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Singularities

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ABSTRACT. This is the report of the Oberwolfach Workshop on singularity theory held in september 2006. Singularity theory is concerned with the local structure of maps and spaces that occur in many situations in mathematics. It uses methods from algebra, topology and algebraic geometry for their study.

Mathematics Subject Classification (2000): 14Bxx, 32Sxx, 58Kxx.

Introduction by the Organisers

The Oberwolfach workshop *Singularity Theory* was held in the week September 10th–September 15th, 2006. It was organised by J. Steenbrink (Nijmegen), D. van Straten (Mainz) and V. Vassiliev (Moscow) and continued the sequence of workshops *Singularitäten* that take place regularly at Oberwolfach.

The morning sessions consisted of three talks, which were complemented with two talks in the afternoon. So by taking the traditional Oberwolfach hike on wednesday into account, a total of 23 full hour talks was presented during the workshop.

A broad variety of topics was covered by the speakers. Normal surface singularities continue to pose some big questions which were adressed in the talks of A. Nemethi and A. Melle. The theory of motivic integration has brought the arc structure of singularities into the focus of current interest. A. Reguera and P. Popescu-Pampu reported on recent progress related to the original Nash-problem. F. Loeser reported on the first steps in non-archimedian local analytic geometry, A. Parushinski talked on conormal geometry. There has been an increased activity around global aspects of singularity theory, characteristic classes, Thom-polynomials and their applications. These topics were adressed by M. Kazarian, M. Weiss J. Schürmann and V. Sedykh. Applications of ideas from singularity

theory to the geometry of the fundamental group were presented in the talks by A. Dimca, V. Kulikov and A. Libgober. S. Shadrin presented a conjectural method to produce tautological relations in the cohomology of the moduli space $\mathcal{M}_{g,n}$ and O. Tommasi reported on the rational cohomology of \mathcal{M}_4 . A. Gorinov talked about bounds for the size of the automorphism group of complete intersections that arise from the computation of the cohomology of the discriminant-complement. The recent interest in the Hodge/Twistor structures associated to isolated hypersurface germs and tame polynomials on affine varieties, reported on by C. Sevenheck and C. Hertling, stems partly from the interpretation of quantum cohomology via mirror symmetry in terms of so-called Landau-Ginsburg models. D. Mond talked about his idea to relate these to the theory of functions on a free divisor. J. Christophersen showed us a Calabi-Yau space appearing in the deformation theory of a degenerate abelian surface. A. Gabrielov talked about the maximally inflected rational curves and its relation to the Bethe Ansatz. A. Takahashi presented his results for exceptional collections in the category maximal Cohen-Macaulay modules on hypersurface singularities and its mirror interpretation in the case of the exceptional singularities.

An informal concert by J. Steenbrink on piano und M. Weiss on flute formed the musical coda of the workshop.

The organisers would like to use this occasion to thank the Oberwolfach staff for their efficient handling of boundary conditions, which created the unique Oberwolfach atmosphere that helped to make the workshop a success.

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Abstracts

Invariants of Newton non-degenerate surface singularities

ANDRÁS NÉMETHI

(joint work with Gábor Braun)

In general, it is a rather challenging task to connect the analytic and topological invariants of normal surface singularities. The program which aims to recover different discrete analytic invariants from the abstract topological type of the singularity (i.e. from the oriented homeomorphism type of the link K , or from the resolution graph) can be considered as the continuation of the work of Artin, Laufer, Tomari, S. S.-T. Yau (and the author) about rational and elliptic singularities, it includes the efforts of Neumann and Wahl to recover the possible equations of the universal abelian covers [7], and of the author and Nicolaescu about the possible connections of the geometric genus with the Seiberg–Witten invariants of the link [6]. See [5] for a review of this program.

In order to have a chance for this program, one has to consider a topological restriction (maybe, the weakest one for which we still hope for positive results is when we ask for the link to be a rational homology sphere, *RHS* in the sequel), and also restriction about the analytic type of the singularity. [3] shows that the Gorenstein condition, in general, is not sufficient; but even for hypersurface singularities one may expect pathologies.

For isolated hypersurface singularities a famous conjecture was formulated by Zariski [9], which predicts that the multiplicity is determined by the *embedded* topological type. For hypersurface germs with *RHS* links, Mendris and the author in [4] formulated (and verified for suspension singularities) even a stronger conjecture, namely that already the abstract link determines the embedded topological type, the multiplicity and equivariant Hodge numbers (of the vanishing cohomology).

The goal of this talk is to present the validity of this stronger conjecture for isolated singularities with non-degenerate Newton principal part (*NNPP* in the sequel; for its definition see [2]). In fact, we proved that from the link (provided that it is a *RHS*) one can recover the Newton boundary (up to a natural ambiguity — see below, and up to a permutation of coordinates); hence, in fact, the ‘equation of the germ’ (up to an equisingular deformation). This is the maximum what we can hope for.

First, we define an equivalence relation \sim of Newton boundaries: two diagrams Γ_1 and Γ_2 are equivalent if both define isolated singularities and $\Gamma_1 \subset \Gamma_2$. (This, at the level of germs, can be described by a linear deformation.) Although the structure of an equivalence class is not immediate from the definition, one can identify in each class an easily recognizable representative; we call it *d-minimal* representative.

Next, we notice that there is an algorithm, given by toric resolution, for details see Oka's article [8], which provides a possible resolution graph $G(\Gamma)$ (or, equivalently, a plumbing graph of the link) from the Newton boundary Γ . (Equivalent graphs provide plumbing graphs related by blowing ups/downs, hence the same link.) Our main result says that Oka's algorithm essentially can be inverted:

Theorem. *Assume that the Newton diagrams Γ_1 and Γ_2 determine isolated singularities with NNPP whose links are RHS. Assume that the good minimal resolution graphs associated with $G(\Gamma_1)$ and $G(\Gamma_2)$ are the same. Then (up to a permutation of coordinates) $\Gamma_1 \sim \Gamma_2$. In particular, from the link K , one can identify the \sim -equivalence class of the Newton boundary (up to a permutation of coordinates), or even the d -minimal representative of this class.*

In fact, we proved an even stronger result: one can recover the corresponding class of Newton diagrams (or its distinguished representative) already from the orbifold diagram G^o associated with the good minimal resolution graph. This diagram, a priori, contains less information than the resolution graph, it codifies only its shape and some subgraph-determinants. (G^o , although has different decorations, it is equivalent with $\det \Gamma$ together with the 'splice diagram' considered in [7].)

Since under the deformations corresponding to \sim most of the invariants of the germs are stable, one has the following

Corollary. *Let f be an isolated germ with NNPP whose link is RHS. Then the oriented topological type of its link determines completely its Milnor number, geometric genus, spectral numbers, multiplicity, and finally, its embedded topological type.*

Such a statement is highly non-trivial for any of the above invariants. For the history of the problem regarding the Milnor number and the geometric genus the reader is invited to consult [5]. Here we emphasize only the following:

- Regarding the embedded topological type, notice that the result shows that if a RHS 3-manifold can be embedded into S^5 as the embedded link of an isolated hypersurface singularity with NNPP, then this embedding is *unique*. (Notice the huge difference with the case of plane curves, and also with the higher dimensional case, when already the Brieskorn singularities provide a big variety of embeddings $S^{2n-1} \subset S^{2n+1}$, $n \neq 2$.)

- Moreover, such a link can be realized by a germ f with NNPP essentially in a unique way up to a sequence of linear μ -constant deformations (corresponding to \sim) and permutation of coordinates.

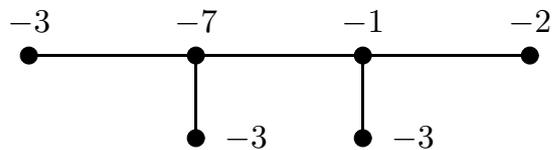
Regarding the main theorem, some more comments are in order.

- The assumption that the link is RHS is necessary: the germs $\{x^a + y^b + z^c = 0\}$ with exponents (3,7,21) and (4,5,20) share the same minimal resolution graph.

- The proof of the main theorem is, in fact, a *constructive algorithm* which provides the d -minimal representatives of the corresponding class of diagrams from the orbifold diagram G^o . Hence, one may check effectively that an arbitrary

resolution graph can or cannot be realized by a hypersurface singularity with *NNPP*. Indeed, if one runs our algorithm and it fails, then it definitely is not of this type. If the algorithm goes through and provides some candidate for a Newton diagram, then one has to compute the graph (orbifold diagram) of this candidate (by Oka's procedure) and compare with the initial one. If they agree then the answer is yes; if they are different, the answer again is no (this may happen since our algorithm uses only a part of the information of G^o).

E.g., one can check that the following resolution graph cannot be realized by an isolated singularity with *NNPP* (although it can be realized by a suspension $\{z^2 + g(x, y) = 0\}$, where g is an irreducible plane curve singularity with Newton pairs (2,3) and (1,3)).



We mention that, in general, there is no procedure which would decide if a graph can be the resolution graph of a hypersurface isolated singularity (this is one of the open problems asked by Laufer [1], p. 122; for suspension singularities is solved in [4]).

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Higher dimensional singularities with bijective Nash map

PATRICK POPESCU-PAMPU

(joint work with Camille Plénat)

Let X be a reduced complex algebraic variety. An *arc* contained in X is a germ of formal map: $(\mathbf{C}, 0) \rightarrow X$.

In a preprint written around 1966, published later as [2], Nash defined the associated *arc space* X_∞ of X , whose points represent the arcs contained in X . By looking at the Taylor expansions of the functions on X with respect to the parameter of the arc and to their truncations at all the orders, Nash constructed this space as a projective limit of algebraic varieties of finite type over X .

If one associates to a formal arc the point of X where it is based, that is the image of $0 \in \mathbf{C}$, one gets a natural map: $\alpha : X_\infty \rightarrow X$. If Y is a closed subvariety of X , denote by: $(X, Y)_\infty := \alpha^{-1}(Y)$ the *space of arcs on X based at Y* .

Nash was thinking of the spaces X_∞ and $(X, Y)_\infty$ for varying $Y \subset \text{Sing}(X)$ as tools for studying the structure of X in the neighborhood of its singular set. Indeed, the main object of his paper was to state a program for comparing the various resolutions of the singularities of X .

In the sequel we restrict to the case where $(X, 0)$ is a germ of a normal complex analytic variety and $\text{Sing}(X) = \{0\}$.

The space $(X, 0)_\infty$ of arcs on X based at 0 is a relative subscheme over X of X_∞ . As it projects onto 0 , we see that it is in fact a true scheme (but not of finite type over \mathbf{C}). This implies that it makes sense to speak about the set $\mathcal{C}(X, 0)_\infty$ of its irreducible components.

Denote by $\pi : \tilde{X} \rightarrow X$ a resolution of X . The *exceptional set* $\text{Exc}(\pi) := \pi^{-1}(0)$ is not assumed to be of pure codimension 1, that is, the resolution is not necessarily divisorial.

An irreducible component of $\text{Exc}(\pi)$ is called *an essential component of π* if it corresponds to an irreducible component of the exceptional set of any other resolution of X . In other words, if its birational transform is an irreducible component of the exceptional set in any resolution. An equivalence class of such essential components over all the resolutions of X is called an *essential divisor over $(X, 0)$* . If we denote by $\mathcal{E}(X, 0)$ the set of essential divisors over $(X, 0)$, the essential components of the given resolution morphism π are in a canonical bijective correspondence with the elements of $\mathcal{E}(X, 0)$.

