^{2i}\varphi(1-\varkappa^2\sin\varphi)^{2i-1}d\varphi$$
$$\cdot\int_0^{\frac{1}{2}\pi}\sin^{2i}\varphi\cos^{2i}\varphi(1-\varkappa^2\sin\varphi)^{2i'-2i-1}d\varphi;$$

if $i \geq i'$,

$$\frac{4}{\pi^2}\frac{1^2 3^2 \cdots (2i-1)^2}{(2i+2i'-1)(2i+2i'-3)\cdots(2i-2i'+1)}\int_0^\pi\int_0^\pi \frac{\cos ix \cos i'x dx dx'}{(l+2l'\cos x+2l''\cos x')^{\frac{1}{2}}}$$

$$= \frac{(-1)^{i+i'}}{\sqrt{l'+l''}+\sqrt{l''}}\int_0^{\frac{1}{2}\pi}\sin^{2i-2i'}\varphi\cos^{2i'}\varphi(1-\varkappa^2\sin\varphi)^{2i'-1}d\varphi$$
$$\cdot\int_0^{\frac{1}{2}\pi}\sin^{2i'}\varphi\cos^{2i'}\varphi(1-\varkappa^2\sin\varphi)^{2i-2i'-1}d\varphi.$$

Since $\lambda = \frac{1-\varkappa}{1+\varkappa}$, the modulus λ results from the modulus $\varkappa'$ by Landen's transformation. If

$$l+2l'\cos x+2l''\cos x' = 1+2a(\cos^2(\frac{1}{2}I)\cos x+\sin^2(\frac{1}{2}I)\cos x')+a^2,$$

in the case $a=1$ one finds

$$\varkappa = \tan\left(45^\circ - \frac{1}{4}I\right), \quad \lambda = \tan\left(\frac{1}{4}I\right),$$

$$\frac{1}{\sqrt{l' + l''} + \sqrt{l''}} = \frac{1}{1 + \sin(\frac{1}{2}I)}, \quad \frac{1}{\sqrt{l' + l''} + \sqrt{l'}} = \frac{1}{1 + \cos(\frac{1}{2}I)}.$$

The formulas concern the case, in which the mean distances of two planets to the sun are equal. In this case the ordinary expansions in powers of the inclination fail.

From the preceding formulas, which were rather difficult to find, many other and very memorable ones follow; we will discuss them all on another occasion. If $i = i'$, two representations of the double integral in terms of simple integrals result, which, by means of the substitution

$$\cos\varphi\Delta(\lambda, \varphi) = \sin 2\psi,$$

are reduced to each other.

15.

If in formula (7) we substitute its expansion into a series of cosines of multiples of $2x$ for $\sin^{2i} x$, it results

$$(37) \qquad \int_0^\pi f(\cos x)\cos ixdx$$

$$= \frac{1}{2\cdot 4\cdot 6\cdots 2i}\int_0^\pi f^{(i)}(\cos x)\left[1 - 2\frac{i}{i+1}\cos 2x + 2\frac{i(i-1)}{(i+1)(i+2)}\cos 4x - \cdots\right]dx.$$

If the single integral signs are again transformed by the same (37), having successively put $i = 2, 4, 6, \cdots$, it results

$$2\cdot 4\cdot 6\cdots 2i\int_0^\pi f(\cos x)\cos ixdx$$

$$= \int_0^\pi dx\left[f^{(i)} - 2\frac{i}{i+1}\frac{f^{(i+2)}}{2\cdot 4}\left(1 - \frac{4}{3}\cos 2x + \frac{1}{3}\cos 4x\right)\right.$$

$$+2\frac{i(i-1)}{(i+1)(i+2)}\frac{f^{(i+4)}}{2\cdot 4\cdot 6\cdot 8}\left(1-\frac{8}{5}\cos 2x+\frac{4}{5}\cos 4x-\frac{8}{5\cdot 7}\cos 6x+\frac{1}{5\cdot 7}\cos 8x\right)+\cdots\Bigg].$$

Having repeated this transformation, we get to an infinite series, by which we can represent the propounded integral,

$$(38)\qquad \int_0^\pi f(\cos x)\cos ix dx = \int_0^\pi dx(\alpha f^{(i)}-\beta f^{(i+2)}+\gamma f^{(i+4)}-\delta f^{(i+6)}+\cdots),$$

where $f^{(m)}$ denotes the value of $\dfrac{d^m f(z)}{dz^m}$ for $z=\cos x$, and α, β, γ, δ, $\cdots$ are constant numbers.

Let $f(z)=\cos(\varkappa z)$, i an even number, from (38) we will have

$$\int_0^\pi \cos(\varkappa\cos x)\cos ix dx$$

$$=(-1)^{\frac{1}{2}i}\varkappa^i\int_0^\pi dx\cos(\varkappa\cos x)[\alpha+\beta\varkappa^2+\gamma\varkappa^4+\delta\varkappa+\cdots].$$

Let $f(z)=\sin(\varkappa z)$, i and odd number, from (38) it will be

$$\int_0^\pi \sin(\varkappa\cos x)\cos ix dx$$

$$=(-1)^{\frac{1}{2}(i-1)}\varkappa^i\int_0^\pi dx\cos(\varkappa\cos x)[\alpha+\beta\varkappa^2+\gamma\varkappa^4+\delta\varkappa+\cdots].$$

Hence for either even or odd i from § 9

$$(39)\qquad \alpha+\beta\varkappa^2+\gamma\varkappa^4+\delta\varkappa^6+\cdots=\frac{I_n^{(i)}}{\varkappa^i I_n^{(0)}}$$

$$=\frac{1}{2\cdot 4\cdot 6\cdots 2i}\frac{1-\dfrac{\varkappa^2}{2\cdot(2i+2)}+\dfrac{\varkappa^4}{2\cdot 4\cdot(2i+2)(2i+4)}-\dfrac{\varkappa^6}{2\cdot 4\cdot 6\cdot(2i+2)(2i+4)(2i+6)}+\cdots}{1-\dfrac{\varkappa^2}{2^2}+\dfrac{\varkappa^4}{2^2\cdot 4^2}-\dfrac{\varkappa^6}{2^2\cdot 4^2\cdots 6^2}+\cdots},$$

from which formula one can determine the numbers α, β, γ, δ, $\cdots$.

9th of July 1835