Let \mathcal{K} be an element of $\mathcal{C}(X, 0)_\infty$. For each arc represented by a point of \mathcal{K} , one can consider the intersection point with $\text{Exc}(\pi)$ of its strict transform on \tilde{X} . For an arc generic with respect to the Zariski topology of \mathcal{K} , this intersection point is situated on a unique irreducible component of $\text{Exc}(\pi)$; moreover, this component is essential (Nash [2]). In this manner one defines a map:

$$\mathcal{N}_{X,0} : \mathcal{C}(X, 0)_\infty \rightarrow \mathcal{E}(X, 0)$$

which is called *the Nash map* associated to $(X, 0)$. Nash proved that the map $\mathcal{N}_{X,0}$ is always *injective* (which shows in particular that $\mathcal{C}(X, 0)_\infty$ is a finite set) and he

asked:

When is the map $\mathcal{N}_{X,0}$ bijective?

In [3] we listed the classes of normal isolated *surface* singularities for which the Nash map was proved to be bijective. In higher dimensions, among normal germs with isolated singularities, the bijectivity of \mathcal{N} was proved till now only for the germs which have resolutions with irreducible exceptional set (for trivial reasons) and for the germs of normal toric varieties (by Ishii and Kollár in [1]).

No surface or 3-fold germ is known for which the Nash map is not bijective. But Ishii and Kollár [1] constructed such a germ in dimension 4.

I explained first a criterion in arbitrary dimension for an exceptional divisor to be in the image of the Nash map :

Theorem *Let $\pi : \tilde{X} \rightarrow X$ be a divisorial projective resolution of $(X,0)$. Consider an irreducible component E_i of $\text{Exc}(\pi)$. Suppose that for any other component E_j , there exists an effective integral divisor F_{ij} on \tilde{X} whose support coincides with $\text{Exc}(\pi)$, in which the coefficient of E_i is strictly less than the coefficient of E_j and such that the line bundle $\mathcal{O}_{\text{Exc}(\pi)}(-F_{ij})$ is ample. Then E_i is an essential component contained in the image of the Nash map.*

Combining the previous criterion with Kleiman's criterion of ampleness, I explained our *construction of an infinite family of examples of 3-dimensional singularities with bijective Nash map*. Namely, we construct algebraic 3-dimensional manifolds which contain two smooth ruled surfaces intersecting transversely along a curve C , and such that the union of the two surfaces may be contracted (by Grauert's criterion). We adjust the parameters of the construction such that the previous theorem may be applied.

I explained how to determine among the germs constructed by us, which ones are isomorphic to germs of toric varieties, establishing like this the intersection of our class of examples with the classes known before. Namely, such a germ is toric if and only if the curve C is rational.

Details may be found in [4], a work which generalizes the surface case, treated in [3].

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Algorithms to produce tautological relations in $\overline{\mathcal{M}}_{g,n}$

SERGEI SHADRIN

1. The tautological ring of the moduli space of curves $\overline{\mathcal{M}}_{g,n}$ is the subring of its cohomology ring that contains all geometrically natural classes. It is known that the tautological ring is spanned by all natural strata equipped with ψ and κ classes (ψ - κ -strata). But these ψ - κ -strata are not linearly independent in the cohomology of $\overline{\mathcal{M}}_{g,n}$. Relations among them are called *tautological relations*.

Tautological relations have an interpretation as universal differential equations for Gromov-Witten potentials. The first example of such differential equation is WDVV. So, we can try to look for the universal equations. And this point of view really simplifies the problem, since there exist several alternative formal approaches to the GW theory (or rather the theory of genus expansion of Frobenius manifolds).

In two of these approaches there exist direct algorithms how to produce the universal equations: Y.-P. Lee algorithm [1] and our own one [3], which is a byproduct of some calculations in dGBV-algebras [2]. Both of them use only finite dimensional linear algebra. In this talk we discuss these two algorithms and compare them. We announce the result that these two algorithms are almost equivalent.

2. I'd like to thank A. Gorinov, M. Kazarian, and O. Tommasi for the fruitful discussion of these algorithms during the workshop.

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New results on the fundamental groups of algebraic varieties

ALEXANDRU DIMCA

(joint work with Stefan Papadima, Alexander Suciu)

In a series of papers we have investigated some of the characteristic properties of the fundamental groups of smooth complex algebraic varieties.

In the first one [2] we have shown that for a 1-formal finitely presented group G the tangent cone at the origin of its k -th characteristic variety (associated to the first cohomology group of G with rank one local system coefficients) coincide to the k -th resonance variety of G (which is computable in terms of the cohomology algebra of G in degrees up to 2). As a result we have obtained new restrictions on the quasi-projective groups G , i.e. the groups which may occur as the fundamental groups of smooth complex quasi-projective varieties. In particular this allows us to list all the right-angled Artin groups which are quasi-projective.

In the second paper [3] we have listed all the quasi-projective groups in the class of Bestvina-Brady groups, see [1] for this important class of groups. In particular

some groups on this list provided a negative answer to a question by Kollár in [5] on the existence of quasi-projective classifying spaces (up to commensurability) of quasi-projective groups

Finally, in the third paper [4], we construct for each integer $n \geq 2$ an irreducible, smooth, complex projective variety M of dimension n , whose fundamental group has infinitely generated homology in degree $n + 1$ and whose universal cover is a Stein manifold, homotopy equivalent to an infinite bouquet of n -dimensional spheres. This non-finiteness phenomenon is also reflected in the fact that the homotopy group $\pi_n(M)$, viewed as a module over $\mathbb{Z}\pi_1(M)$, is free of infinite rank. As a result, we give a negative answer to the above question of Kollár for the fundamental groups of smooth projective varieties. To obtain our examples, we develop a complex analog of a method in geometric group theory due to Bestvina and Brady.

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***C*-groups: their realizations and properties**

VIKTOR S. KULIKOV

A class \mathcal{C} of C -groups and its subclass \mathcal{H} of Hurwitz C -groups (see definitions below) play very important role in geometry of codimension two submanifolds. For example, it is well known that a link group (given by Wirtinger presentation) is a C -group. More generally, the group of a linked n -manifold, $n \geq 2$, (that is, the fundamental group $\pi_1(S^{n+2} \setminus V)$ of the complement of a closed oriented n -manifold V without boundary in the $(n + 2)$ -dimensional sphere S^{n+2}) given by generalized Wirtinger presentation, is also a C -group. In the same way, if $H \subset \mathbb{C}\mathbb{P}^2$ is an algebraic or, more generally, Hurwitz (resp., pseudo-holomorphic) curve of degree m (see the definition of Hurwitz curves in [4] or in [3]), then the Zariski – van Kampen presentation of $\pi_1 = \pi_1(\mathbb{C}\mathbb{P}^2 \setminus (H \cup L))$ defines on π_1 a structure of a Hurwitz C -group of degree m , where L is a line at "infinity" (that is, L is a fiber of linear projection $\text{pr} : \mathbb{C}\mathbb{P}^2 \rightarrow \mathbb{C}\mathbb{P}^1$ and it is in general position with respect to H ; if H is a pseudo-holomorphic curve, then pr is given by a pencil of pseudo-holomorphic lines). Similarly, any fundamental group $\pi_1(\Delta^2 \setminus (C \cap \Delta^2))$, where $\Delta^2 = \{|z| < 1\} \times \{|w| < 1\} \subset \mathbb{C}^2$ is a bi-disc and $C \subset \mathbb{C}^2$ is an algebraic curve

such that the restriction of $\text{pr}_1 : \Delta^2 \rightarrow \{|z| < 1\}$ to $C \cap \Delta^2$ is a proper map, is a C -group.

By definition, a C -group is a group together with a finite presentation

$$(1) \quad G_W = \langle x_1, \dots, x_m \mid x_i = w_{i,j,k}^{-1} x_j w_{i,j,k}, w_{i,j,k} \in W \rangle,$$

where $W = \{w_{i,j,k} \in \mathbb{F}_m \mid 1 \leq i, j \leq m, 1 \leq k \leq h(i, j)\}$ is a collection (it is possible that $w_{i_1, j_1, k_1} = w_{i_2, j_2, k_2}$ for $(i_1, j_1, k_1) \neq (i_2, j_2, k_2)$) consisting of elements of the free group \mathbb{F}_m generated by free generators x_1, \dots, x_m and $h : \{1, \dots, m\}^2 \rightarrow \mathbb{Z}$ is some function. Such a presentation is called a C -presentation (C , since all relations have the form of conjugations). Let $\varphi_W : \mathbb{F}_m \rightarrow G_W$ be the canonical epimorphism. The elements $\varphi_W(x_i) \in G$, $1 \leq i \leq m$, and the elements conjugated to them are called the C -generators of G . Let $f : G_1 \rightarrow G_2$ be a homomorphism of C -groups. It is called a C -homomorphism if the images of the C -generators of G_1 under f are C -generators of the C -group G_2 . C -groups are considered up to C -isomorphisms.

A C -group G is called a *Hurwitz C -group of degree m* if there are C -generators x_1, \dots, x_m generating G such that the product $x_1 \dots x_m$ belongs to the center of G . Note that the degree of a Hurwitz C -group G is not defined canonically and depends on the choice of its C -generators. Denote by \mathcal{H} the class of all Hurwitz C -groups.

It is easy to show that G/G' is a finitely generated free abelian group for any C -group G , where $G' = [G, G]$ is the commutator subgroup of G . A C -group G is called *irreducible* if $G/G' \simeq \mathbb{Z}$ and we say that G consists of k *irreducible components* if $G/G' \simeq \mathbb{Z}^k$. If a Hurwitz C -group G is realized as the fundamental group $\pi_1(\mathbb{C}\mathbb{P}^2 \setminus (H \cup L))$ of the complement of some Hurwitz curve H , then the number of irreducible components of G is equal to the number of irreducible components of H . Similarly, if a C -group G consisting of k irreducible components is realized as the group of a linked n -manifold V , that is, $G = \pi_1(S^{n+2} \setminus V)$, then the number of connected components of V is equal to k .

Theorem 1. ([5]) *Any C -group G can be realized as a group of a linked n -manifold, $n \geq 2$, in the $(n + 2)$ -dimensional sphere S^{n+2} , that is, the set \mathcal{C} coincides with the set of the groups of linked n -manifolds for any $n \geq 2$.*

The set \mathcal{C} of C -groups is rather broad. For example, for any finitely presented group $G = \langle x_1, \dots, x_m \mid r_i(\bar{x}) = 1, 1 \leq i \leq n \rangle$ there is a C -group $\bar{G} = \langle x_1, \dots, x_m \mid [r_i(\bar{x}), x_j] = 1, 1 \leq i \leq n, 1 \leq j \leq m \rangle$ such that \bar{G} dominates G , that is, there is the canonical epimorphism $h : \bar{G} \rightarrow G$. In particular, the Baumslag – Solitar group

$$\begin{aligned} \tilde{G} = & \langle a, t \mid t^{-1} a^2 t = a^3 \rangle \simeq \langle a, t, x_1, x_2 \mid x_1 = t, x_2 = at, a^2 t a^{-2} = at \rangle \simeq \\ & \langle x_1, \dots, x_5 \mid x_3 = x_1^{x_2}, x_3 = x_4^{x_1}, x_5 = x_4^{x_2}, x_5 = x_2^{x_1} \rangle \end{aligned}$$

is an irreducible C -group. It is a non-Hopfian group ([2]). Applying this example, an example of non-Hopfian Hurwitz C -group consisting of two irreducible components was constructed in [8].

Let \mathcal{C}_{Br} be a set of the groups which possess presentations of the form

$$G_B = \langle x_1, \dots, x_m \mid x_i^{-1} b(x_i) = \mathbf{1}, i = 1, \dots, m, b \in B \rangle$$

for some $m \in \mathbb{N}$ and a finitely generated subgroup B of the braid group Br_m , where $b(x_i)$ is the image of x_i under the standard action of $b \in Br_m$ on \mathbb{F}_m . It is easy to see that for any $B \subset Br_m$ the group G_B is a C -group.

Theorem 2. ([6]) *We have $\mathcal{C}_{Br} = \mathcal{C}$. Moreover, any C -group G is C -isomorphic to some $G_B \in \mathcal{C}_{Br}$ such that the group $B \subset Br_m$ is generated by elements conjugated to the standard generator of the group Br_m .*

Recall also Artin's theorem

Theorem 3. ([1]) *The set of the link groups $\{\pi_1(S^3 \setminus L)\}$ coincides with the set $\{G_B \in \mathcal{C}_{Br} \mid B = \langle b \rangle$ cyclic subgroups of of the braid groups $\}$.*

By Theorem 2.1 in [4], any braid monodromy factorization with factors, conjugated to the standard generator of the group Br_m , can be realized as a braid monodromy factorization of a non-singular algebraic curve $C \subset \mathbb{C}^2$ in the bi-disc Δ^2 . Therefore we have

Corollary 4. *The set of the fundamental groups $\{\pi_1(\Delta^2 \setminus (C \cap \Delta^2))\}$ coincides with \mathcal{C} , where $\Delta^2 = \{|z| < 1\} \times \{|w| < 1\} \subset \mathbb{C}^2$ is a bi-disc and $C \subset \mathbb{C}^2$ are non-singular algebraic curves such that the restriction of $pr_1 : \Delta^2 \rightarrow \{|z| < 1\}$ to $C \cap \Delta^2$ is a proper map.*

In [6], the following theorem was proved.

Theorem 5. *Any Hurwitz C -group G of degree m can be realized as the fundamental group $\pi_1(\mathbb{C}P^2 \setminus (H \cup L))$ for some Hurwitz (resp. pseudo-holomorphic) curve H , $\deg H = 2^n m$, with singularities of the form $w^m - z^m = 0$, where n depends on the Hurwitz C -presentation of G . So the class \mathcal{H} coincides with the class $\{\pi_1(\mathbb{C}P^2 \setminus (H \cup L))\}$ of the fundamental groups of the complements of "affine" Hurwitz (resp., of "affine" pseudo-holomorphic) curves.*

A free group \mathbb{F}_n with fixed free generators is a C -group and for any C -group G the canonical C -epimorphism $\nu : G \rightarrow \mathbb{F}_1$, sending the C -generators of G to the C -generator of \mathbb{F}_1 , is well defined. Denote by N its kernel. Note that if G is an irreducible C -group, then N coincides with G' .

Let G be a C -group. The C -epimorphism ν induces the following exact sequence of groups

$$1 \rightarrow N/N' \rightarrow G/N' \xrightarrow{\nu_*} \mathbb{F}_1 \rightarrow 1,$$

where $N' = [N, N]$. The C -generator of \mathbb{F}_1 acts on N/N' by conjugation $\tilde{x}^{-1} g \tilde{x}$, where $g \in N$ and \tilde{x} is one of the C -generators of G . Denote by t this action. The group $A_0(G) = N/N'$ is an abelian group and the action t defines on $A_0(G)$ a structure of Λ -module, where $\Lambda = \mathbb{Z}[t, t^{-1}]$ is the ring of Laurent polynomials with integer coefficients. The Λ -module $A_0(G)$ is called the *Alexander module* of the C -group G . The action t induces an action $h_{\mathbb{C}}$ on $A_{\mathbb{C}} = A_0(G) \otimes \mathbb{C}$ and it is easy to see that its characteristic polynomial $\det(h_{\mathbb{C}} - t\text{Id}) \in \mathbb{Q}[t]$. The

polynomial $\Delta(t) = a \det(h_{\mathbb{C}} - t\text{Id})$, where $a \in \mathbb{N}$ is the smallest number such that $a \det(h_{\mathbb{C}} - t\text{Id}) \in \mathbb{Z}[t]$, is called the *Alexander polynomial* of the C -group G . If H is either an algebraic, or Hurwitz, or pseudo-holomorphic curve in $\mathbb{C}\mathbb{P}^2$ (resp., $V \subset S^{n+2}$ is a linked n -manifold, $n \geq 1$) and $G = \pi_1(\mathbb{C}\mathbb{P}^2 \setminus (H \cup L))$ (resp., $G = \pi_1(S^{n+2} \setminus V)$), then the Alexander module $A_0(G)$ of the group G and its Alexander polynomial $\Delta(t)$ are called the *Alexander module and Alexander polynomial* of the curve H (resp., of the linked manifold V). Note that the Alexander module $A_0(H)$ and the Alexander polynomial $\Delta(t)$ of a curve H do not depend on the choice of the generic (pseudo)-line L .

In [3] and [7], properties of the Alexander polynomials of Hurwitz curves were investigated. In particular, it was proved that if H is a Hurwitz curve of degree d consisting of k irreducible components and $\Delta(t)$ is its Alexander polynomial, then $\Delta(t)$ has the following properties

- (i) $\Delta(t) \in \mathbb{Z}[t]$;
- (ii) $\Delta(0) = \pm 1$;
- (iii) $\Delta(t)$ is a divisor of the polynomial $(t-1)(t^d-1)^{d-2}$, in particular, the roots of $\Delta(t)$ are d -th roots of unity;
- (iv) the action of $h_{\mathbb{C}}$ on $(N/N') \otimes \mathbb{C}$ is semisimple;
- (v) the multiplicity of the root $t=1$ of $\Delta(t)$ is equal to $k-1$;
- (vi) if $k=1$, then $\Delta(1) = 1$ and $\deg \Delta(t)$ is an even number,

and, moreover, a polynomial $P(t) \in \mathbb{Z}[t]$ is the Alexander polynomial of an irreducible Hurwitz curve iff the roots of $P(t)$ are roots of unity and $P(1) = 1$.

Let $G = \pi_1(\mathbb{C}\mathbb{P}^2 \setminus (H \cup L))$ be the fundamental group of the complement of an "affine" Hurwitz curve of degree m (resp., $G = \pi_1(S^{n+2} \setminus V)$ is the group of a linked n -manifold, $n \geq 1$). The homomorphism $\nu : G \rightarrow \mathbb{F}_1$ defines an infinite unramified cyclic covering $f_{\infty} : X_{\infty} \rightarrow \mathbb{C}\mathbb{P}^2 \setminus (H \cup L)$ (resp., $f_{\infty} : X_{\infty} \rightarrow S^{n+2} \setminus V$). We have $H_1(X_{\infty}, \mathbb{Z}) = N/N'$ and the action of t on $H_1(X_{\infty}, \mathbb{Z})$ coincides with the action of a generator h of the covering transformation group of the covering f_{∞} .

For any $k \in \mathbb{N}$ denote by $\text{mod}_k : \mathbb{F}_1 \rightarrow \mu_k = \mathbb{F}_1/\{t^k\}$ the natural epimorphism to the cyclic group μ_k of degree k . The covering f_{∞} can be factorized through the cyclic covering $f'_k : X'_k \rightarrow \mathbb{C}\mathbb{P}^2 \setminus (H \cup L)$ (resp., $f'_k : X'_k \rightarrow S^{n+2} \setminus V$) associated with the epimorphism $\text{mod}_k \circ \nu$, $f_{\infty} = f'_k \circ g_k$. Since a Hurwitz curve H has only analytic singularities, the covering f'_k can be extended (see [3]) to a finite map $\tilde{f}_k : \tilde{X}_k \rightarrow X$ branched along H and, maybe, along L . Here \tilde{X}_k is a closed four dimensional variety locally isomorphic over a singular point of $H \cup L$ to a complex analytic singularity. One can resolve the singularities of \tilde{X}_k and obtain a smooth manifold \bar{X}_k , $\dim_{\mathbb{R}} \bar{X}_k = 4$. Let $\sigma : \bar{X}_k \rightarrow \tilde{X}_k$ be a resolution of the singularities, $E = \sigma^{-1}(\text{Sing } \tilde{X}_k)$ the proper transform of the set of singular points of \tilde{X}_k , and $\bar{f}_k = \tilde{f}_k \circ \sigma$. The action h induces an action \bar{h}_k on \bar{X}_k and an action t on $H_1(\bar{X}_k, \mathbb{Z})$.

Similarly, the covering $f'_k : X'_k \rightarrow S^{n+2} \setminus V$ can be extended to a smooth map $f_k : X_k \rightarrow S^{n+2}$ branched along V , where X_k is a smooth compact $(n+2)$ -manifold, and the action h induces actions h_k on X_k and h_{k*} on $H_1(X_k, \mathbb{Z})$. The action h_{k*} defines on $H_1(X_k, \mathbb{Z})$ a structure of Λ -module.

In [3], it was shown that for any Hurwitz curve H , a covering space \overline{X}_k can be embedded as a symplectic submanifold to a complex projective rational 3-fold on which the symplectic structure is given by an integer Kähler form, and it was proved that the first Betti number $b_1(\overline{X}_k) = \dim_{\mathbb{C}} H_1(\overline{X}_k, \mathbb{C})$ of \overline{X}_k is equal to $r_{k, \neq 1}$, where $r_{k, \neq 1}$ is the number of roots of the Alexander polynomial $\Delta(t)$ of the curve \overline{H} which are k -th roots of unity not equal to 1.

Let M be a Noetherian Λ -module. We say that M is $(t-1)$ -invertible if the multiplication by $t-1$ is an automorphism of M . A Λ -module M is called t -unipotent if for some $n \in \mathbb{N}$ the multiplication by t^n is the identity automorphism of M . The smallest $k \in \mathbb{N}$ such that

$$t^k - 1 \in \text{Ann}(M) = \{f(t) \in \Lambda \mid f(t)v = 0 \text{ for } \forall v \in M\}$$

is called the *unipotence index* of t -unipotent module M .

Let M be a Noetherian $(t-1)$ -invertible Λ -module. A t -unipotent Λ -module $A_n(M) = M/(t^n - 1)M$ is called the k -th derived Alexander module of M and if M is the Alexander module of an irreducible C -group G (resp., of a knotted n -manifold V , resp., of an irreducible Hurwitz curve H), then $A_k(M)$ is called the k -th derived Alexander module of G (resp., of V , resp., of H) and it will be denoted by $A_k(G)$ (resp., $A_k(V)$, resp., $A_k(H)$)

Theorem 6. *A Λ -module M is the Alexander module of a knotted n -manifold for $n \geq 2$ if and only if it is a Noetherian $(t-1)$ -invertible Λ -module.*

Theorem 7. *Let V be a knotted n -manifold, $n \geq 1$, and $f_k : X_k \rightarrow S^{n+2}$ the cyclic covering branched along V . Then $H_1(X_k, \mathbb{Z})$ is isomorphic to the k -th derived Alexander module $A_k(V)$ of V as a Λ -module.*

Similar statements hold in the case of algebraic and, more generally, of Hurwitz (resp., pseudo-holomorphic) curves. Namely, we have the following theorems.

Theorem 8. *A Λ -module M is the Alexander module of an irreducible Hurwitz (resp., pseudo-holomorphic) curve if and only if it is a Noetherian $(t-1)$ -invertible t -unipotent Λ -module. In particular, the Alexander module of an irreducible algebraic plane curve is a Noetherian $(t-1)$ -invertible t -unipotent Λ -module.*

The unipotence index of the Alexander module $A_0(H)$ of an irreducible plane algebraic (resp., Hurwitz or pseudo-holomorphic) curve H is a divisor of $\deg H$.

Corollary 9. *The Alexander module $A_0(H)$ of an irreducible plane algebraic (resp., Hurwitz or pseudo-holomorphic) curve H is finitely generated over \mathbb{Z} , that is, $A_0(H)$ is a finitely generated abelian group.*

A finitely generated abelian group G is the Alexander module $A_0(H)$ of some irreducible Hurwitz or pseudo-holomorphic curve H if and only if there are an integer m and an automorphism $h \in \text{Aut}(G)$ such that $h^m = \text{Id}$ and $h - \text{Id}$ is also an automorphism of G .

Theorem 10. *Let H be an algebraic (resp., Hurwitz or pseudo-holomorphic) irreducible curve in \mathbb{CP}^2 , $\deg H = m$, and $\bar{f}_k : \bar{X}_k \rightarrow \mathbb{CP}^2$ be a resolution of singularities of the cyclic covering of degree $\deg \bar{f}_k = k$ branched along H and, maybe, alone the line "at infinity" L . Then*

$$\begin{aligned} H_1(\bar{X}_k \setminus E, \mathbb{Z}) &\simeq A_k(H), \\ H_1(\bar{X}_k, \mathbb{Q}) &\simeq A_k(H) \otimes \mathbb{Q}, \end{aligned}$$

where $A_k(H)$ is the k -th derived Alexander module of H and $E = \sigma^{-1}(\text{Sing } \tilde{X}_k)$.

It should be noticed that in general case of Hurwitz curves the epimorphism $H_1(\bar{X}_k \setminus E, \mathbb{Z}) \simeq A_k(H) \rightarrow H_1(\bar{X}_k, \mathbb{Z})$, induced by the embedding $\bar{X}_k \setminus E \hookrightarrow \bar{X}_k$, is not necessary to be an isomorphism.

Corollary 11. *Let H be an algebraic (resp., Hurwitz or pseudo-holomorphic) irreducible curve in \mathbb{CP}^2 , $\deg H = m$, and $\bar{f}_k : \bar{X}_k \rightarrow \mathbb{CP}^2$ be a resolution of singularities of the cyclic covering of degree $\deg f_k = k$ branched along H and, maybe, alone the line "at infinity". Then*

- (i) *the first Betti number $b_1(\bar{X}_k)$ of \bar{X}_k is an even number;*
- (ii) *if $k = p^r$, where p is prime, then $H_1(\bar{X}_k, \mathbb{Q}) = 0$;*
- (iii) *if k and m are coprime, then $H_1(\bar{X}_k, \mathbb{Z}) = 0$;*
- (iv) *$H_1(\bar{X}_2, \mathbb{Z})$ is a finite abelian group of odd order.*

In [3], it was shown that for any $k \in \mathbb{N}$, there is a Hurwitz curve \bar{H}_k consisting of two irreducible components such that the resolution of singularities $\bar{X}_{k,6}$ of a cyclic covering of the projective plane, branched along \bar{H}_k , has the first Betti number $b_1(\bar{X}_{k,6}) = k$. Recall that $b_1(\bar{X}_n)$ is always even if \bar{H} is a plane algebraic curve, hence such \bar{H}_k cannot be algebraic if k is odd.

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Higher elliptic genera

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(joint work with Lev Borisov)

Novikov's conjecture states that for a C^∞ manifold M with a fundamental group π , the classifying map $f : M \rightarrow B\pi$ and a class $\alpha \in H^*(B\pi, \mathbf{Q})$ the Novikov's higher signatures $(L(M) \cup f^*(\alpha))[M]$ are homotopy invariants (cf. [5] for a survey of enormous amount of work and references related to this conjecture). Here $L(M)$ is the total Hirzebruch's L -class and $[M] \in H_{\dim M}(M, \mathbf{Q})$ is the fundamental class of M . In [6] J.Rosenberg asked if for a projective manifold, the higher Todd genera $Td_\alpha = (Td(X) \cup f^*(\alpha))[X]$, where $Td(X) \in H^*(X, \mathbf{Q})$ is the total Todd class, are *birational* invariants of X . The same class α can be used to calculate higher genera for any pair of birational manifolds since the fundamental group is a birational invariant. This question was answered in [2], where it was shown also that the higher Todd genera are the only birational invariants which are at the same time the invariants of complex cobordisms of manifolds with given fundamental group.

This talk is a report on the work [3] where we considered the invariants of cobordisms of manifolds with given fundamental group π which are the invariants of K -equivalences. Recall that two (projective) manifolds X_1, X_2 are called K -equivalent if there is a smooth manifold \tilde{X} and a diagram:

$$(1) \quad \begin{array}{ccc} & \tilde{X} & \\ \phi_1 \swarrow & & \searrow \phi_2 \\ X_1 & & X_2 \end{array}$$

where ϕ_1 and ϕ_2 are birational morphisms and $\phi_1^*(K_{X_1})$ is linearly equivalent to $\phi_2^*(K_{X_2})$.

Our main results describe the complete set of invariants of cobordisms of the maps of $X \rightarrow B\pi$ of almost complex manifolds which also are the invariants of K -equivalence as in (1). These invariants are given by the higher elliptic genera. In terms of the characteristic classes they can be described as follows. Let x_i be the Chern roots of X i.e. the total Chern class of the tangent bundle of X is given by $c(X) = \prod(1 + x_i)$. Let the class $\mathcal{E}LL(X)$ be given by

$$(2) \quad \mathcal{E}LL(X) = \prod_i x_i \frac{\theta(\frac{x_i}{2\pi i} - z, \tau)}{\theta(\frac{x_i}{2\pi i}, \tau)}$$

where (with $q = e^{2\pi i \tau}$)

$$(3) \quad \theta(z, \tau) = q^{\frac{1}{8}} (2\sin \pi z) \prod_{l=1}^{l=\infty} (1 - q^l) \prod_{l=1}^{l=\infty} (1 - q^l e^{2\pi i z}) (1 - q^l e^{-2\pi i z})$$

is the classical theta function. One can view $\theta(z, \tau)$ and $\mathcal{E}LL(X)$ as power series in q and $y = e^{2\pi i z}$, with the coefficients of the latter being the cohomology (or

Chow group) classes of X . In terms of $\mathcal{E}LL(X) \in H^*(X, \mathbf{Q})$ given by (2), the higher elliptic genus is defined as:

$$(4) \quad Ell_\alpha(X) = (\mathcal{E}LL(X) \cup f^*(\alpha))[X]$$

The elliptic genus (4) specializes into Novikov's higher signature and the Todd higher genus for $q = 0$ and the appropriate values of y . In fact we work in the category of triples (X, D, G) where X is projective, D is a divisor with normal crossings and G a finite group of biholomorphic automorphisms of X leaving D invariant and such that the D is G -normal (the same category as the one considered by V. Batyrev in [1] and used in [4]) and define twisted genus $Ell_\alpha(X, D, G)$ using the elliptic class $\mathcal{E}LL(X, D, G)$ constructed in [4]. $Ell_\alpha(X, D, G)$ becomes $Ell_\alpha(X)$ if G is trivial and $D = \emptyset$.

Our main results are the following:

Theorem 1 *Let X be a SU -manifold, $d = \dim X$, $\pi = \pi_1(X)$ and $\alpha \in H^{2k}(\pi, \mathbf{Q})$. Then the higher elliptic genus $(\mathcal{E}LL(X) \cup f^*(\alpha))[X]$ is a Jacobi form having index $\frac{d}{2}$ and weight $-k$. It has the Novikov signature, the higher Todd genus and higher \hat{A} -genus as specializations.*

More generally, let X, D be a Kawamata log-terminal G -normal pair where G is a finite group. If $m(K_X + D)$ is a trivial Cartier divisor, n is the order of the image $G \rightarrow \text{Aut}H^0(X, m(K_X + D))$ and $\alpha \in H^k(\pi_1(X), \mathbf{Q})$ as above then $\mathcal{E}LL_\alpha(X, D, G, z, \tau)$ is a Jacobi form having weight $-k$ and the index $\frac{\dim X}{2}$ for the subgroup of Jacobi group acting on $\mathbf{C} \times H$ and generated by:

$$(z, \tau) \rightarrow (z + mn\tau, \tau), (z, \tau) \rightarrow (z + mn, \tau), (z, \tau) \rightarrow (z, \tau + 1), (z, \tau) \rightarrow \left(\frac{z}{\tau}, -\frac{1}{\tau}\right)$$

Theorem 2 *For any $\alpha \in H^*(B\pi, \mathbf{Q})$ the higher elliptic genus $(\mathcal{E}LL(X) \cup f^*(\alpha), [X])$ is an invariant of K -equivalence. Moreover, if (X, D, G) and (\hat{X}, \hat{D}, G) are G -normal and Kawamata log-terminal and if $\phi : (\hat{X}, \hat{D}) \rightarrow (X, D)$ is G -equivariant such that*

$$\phi^*(K_X + D) = K_{\hat{X}} + \hat{D}$$

then

$$Ell_\alpha(\hat{X}, \hat{D}, G) = Ell_\alpha(X, D, G)$$

In particular the higher elliptic genera (and hence the higher signatures and \hat{A} -genus) are invariant for crepant birational morphisms and the specialization into the Todd class is birationally invariant (providing a proof asked for in [6]).

The proof of theorem 2 is a consequence of the push forward formulas for the elliptic classes $\mathcal{E}LL(X, D, G)$ proven in [4].

Another properties of the higher elliptic genus discussed in [3] include

a) existence of the higher elliptic genera for varieties with \mathbf{Q} -Gorenstein and Kawamata log-terminal singularities.

b) McKay correspondence for higher elliptic genera extending to non simply connected case the results of [4].

c) calculation of the quotient of cobordisms $\Omega^{SU}(B\pi)/I_\pi$ where I_π is the ideal generated by the differences of manifolds differ by classical flops from [7] i.e. a proof of the isomorphism:

$$(5) \quad \text{Hom}(\Omega_d^{SU}(B\pi)/I_\pi \cap \Omega_d^{SU}(B\pi), \mathbf{Q}) = \bigoplus_{k \in 2\mathbf{Z}} H^k(B\pi, \text{Jac}_{-k, \frac{d}{2}})$$

(where $\text{Jac}_{*,*}$ is the bigraded ring of Jacobi forms).

In particular, the higher elliptic genera are the only cobordism and K-equivalence invariants.

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Pontryagin classes of euclidean bundles and singularity theory

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A euclidean bundle is a fiber bundle with fibers homeomorphic to \mathbb{R}^n , for some fixed n . Such a bundle has an associated principal bundle with structure group $\text{TOP}(n)$, the topological group of homeomorphisms of \mathbb{R}^n . Conversely, the classifying space $B\text{TOP}(n)$ carries a universal euclidean bundle with fiber \mathbb{R}^n .

The homological and homotopical properties of $\text{TOP}(n)$ and $B\text{TOP}(n)$ are important to differential topologists because of a theorem due to Morlet [6], [5], which says that the topological group $\text{Diff}(D^n)$ of diffeomorphisms of the disk D^n , fixing the boundary S^{n-1} pointwise, is homotopy equivalent to

$$\Omega^{n+1}(\text{TOP}(n)/\text{O}(n)).$$

(Here $\Omega^i(X)$ should be read as: space of based maps from an i -sphere to the pointed space X .) In greater detail, smoothing theory, in the form developed by Morlet, expresses the “difference” between $\text{Diff}(D^n)$ and its non-differentiable variant $\text{Homeo}(D^n)$ in terms of the appropriate structure groups of the tangent bundles, $\text{GL}(n) \simeq \text{O}(n)$ and $\text{TOP}(n)$, while the Alexander trick shows that $\text{Homeo}(D^n)$ is contractible.

A calculation of the rational cohomology of $BTOP(n)$ or $TOP(n)$ would therefore almost mechanically lead to a calculation of the rational cohomology of $\text{Diff}(D^n)$. Unfortunately, except in cases where n is very small, $H^i(BTOP(n); \mathbb{Q})$ is only well understood for $i < 4n/3$ approximately [2], [4]. It follows from these calculations that for odd n , not too small, the inclusion $BO(n) \rightarrow BTOP(n)$ is *not* a rational homotopy equivalence. By contrast it is well known that the inclusion of BO in $BTOP$ is a rational homotopy equivalence, where $BO = \bigcup_n BO(n)$ and $BTOP = \bigcup_n BTOP(n)$; therefore $H^*(BTOP; \mathbb{Q}) = \mathbb{Q}[p_1, p_2, p_3, \dots]$ where $p_i \in H^{4i}(BTOP; \mathbb{Q})$ is the i -th rational Pontryagin class due to Thom and Novikov, extending the better known $p_i \in H^{4i}(BO; \mathbb{Q})$. Over the years I have developed an obsession with the following conjectures:

Conjecture A. *For all n , the relation $e^2 = p_n$ holds in $H^{4n}(BSTOP(2n); \mathbb{Q})$, where e is the Euler class.*

Conjecture B. *For all n , the restrictions $H^*(BTOP; \mathbb{Q}) \rightarrow H^*(BTOP(n); \mathbb{Q})$ and $H^*(BTOP; \mathbb{Q}) \rightarrow H^*(BO(n); \mathbb{Q})$ have the same kernel.*

Despite appearances, conjecture B easily implies conjecture A. The relation $e^2 = p_n$ certainly holds in $H^{4n}(BSO(2n); \mathbb{Q})$, which motivates conjecture A but does not make it trivial. A proof of conjecture B which I outlined in my talk collapsed only two days later, thereby demonstrating the debilitating nature of obsessions. There is no point in summarizing the collapsed proof here. But I will try to explain what these conjectures have to do with singularity theory.

There is something like a theory of characteristic classes specially for homotopy theorists. It is called *orthogonal calculus* and fits into the general pattern of a functor calculus. I am partly responsible for it [9]. It starts from the idea that if you want to make good characteristic classes in the cohomology, perhaps generalized cohomology, of some space X , you need a filtration of X indexed by the finite dimensional linear subspaces V of \mathbb{R}^∞ . So you need subspaces $X_V \subset X$, depending naturally on V , so that $X_V \subset X_W$ if $V \subset W$, and also depending continuously on V in case V wants to vary continuously in \mathbb{R}^∞ . A good example is $X = BO$ with the subspaces $X_V = BO(V)$. Another example is $X = BTOP$ with the subspaces $X_V = BTOP(V)$.

After that first idea, the theory proceeds by looking for “rates of change”. For example it looks at the relative homotopy groups of the inclusions $X_V \rightarrow X_W$ where $V \subset W$ and $\dim(W/V) = 1$; these would be filed under “first rate of change” aspects, and the relative homotopy groups of inclusions $X_V \rightarrow X_W$ where $\dim(W/V) = 2$ would be filed under “second rate of change” aspects, roughly, and so on. This works well when $X = BO$ and $X_V = BO(V)$. After much tidying up, one finds that, in the case $X = BO$ etc., the first rate of change aspects (all thrown in one pot) roughly amount to the traditional Stiefel-Whitney classes, and the second rate of change aspects (all thrown in one pot) amount to the traditional Pontryagin classes. The tidying up is much harder in the case $X = BTOP$ and $X_V = BTOP(V)$. The first rate of change aspects are nevertheless well understood. For the second rate of change aspects, which are not well understood in this

example, there is an “ansatz” based on the following lemma, again a consequence of Morlet’s version of smoothing theory. Let $\text{reg}(D^n \times D^2, D^2)$ be the space of smooth regular (i.e., nonsingular) maps from $D^n \times D^2$ to D^2 which agree with the projection near the boundary of $D^n \times D^2$.

Lemma. *There is a homotopy fibration sequence*

$$\text{reg}(D^n \times D^2, D^2) \xrightarrow{\delta} \Omega^{n+2}(\text{O}(n+2)/\text{O}(n)) \longrightarrow \Omega^{n+2}(\text{TOP}(n+2)/\text{TOP}(n)) .$$

(Homotopy fibration sequences have long exact sequence of homotopy groups, and so on.) The first map, δ , is obtained by associating to a regular map f in $\text{reg}(D^n \times D^2, D^2)$ its derivative df , viewed as a map from $D^n \times D^2$ modulo boundary to the coset space $\text{GL}(n+2)/\text{GL}(n)$; note also that $D^n \times D^2$ modulo boundary can be identified with S^{n+2} and $\text{GL}(n+2)/\text{GL}(n)$ is homotopy equivalent to $\text{O}(n+2)/\text{O}(n)$. The second map is an inclusion.

Conjecture C. *For odd n , the map δ is rationally nullhomotopic, by a nullhomotopy satisfying certain sensible conditions listed below.*

For clarification, if n is odd, then the target of δ is rationally homotopy equivalent to an Eilenberg-Mac Lane space $K(\mathbb{Q}, n-1)$; if n is even, that is not the case.

Proposition. *Conjecture C implies conjectures A and B.*

The extra conditions on the nullhomotopy in conjecture C are as follows.

- (i) Source and target of δ come with an action of $\text{SO}(2)$, and δ respects the actions. The nullhomotopy should also respect the actions, in a derived sense. Strictly speaking, we only ask for an equivariant nullhomotopy of δh , where h is an $\text{SO}(2)$ -map from a free $\text{SO}(2)$ -space to $\text{reg}(D^n \times D^2, D^2)$ which is an ordinary homotopy equivalence.
- (ii) The space $\text{reg}(D^n \times D^2, D^2)$ has a subspace $\text{reg}_1(D^n \times D^2, D^2)$, consisting of those $f \in \text{reg}(D^n \times D^2, D^2)$ for which $q_2 f = q_{n+2}$, where $q_2 : D^2 \rightarrow \mathbb{R}$ and $q_{n+2} : D^n \times D^2 \rightarrow \mathbb{R}$ are the projections to the last coordinate. The map δ takes this subspace to the subspace $\Omega^{n+2}(\text{O}(n+1)/\text{O}(n))$ of $\Omega^{n+2}(\text{O}(n+2)/\text{O}(n))$, which is rationally contractible since n is odd. Therefore $\delta \mid \text{reg}_1(D^n \times D^2, D^2)$ already has a preferred rational nullhomotopy, and the nullhomotopy of δ that we are asking for should extend that one.

Obsessions notwithstanding, the proposition seems safe. The idea of the proof is that a high-quality rational nullhomotopy for δ implies, modulo the lemma and a lot of orthogonal calculus, that the inclusions $\text{BO}(n) \rightarrow \text{BTOP}(n)$ are in some rational homotopy sense split injective and that the induced maps in cohomology are consequently split surjective, and naturally so. The splitting in cohomology can be shown to respect Euler classes and Pontryagin classes.

As regards conjecture C, it is not hard to find a nullhomotopy for δ without paying attention to conditions (i) and (ii). For that it is not even necessary to go rational. On the other hand, without going rational, it is not possible to make a nullhomotopy for δ satisfying (i). If such a nullhomotopy existed, orthogonal calculus could

process it and produce classes p_i in the *integral* cohomology of $BTOP$ extending the $p_i \in H^*(BO; \mathbb{Z})$. But that is impossible. The order of $p_6 \in H^{24}(BO; \mathbb{Z})$ in the cokernel of $H^{24}(BTOP; \mathbb{Z}) \rightarrow H^{24}(BO; \mathbb{Z})$ is divisible by 691, for example.

Now it should be clear that conjecture C, being a conjecture about certain spaces of regular maps, can also be reformulated as a conjecture about certain spaces of singular maps. More generally, suppose that M and N are smooth manifolds and let $\text{reg}(M, N)$ be the space of regular smooth maps from M to N . (There may be boundaries and there may be boundary conditions.) A well-tryed method for investigating the homotopy theoretic properties of $\text{reg}(M, N)$ consists in introducing a suitable class \mathfrak{A} of “moderate singularities” and the space $\text{gen}(M, N)$ of smooth maps from M to N having all their singularities in \mathfrak{A} (and satisfying the same boundary conditions as the maps in $\text{reg}(M, N)$, where applicable). Ideally the class \mathfrak{A} should be small enough so that the geometry of these singular maps can be understood, but large enough so that the homotopy/homology properties of $\text{gen}(M, N)$ can be understood. In that case, calculating the homological/homotopical properties of $\text{reg}(M, N)$ becomes essentially equivalent to understanding the homological/homotopical properties of the pair $(\text{gen}(M, N), \text{reg}(M, N))$. The pair is more accessible than the space $\text{reg}(M, N)$ on its own because one has the geometry of those singular maps to play with.

A classical application of this method which goes back as far as [1] begins with $N = I$ and $M = L \times I$, where I means $[0, 1]$ and L is some smooth compact manifold. We add a boundary condition, letting $\text{reg}(L \times I, I)$ consist of the regular maps $L \times I \rightarrow I$ which agree with the projection on the boundary $\partial(L \times I)$. The allowed singularities which we need to specify in order to define $\text{gen}(L \times I, I)$ are the Morse singularities and the birth-death singularities (where two Morse singularities collide). A theorem due to K Igusa [3], later essentially reproved and vastly generalized by V Vassiliev [7], [8], implies that $\text{gen}(L \times I, I)$ is “nearly” homotopy equivalent to the space of sections (subject to obvious boundary conditions) of an appropriate subbundle of the bundle of 3-jets $J^3(L \times I, I) \rightarrow L \times I$. (In Igusa’s version, the comparison map is highly connected; in Vassiliev’s version, it induces an isomorphism in integral homology.) This makes $\text{gen}(L \times I, I)$ quite accessible, so that the general method outlined above cannot fail. Indeed Igusa used it to prove many wonderful things about $\text{reg}(L \times I, I)$.

I hope that the same “well-tryed” method can be used to shed light on the homotopical and homological properties of $\text{reg}(D^n \times D^2, D^2)$. It is not too difficult to decide what $\text{gen}(D^n \times D^2, D^2)$ should be, in other words, what the class \mathfrak{A} of moderate singularities should be in this case. Vassiliev’s theorem suggests a choice or a few preferred choices. Unfortunately, “playing with the geometry of those singular maps” is not so straightforward in this case. At any rate I have not yet played enough to bump into a proof of conjecture C. But I do not see any strong reason to give up, other than exhaustion.

To conclude, here is a kind of apology. All by itself, the plan to develop a theory of generic smooth functions from smooth manifolds into the plane \mathbb{R}^2 or the disk D^2 , analogous to Morse theory, is brave but overwhelming. It may be that coupling

it with a clear-cut task, such as proving conjecture C and therefore conjectures A and B , will do some good.

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Thom polynomials for maps of curves with isolated singularities

MAXIM KAZARIAN

(joint work with Sergei Lando)

By a family of maps of curves we mean a diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow p & \swarrow q \\ & & B \end{array}$$

where X , Y , and B are complex nonsingular varieties of dimensions $\dim X = \dim Y = \dim B + 1$. For each $b \in B$, the diagram determines the map of curves $f_b : X_b \rightarrow Y_b$. We say that f acquires singularity of type A_m at a point $x \in X$ if the fibers of p and q are smooth at the points x and $f(x)$, respectively, and the map f_b can be written in local coordinates in the form $z \mapsto z^{m+1}$. We say that the map f acquires multisingularity of type A_{m_1, \dots, m_r} at a point $y \in Y$ if there exists a tuple of pairwise different preimages x_1, \dots, x_r of y with singularity types A_{m_1}, \dots, A_{m_r} . Denote by $A_m(X)$ and $A_{m_1, \dots, m_r}(Y)$, respectively, the closures of the corresponding (multi)singularity loci. We discuss the problem of finding the (Poincaré dual) cohomology classes of these loci. We make the following assumptions about the map f :

- the fibers of q are smooth;
- the fibers of p are nodal complex curves, that is, the only singularities they can acquire are points of simple selfintersection (nodes);

- the map f is proper and the restriction of p to the critical set of f is proper;
- the family is generic in the following sense: its restriction to each fiber acquires isolated singularities only and their deformations provided by the family are versal (in particular, singular fibers of p , if any, form a subvariety of codimension 1 in B).

We do not assume that the fibers of p and q are compact. The properness condition means that we fix the asymptotical behavior of the map near “poles” which implies that the critical points of f_b “never move to infinity”. The genericity condition is crucial. Unfortunately, this condition does not hold in all families appearing in applications. However, in many cases it is satisfied.

The main tool of the study is the theory of universal residual polynomials in characteristic classes developed mainly by R. Thom for singularities and M. Kazarian for multisingularities. The universal polynomials for singularities are usually expressed in terms of the Chern classes of (the tangent bundles over) the manifolds under study. In particular, the number of basic classes grows as the complexity, whence the codimension, of the singularity grows. In the case of families of functions on curves, however, one can manage with finitely many basic classes, whatever is the codimension. In particular, if the functions in the family acquire only isolated singularities, then the following four basic cohomology classes on X are sufficient [1]: ψ , ν , ν_1 , and ν_2 , where

$$\begin{aligned}\psi &= f^*(c_1(Y)) - p^*(c_1(B)) = f^*(c_1(Y) - q^*(c_1(B))); \\ \nu &= c_1(X) - p^*(c_1(B)),\end{aligned}$$

and ν_1, ν_2 are the Chern roots of the normal bundle to the codimension 2 subvariety Δ in X formed by the singular points on the fibers X_b . The latter two classes are defined well only on Δ ; their precise meaning is clarified in [1].

Theorem 1 [1]. *There exists a generating function*

$$\mathcal{R}(t_1, t_2, t_3, \dots) = \sum R_{m_1, \dots, m_r} t_{m_1} t_{m_2} \dots t_{m_r}$$

with polynomial coefficients $R_{m_1, \dots, m_r} = R_{m_1, \dots, m_r}(\psi, \nu, \nu_1, \nu_2)$ such that for an arbitrary general family of holomorphic mappings of curves to curves over a base B admitting only isolated singularities, the classes $[A_m(X)] \in H^*(X)$ (respectively, $[A_{m_1, \dots, m_r}(Y)] \in H^*(Y)$) coincide with the polynomials R_m (respectively, with the coefficients of $t_1 t_2 \dots t_r$ in the exponent $\exp f_*(\mathcal{R})$ of the push-forward $f_*(\mathcal{R})$ of \mathcal{R}).

Due to universal relations between the basic classes, the series \mathcal{R} can be represented in the form

$$\mathcal{R}(\psi, \nu, \nu_1, \nu_2; t) = \mathcal{R}_A(\psi, \nu; t) + \mathcal{R}_I(\psi, \nu_1, \nu_2; t) - \mathcal{R}_0(\psi; t).$$

where $\mathcal{R}_A = \mathcal{R}|_{\nu_1=\nu_2=0}$, $\mathcal{R}_I = \mathcal{R}|_{\nu=0}$, and $\mathcal{R}_0 = \mathcal{R}_A|_{\nu=0} = \mathcal{R}_I|_{\nu_1=\nu_2=0}$. The series \mathcal{R}_A and \mathcal{R}_I are called the A -contribution and I -contribution to \mathcal{R} , respectively. The main result of the present talk consists in explicit computation of these

contributions. Define the series \mathcal{M}_A depending on the additional variable z :

$$\mathcal{M}_A(\psi, \nu; z; t) = 1 + (\psi - \nu)Q_1 + (\psi - \nu)(\psi - 2\nu)Q_2 + (\psi - \nu)(\psi - 2\nu)(\psi - 3\nu)Q_3 + \dots,$$

where the rational functions $Q_n = Q_n(\nu; z; t)$ are defined by the expansion

$$\begin{aligned} 1 + Q_1 h + Q_2 h^2 + \dots &= \exp\left(\frac{t_1}{\nu} h^2 + \frac{t_2}{\nu} h^3 + \frac{t_3}{\nu} h^4 + \dots\right) / (1 - zh) \\ &= 1 + zh + \left(\frac{t_1}{\nu} + z^2\right) h^2 + \left(\frac{t_2}{\nu} + \frac{t_1}{\nu} z + z^3\right) h^3 + \dots \end{aligned}$$

Denote also $\mathcal{N}_A(\psi, \nu; t) = \mathcal{M}_A(\psi + \nu, \nu; 0; t)$.

Theorem 2 [2]. *The A -contribution \mathcal{R}_A to the function \mathcal{R} is given by the equation*

$$\frac{\psi}{\nu} \mathcal{R}_A(\psi, \nu; t) = \log \mathcal{N}_A(\psi, \nu; t).$$

Theorem 3 [2]. *The I -contribution \mathcal{R}_I to the function \mathcal{R} is given by the equation*

$$\frac{\psi(\nu_1 + \nu_2)}{\nu_1 \nu_2} \mathcal{R}_I = \log(\mathcal{N}'_I + \mathcal{N}''_I),$$

where

$$\mathcal{N}'_I(\psi, \nu_1, \nu_2; t) = \mathcal{N}_A(\psi, \nu_1; t) \mathcal{N}_A(\psi, \nu_2; t)$$

and \mathcal{N}''_I is the result of replacement of each monomial z^n by $(n+1)t_n$ in the product

$$\psi z \mathcal{M}_A(\psi, \nu_1; z; t) \mathcal{M}_A(\psi, \nu_2; z; t).$$

The statements of Theorems 2 and 3 imply that the coefficients of the series \mathcal{R}_A and \mathcal{R}_I are polynomial rather than rational functions in the basic classes. We have no algebraic proof of these peculiar facts. The universal formulas of the theorems are obtained by studying the equivariant cohomology of finite-dimensional approximations of the universal unfoldings of singularities of types A_∞ and $I_{\infty, \infty}$. These mappings are not proper. This results in a fact that the corresponding universal expression is rational rather than polynomial. It becomes polynomial when specified to the unfoldings of finite singularities A_n and $I_{k, l}$.

Among the corollaries of our results we obtain new closed formulas for some families of Hurwitz numbers enumerating ramified coverings of the sphere.

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Invariants of maximally inflected real rational quintics

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The B. & M. Shapiro conjecture, recently proved by Mukhin, Tarasov, and Varchenko, implies that a maximally inflected rational projective curve (i.e., a curve with all its inflection points real) is itself real. The number of rational curves of degree $n - 1$ in \mathbb{P}^{m-1} with prescribed generic simple inflection points, computed with the Schubert Calculus, equals the degree $d(m, n - m)$ of the Grassmann variety $G(m, n)$ in its Plücker embedding. The Shapiro conjecture then implies that there are exactly $d(m, n - m)$ connected components of the set of real maximally inflected rational curves with simple inflection points, linearly ordered compatibly with their natural cyclic order.

For $m = 2$, the “curve” is a rational function f , and its “inflection points” are the critical points of f . The Shapiro conjecture in this case was proved by Eremenko and Gabrielov. The proof was based on combinatorial invariants of real rational functions distinguishing all $d(2, n - 2)$ connected components. Such invariants are unknown even for the planar curves ($m = 3$). Invariants of planar maximally inflected rational curves were studied by Kharlamov and Sottile, and a complete classification was obtained for the curves of degree 4. Here we suggest invariants of the planar rational maximally inflected planar quintics ($m = 3$, $n = 6$) distinguishing the $d(3, 3) = 42$ connected components.

The invariants are based on the geometry of the curve dual to a quintic. For a smooth maximally inflected quintic, its dual is a curve of degree 8, with 9 cusps and at most 12 self-intersection. Since a quintic cannot have a triple tangent, its dual curve cannot have triple self-intersection points. Accordingly, Reidemeister moves of type 3 cannot occur under on the dual curve under a deformation of a quintic. For a smooth maximally inflected quintic, Reidemeister moves of type 1 are also forbidden for its dual curve. One can show then that certain connected components of the complement to the dual curve cannot disappear under any deformation. Selecting a point in such a component, one can define a real rational function (of degree 8) by combining the dual curve to a quintic with the central projection from that point. The invariants of these real rational functions are sufficient to distinguish the 42 components of the space of maximally inflected quintics with a fixed linear order of its inflection points. If that linear order is not fixed, the invariants distinguish 6 orbits of smooth maximally inflected quintics with simple inflection points.

Variation of twistor structures and hypersurface singularities

CHRISTIAN SEVENHECK

(joint work with Claus Hertling)

We are interested in applications of the theory of (mixed) twistor structures to hypersurface singularities. The relation between these subjects stems from a construction which has its origin in [CV91] and which was formalized in [Her03] under the name TERP-structure. Roughly speaking, given a hypersurface singularity or a tame function on an affine manifold, a Fourier-Laplace transformation of the Brieskorn lattice produces a holomorphic bundle H on \mathbb{C} equipped with a flat connection ∇ having a pole of order at most 2 at zero, and a real subbundle H'_R of maximal rank on \mathbb{C}^* . This allows to construct a \mathbb{P}^1 -bundle: for a section s of H near zero, consider the flat shift of \bar{s} to infinity along the circular involution $z \mapsto \bar{z}^{-1}$. These sections define an extension to infinity which produces a twistor (i.e., a holomorphic \mathbb{P}^1 -bundle). The notion and a good part of the theory of twistor structures goes back to the work of Simpson ([Sim88], [Sim88], [Sim92], [Sim97]).

The same construction makes sense in the relative case (e.g., given an unfolding of a singularity), then a variation of twistor structures is obtained. The striking feature of these variations is that they are holomorphic in the \mathbb{P}^1 -direction, but depend only smoothly (or real analytically) on the parameters.

There is another ingredient in a TERP-structure, which plays the role of a polarization. It can be shown that the Fourier-Laplace transformation of the Brieskorn lattice comes equipped with a pairing between opposite fibres, i.e., a form $P : H_z \times H_{-z} \rightarrow \mathbb{C}$ for all $z \in \mathbb{C}^*$ with meromorphic behavior at zero. In the case of local singularities, it is essentially the form one gets from K. Saito's higher residue pairings. The twistor constructed above is naturally equipped with a hermitian form h derived from P .

By the classical construction of Steenbrink and Scherk ([SS85]), the Brieskorn lattice allows to define a Hodge filtration on the cohomology of the Milnor fibre of the singularity making up a polarized mixed Hodge structure. Up to a twist, this also works if one starts with the Fourier-Laplace-transformed object, at least in the local case. In the case of tame functions, it was shown in [Sab] that a modified construction also gives rise to a mixed Hodge structure. In this way TERP/twistor structures are considered as generalization of Hodge structures: Any TERP-structure gives rise to a filtration on the space of nearby cycles, and we can identify a particular class (those generated by "elementary sections") which are in fact equivalent to (i.e., can be reconstructed from) their induced filtration. The nice feature of this geometric point of view is that if we look at the twistor corresponding to a such a filtration, then it is very easy to tell whether this filtration gives rise to a Hodge structure: This is precisely the case if the twistor is a trivial (or more generally semi-stable) bundle on \mathbb{P}^1 , these twistors are called pure, and pure polarized if the form h is positive definite. For the semi-universal unfolding of a hypersurface singularity, the corresponding variation of twistor structures

is pure outside a real analytic hypersurface and this complement has connected components on which the signature of h is constant.

As in classical Hodge theory, we are interested in studying degenerations of twistor structures, which leads to the notion of a (polarized) mixed twistor: a bundle on \mathbb{P}^1 equipped with an increasing filtration of subbundles such that the quotients are semi-stable of appropriate weight.

Pursuing this analogy further, one might ask for a generalization of nilpotent orbits of Hodge structures which are considered in the work of Schmid ([Sch73]). It turns out that this leads to a particularly important class of variations of twistor/TERP structures, which are obtained by a rather simple procedure: Start with a single TERP-structure (without parameters) and rescale the coordinate on \mathbb{C} . This yields a variation on \mathbb{C}^* (the space of the rescaling parameter) and similarly one gets a variation of twistor structures. If these twistors are pure polarized for sufficiently small parameters, then this variation is called nilpotent orbit of TERP/resp. twistor structures.

There is a classical correspondence due to Cattani, Kaplan and Schmid [CKS86] between nilpotent orbits of Hodge structures and polarized (limit) mixed Hodge structures. The following two results, taken from [HS1] are the appropriate generalizations to TERP/twistor structures.

The first one concerns the case of TERP-structures where (H, ∇) has a regular singularity at zero. We denote by $\pi_r : \mathbb{C} \rightarrow \mathbb{C}; z \mapsto r \cdot z$ for $r \in \mathbb{C}^*$ the rescaling map.

Theorem 1. *A regular singular TERP structure (H, ∇, H'_R, P) induces a nilpotent orbit (i.e., $\pi_r^*(H, \nabla, H'_R, P)$ is pure polarized for all $|r| \ll 1$) if and only if the filtration F^\bullet induced by (H, ∇) on $H^\infty := \psi_z(H, \nabla)$ gives rise to a polarized mixed Hodge structure.*

One direction of this theorem is proved in [Her03]. The other direction, which can be found in [HS1], uses a fundamental result of Mochizuki ([Moc03]) which states that for a variation of polarized twistor structures on a complement of a normal crossing divisor with tame behavior, there exists a limit object which is a polarized mixed twistor.

In the general (i.e., irregular case), we can prove one part of this correspondence. The main ingredients are the decomposition theorem for irregular connections and a precise discussion of the associated Stokes structure. It turns out that under a compatibility condition between the Stokes and the real structure, each regular singular piece that appears in the decomposition induces a filtration as in the regular singular case. The result is then as follows.

Theorem 2. *Consider any (possibly irregular) TERP-structure (H, ∇, H'_R, P) . Suppose that the Stokes structure is compatible with the real structure defined by H'_R . Suppose further that the filtrations defined by the regular singular pieces give rise to a polarized mixed Hodge structures. Then (H, ∇, H'_R, P) induces a nilpotent orbit.*

The missing converse direction in this irregular case is conjectured to be true.

Let us point out two rather direct applications of these results. Both are concerned with the example alluded to above, that is, functions with isolated singularities.

Corollary 3. *Consider the semi-universal unfolding M of a holomorphic function with isolated critical points. Then for any $t \in M$, the restriction of the variation of TERP structures to the orbit of the Euler field passing through t is a nilpotent orbit of TERP-structures, in particular, if one follows this orbit sufficiently far enough, the variation of twistors is pure polarized.*

The second corollary deals with the global case. As already mentioned, a modified procedure of the Scherk-Steenbrink construction due to Sabbah produces a filtration $F_{S^{\bullet}ab}^{\bullet}$ which gives rise to a mixed Hodge structure, but which is not polarized. Using the above correspondence and a recent, fundamental result of Sabbah ([Sab05a]), we obtain:

Corollary 4. *Let $f : Y \rightarrow \mathbb{A}^1$ be a tame function on an affine manifold. Consider the Hodge filtration $F_{S^{\bullet}ab}^{\bullet}$ on H^{∞} as defined in [Sab]. There exists an automorphism $G \in \text{Aut}(H^{\infty})$ which induces the identity on the quotients of weight filtration such that $G^{-1}F_{S^{\bullet}ab}^{\bullet}$ gives rise to a **polarized** mixed Hodge structure.*

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Twistor structures: curvature of classifying spaces and Stokes structure in the irregular case

CLAUS HERTLING

(joint work with Christian Sevenheck)

Twistor structures are holomorphic vector bundles on \mathbb{P}^1 . Simpson [Sim97] considered families of twistor structures which are equipped with additional structure and which encode harmonic bundles. Special harmonic bundles arise from variations of polarized Hodge structures, but they do not keep all the information of the variation of Hodge structures. Sabbah [Sab05a] generalized them further to polarizable twistor \mathcal{D} -modules. He and Mochizuki [Moc03] used these structures to solve a conjecture of Kashiwara on semisimple \mathcal{D} -modules in the regular singular case.

Shortly after Simpson had started to work on harmonic bundles at the end of the 80'ies, Cecotti and Vafa [CV91][CV93] considered independently variations of twistor structures with richer additional structures. They were interested in supersymmetric quantum field theories and found these structures living on vector bundles on moduli spaces of their field theories. They are richer than harmonic bundles and are true generalizations of (variations of) polarized Hodge structures.

An important class of the field theories considered by Cecotti and Vafa are the Landau-Ginzburg models. There a central object is an unfolding of a tame function with isolated singularities on a quasiprojective manifold. The twistor structures are obtained from oscillating integrals over Lefschetz thimbles and from a certain involution which the physicists call topological-antitopological fusion and which generalizes the complex conjugation in a Hodge structure.

This is close to the Gauss-Manin connection and its Fourier-Laplace transform in the case of germs $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ of holomorphic functions with isolated singularities, which was studied since 1970: Brieskorn, Malgrange, Greuel, K. Saito, Scherk, Varchenko, Pham, M. Saito, Hertling, and many others studied the Brieskorn lattice, an extension to 0 of the cohomology bundle on a punctured disc by sections from differential forms. Pham [Ph85] considered its Fourier Laplace transform, that is, the oscillating integrals over Lefschetz thimbles. These give a holomorphic vector bundle H on \mathbb{C} with additional structure. But he did not consider the topological-antitopological fusion with which the bundle is extended to \mathbb{P}^1 .

In [Her03] the structure in [CV91][CV93] was formalized and called TERP structure (for Twistor Extension Real Pairing), and it was applied to the case of germs with isolated singularities.

In [HS06] these studies were continued. Especially, results on nilpotent orbits of Hodge structures of Schmid [Sch73, (6.16)] and Cattani, Kaplan and Schmid [CKS86, (3.13)] were generalized to TERP structures. This is also related to the renormalization group flow in [CV91][CV93]. It applies immediately to the case of germs f , because the nilpotent orbits are induced by the families $r \cdot f$, $r \in \mathbb{C}^*$.

In [HS] we will study classifying spaces of regular singular TERP structures: such as the Fourier Laplace transforms of the Brieskorn lattices of germs of functions with isolated singularities. The main result in [HS] generalizes the old result of Griffiths and Schmid that a classifying space of polarized Hodge structures has negative holomorphic curvature along the horizontal subbundle. Only, in the case of Brieskorn lattices, the dimensions of the horizontal subspaces are in general not equal for all Brieskorn lattices, so that they do not form a subbundle of the tangent bundle.

We want to apply the result to the period maps from μ -constant families of singularities to classifying spaces of Brieskorn lattices. The classifying spaces had already been introduced in [Her99]. The period map from a μ -constant stratum to a classifying space was studied in [Sai91] and in several papers of Hertling. In [Her02, 12.1] it was proved that the period map from the μ -constant stratum in a semiuniversal unfolding is an embedding; this is an infinitesimal Torelli result and improves the finite-to-one result in [Sai91].

A TERP structure is a holomorphic vector bundle $H \rightarrow \mathbb{C}$ with a flat connection ∇ on $H' := H|_{\mathbb{C}^*}$ with a pole of order ≤ 2 at 0, with a real flat subbundle $H'_{\mathbb{R}}$, and a certain flat pairing $P : H_z \times H_{-z} \rightarrow \mathbb{C}$, $z \in \mathbb{C}^*$, with some compatibility conditions. In the singularity case P comes from the intersection form for Lefschetz thimbles and is related by the Fourier Laplace transform to K. Saito's higher residue pairings.

The physicists' topological-antitopological fusion consists in extending $H \rightarrow \mathbb{C}$ to ∞ , that is, to a twistor $\widehat{H} \rightarrow \mathbb{P}^1$, by copies of the germs of sections at 0. These germs are transported to ∞ by a map $\tau : H_z \rightarrow H_{1/\bar{z}}$, which is composed of the flat shift along the real line $\{\tilde{z} \in \mathbb{C} \mid \arg(\tilde{z}) = \arg(z)\}$ and the complex conjugation in the fibers. The TERP structure is pure if $\widehat{H} \rightarrow \mathbb{P}^1$ is a trivial vector bundle. Then P and τ induce a certain hermitian nondegenerate metric h on $\Gamma(\mathbb{P}^1, \mathcal{O}(\widehat{H}))$. If it is positive definite, the pure TERP structure is called polarized. This generalizes a polarized Hodge structure. The following table compares the notions.

pure pol. TERP structure	polarized Hodge structure
$H' = H _{\mathbb{C}^*}$ and ∇	complex vector space
$H'_{\mathbb{R}} \subset H _{\mathbb{C}^*}$	real subspace
P	polarizing form
extension H of H' to 0	Hodge filtration F^\bullet
weight $w \in \mathbb{Z}$	weight $w \in \mathbb{Z}$
top.-antitop. fusion	complex conjugation, $F^\bullet \mapsto \overline{F^{w-\bullet}}$
germ $\mathcal{O}(\widehat{H})_\infty$ at ∞	$\overline{F^{w-\bullet}}$
$\widehat{H} \rightarrow \mathbb{P}^1$ is trivial	F^\bullet and $\overline{F^{w-\bullet}}$ are opposite filtrations
the metric h is pos. def.	a pos. def. hermitian metric
	from the polarizing form and a Weil twist

The above mentioned result of Cattani, Kaplan and Schmid on nilpotent orbits of Hodge structures implies in the case of a germ f of a singularity, that Steenbrink's filtration $F_{St}^\bullet(r \cdot f)$ of a polarized mixed Hodge structure of f is for $|r| \gg 1$

simultaneously part of a polarized **pure** Hodge structure. In [Her03] and [HS06] this is greatly generalized to the theorem that for any fixed deformation F_t of f the TERP structure $\text{TERP}(r \cdot F_t)$ is pure and polarized for $|r| \gg 1$.

If F_t is not a μ -constant deformation of f , this involves a discussion of the Stokes structure of the TERP structure. This Stokes structure comes from the Lefschetz thimbles and thus from distinguished bases of vanishing cycles. The topological studies of A'Campo, Brieskorn, Ebeling, Gabrielov, Gusein-Zade, Looijenga and others on distinguished bases may now become relevant for the TERP structures.

In [HS06] also the notion of mixed TERP structures is defined. It generalizes polarized mixed Hodge structures. In the irregular case its definition involves a compatibility condition between Stokes structure and real structure. In the singularity case the TERP structures are mixed.

Finally, a mathematical proof has been given recently by Sabbah [Sab05b] for a result of Cecotti and Vafa for tame functions f (but not germs): $\text{TERP}(f)$ is pure and polarized. This result has similarity with the fundamental fact that the cohomology of Kähler manifolds carries Hodge structures. It might become just as fundamental in the theory of singularities and their Hodge structures.

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Motivic characteristic classes for singular spaces

JÖRG SCHÜRMAN

This is about joint work with J.-P. Brasselet and S.Yokura. In the papers [6, 7] we study some new theories of characteristic homology classes of singular complex algebraic (or compactifiable analytic) spaces.

We introduce a *motivic Chern class transformation*

$$mC_* : K_0(\text{var}/X) \rightarrow G_0(X) \otimes \mathbb{Z}[y],$$

which generalizes the total λ -class $\lambda_y(T^*X)$ of the cotangent bundle to singular spaces. Here $K_0(\text{var}/X)$ is the relative Grothendieck group of complex algebraic varieties over X as introduced and studied by Looijenga and Bittner [12, 4] in relation to motivic integration, and $G_0(X)$ is the Grothendieck group of coherent sheaves of O_X -modules. A first construction of mC_* is based on resolution of singularities and a suitable “blow-up” relation. In the (complex) algebraic context this “blow-up” relation follows from work of Du Bois [9], based on Deligne’s mixed Hodge theory. Other approaches by work of Guillén and Navarro Aznar [11] (using “only” resolution of singularities) or Looijenga and Bittner (using the “weak factorization theorem” [1, 17]) also apply to the compactifiable complex analytic context. A second more functorial construction of mC_* is based on some results from the theory of algebraic mixed Hodge modules due to M.Saito [14].

We define a natural *Hirzebruch class transformation*

$$T_{y*} : K_0(\text{var}/X) \rightarrow H_*(X) \otimes \mathbb{Q}[y]$$

commuting with proper pushdown, which generalizes the corresponding *Hirzebruch characteristic*. T_{y*} is a homology class version of the motivic measure corresponding to a suitable specialization of the well known Hodge polynomial. This transformation unifies the *Chern class transformation* of MacPherson and Schwartz [13] (for $y = -1$), the *Todd class transformation* in the singular Riemann-Roch theorem of Baum-Fulton-MacPherson [3] (for $y = 0$) and the *L-class transformation* of Cappell-Shaneson [8] (for $y = 1$).

The “stringy versions” of our characteristic classes are finally closely related to the *elliptic classes* of Borisov-Libgober-Totaro [5, 16] and the *stringy Chern classes* of Aluffi and De Fernex-Lupercio-Nevins-Urbe [2, 10]. Moreover, all our results can be extended to varieties over a base field k of characteristic 0.

For more details and an introduction to the theory of characteristic classes of singular spaces we recommend our joint survey paper [15] with S.Yokura.

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New developments on topological equisingularity

JAVIER FERNÁNDEZ DE BOBADILLA

1. TOPOLOGICAL EQUISINGULARITY OF SURFACES IN \mathbf{C}^3

We study topological equisingularity, as defined by Zariski in *Open questions in the theory of singularities*. *Bull. Amer. Math. Soc.* **77** (1971), 481–489. We focus in the problem of providing invariants that capture when a family of hypersurfaces is topologically equisingular a la Zariski. If one considers Whitney equisingularity such numerical invariants were provided by Teissier, but it is well known that Whitney equisingularity is strictly stronger than topological one. Topological equisingularity in higher dimension does not coincide with other notions of equisingularity and controlling it has proved to be very difficult.

In the case of hypersurface singularities with isolated singularities it was proved by Lê-Ramanujam, Timourian and King (except in the surface case) that the topological equisingularity of a family is equivalent to the constancy of the Milnor number. The surface case is still open. It is important here that in the case of

isolated singularities the topology of the Milnor fibration determines the embedded topology of the hypersurface.

In the case of non-isolated singularities Massey (*The Lê varieties I, II*, *Inventiones Mathematicae* **99** (1990) y **104** (1991)) introduced the Le numbers, a set of numerical invariants whose constancy ensures the constancy of the topology of the Milnor fibration of a family of hypersurface singularities with critical set of codimension at least 3. The question whether the constancy of the Le numbers ensures topological equisingularity remained open.

We reported on the following recent results:

- The constancy of the Le numbers implies the constancy the homotopy type of the link of the singularity.
- We have show by counterexamples (that are inspired in the study of non-isolated singularity via deformations) that the constancy of the Le numbers do not imply topological equisingularity. Thus the topology of the Milnor fibration does not determine in general the abstract topology of the central fibre.
- Making variations of the above counterexamples we construct a family of projective hypersurfaces with constant homotopy type but not constant topological type. Such family arises as the family of projectivised tangent cones at the origin of a equisingular (even Whitney equisingular) family of isolated hypersurface singularities. This answers negatively Zariski's question B of the paper cited above.
- The counterexamples above are functions with 3-dimensional critical set. We prove that the constancy of the Le numbers implies topological equisingularity if the critical set of the members of the family are 1-dimensional. However such an implication is not a characterisation of topological equisingularity, since the Le numbers are not topological invariants. We introduce the concept of equisingularity at the critical set and prove that it implies topological equisingularity, and conjecture that it characterises it.
- Using equisingularity at the critical set we construct examples of topologically equisingular families whose critical set experiences drastic changes from the analytic point of view. We explain how to use this to attack Zariski's multiplicity conjecture (Question A of the cited paper).
- Together with T. Gaffney we have shown that the Lê numbers are not topological invariants. However the Lê numbers of the square of a function f defining an isolated singularities provide the same information than the μ^* sequence of Teissier, and hence characterise Whitney equisingularity in a family. We have used the Lê numbers of a square also to obtain a formula of the Euler obstruction of a hypersurface and to reformulate Zariski's multiplicity conjecture (for the case of isolated singularities).
- Together with M. Pe, we apply the techniques developed in the previous paper to prove the following results on surface singularities. We say that a family $f_t : (\mathbf{C}^3, O) \rightarrow \mathbf{C}$ of reduced holomorphic germs is equisingular

at the normalisation if the pairs given by the topological normalisation of the zero set of f_t and the inverse image by the normalisation map of the singular set of $f_t^{-1}(0)$ form a topologically equisingular family. We prove that, if Lê's conjecture holds, equisingularity at the normalisation is equivalent to topological right-equisingularity for the family f_t . Furthermore, if the generic transversal type of the singularities of f_t are curve singularities with smooth branches then the statement does not depend on Lê's conjecture. The case in which the above result is most meaningful is the case of parametrised surfaces. Let $h_t : (\mathbf{C}^2, O) \rightarrow (\mathbf{C}^3, O)$ be a family of surface parametrisations. Assuming Lê's conjecture we prove that the family is topologically A -equisingular if and only if the Milnor number of the curve given by the inverse image by h_t of the singular set of $h_t(\mathbf{C}^2)$ is constant. The result is independent of Lê's conjecture in the case of finitely A -determined map germs. Thus we obtain, as a particular case a recent result of Calleja-Bedregal, Ruas and Houston. We also prove that if the generic Lê numbers at the origin of $h_t(\mathbf{C}^2)$ are independent of t then the family is topologically A -equisingular.

- We give a general definition of the topological stems of hypersurfaces singularities observed by Arnol'd (and later by C.T.C. Wall, Mond and Pelikaan) in the classification of singularities of low modality/codimension.

Linear Free Divisors

DAVID MOND

(joint work with R.-O.Buchweitz, M.Granger, A.Nieto, M.Schulze)

A hypersurface $X_0 \subset \mathbf{C}^n$ is a *free divisor* if the $\mathcal{O}_{\mathbf{C}^n}$ -module $\text{Der}(-\log X_0)$ of germs of vector fields tangent to X_0 at its smooth points is locally free; it is *linear* if $\text{Der}(-\log X_0)_0$ has a basis consisting of vector fields of weight zero, i.e. with linear coefficients. Since the determinant of the matrix of coefficients of a basis of $\text{Der}(-\log X_0)$ is a reduced equation for X_0 (Saito's criterion, [3]), a linear free divisor in \mathbf{C}^n has degree n . Linear free divisors really belong to representation theory. Every weight-zero vector field comes from the infinitesimal action of $\text{Gl}_n(\mathbf{C})$ on \mathbf{C}^n , and it follows that the n -dimensional complex Lie algebra \mathcal{L}_{X_0} , generated by the weight-zero basis of $\text{Der}(-\log X_0)$, lifts to an n -dimensional Lie sub-algebra of $\mathfrak{gl}_n(\mathbf{C})$, which is equal to the Lie algebra of the group G_{X_0} of linear automorphisms of \mathbf{C}^n preserving X_0 ; \mathcal{L}_{X_0} is thus the image of the infinitesimal action of G_{X_0} . At each point of \mathbf{C}^n the space of values of the members of \mathcal{L}_{X_0} coincides with that of the members of $\text{Der}(-\log X_0)$, and it follows that for $p \in \mathbf{C}^n \setminus X_0$ this space coincides with $T_p\mathbf{C}^n$ itself. Thus, the complement of X_0 is a single orbit of G_{X_0} . Reversing the recipe and beginning with an n -dimensional subgroup of $\text{Gl}_n(\mathbf{C})$, we may hope that it has an open orbit whose complement is a linear free divisor. Whether it does is decided by the determinant Δ of the matrix of coefficients of n vector fields generating the infinitesimal action. If Δ is not identically zero then

there is indeed an open orbit whose complement is the divisor $X_0 := \{\Delta = 0\}$. If Δ is moreover a reduced equation for X_0 then X_0 is a linear free divisor, for by Saito's criterion the n vector fields then form a basis for $\text{Der}(-\log X_0)$.

Theorem 5. *If X_0 is a linear free divisor and the group G_{X_0} is reductive, then the global logarithmic comparison theorem (GLTC) holds for X_0 — that is, the de Rham map defines an isomorphism*

$$H^*(\Gamma(\mathbb{C}^n, \Omega^\bullet(\log X_0))) \rightarrow H^*(U; \mathbb{C})$$

where $U = \mathbb{C}^n \setminus X_0$.

This is proved by observing that the weight zero subcomplex of $\Gamma(\mathbb{C}^n, \Omega^\bullet(\log X_0))$ coincides with the complex of Lie algebra cochains with coefficients in \mathbb{C} for the Lie algebra \mathfrak{g}_{X_0} . Both local and global versions of LCT hold for locally weighted homogeneous free divisors; many of the examples described below are not locally weighted homogeneous.

1. EXAMPLES

1.1. Free divisors in spaces of quiver representations. The most familiar linear free divisor is the normal crossing divisor $\{(x_1, \dots, x_n) : x_1 \cdots x_n = 0\}$, with basis of logarithmic vector fields $x_1 \partial / \partial x_1, \dots, x_n \partial / \partial x_n$. This is a special case of a construction suggested by Buchweitz and described in detail in [1], in which linear free divisors appear as discriminants in representation spaces of Dynkin quivers. A quiver Q is just an oriented graph, and is Dynkin if the underlying unoriented graph is a Dynkin diagram of type ADE. A representation of Q is the assignment of a space V_p to each node p and a linear map to each oriented edge. A dimension vector \mathbf{d} specifies the dimensions of the spaces V_p ; the set $\text{Rep}(Q, \mathbf{d})$ of all representations with a given dimension vector is naturally a vector space, on which the quiver group $\text{Gl}_{Q, \mathbf{d}} := \prod_n \text{Gl}(V_n)$ acts in the natural way. In fact $\text{Gl}_{Q, \mathbf{d}}$ acts via its quotient by the 1-dimensional subgroup Z_0 consisting of the same scalar at each vertex. In order that $\text{Rep}(Q, \mathbf{d})$ and $\text{Gl}_{Q, \mathbf{d}}$ should play the role of \mathbb{C}^n and G_{X_0} described in the previous paragraph, we therefore require

$$(1) \quad \dim_{\mathbb{C}} \text{Gl}_{Q, \mathbf{d}} - \dim_{\mathbb{C}} \text{Rep}(Q, \mathbf{d}) \left(= \sum_{n \in N} d_n^2 - \sum_{\alpha \in A} d_{t\alpha} d_{h\alpha} \right) = 1.$$

This equality is not yet sufficient; it is also necessary that $\text{Gl}_{Q, \mathbf{d}}/Z_0$ should act effectively. This occurs if the general representation in $\text{Rep}(Q, \mathbf{d})$ is indecomposable — does not split into a direct sum of subrepresentations. If both conditions hold, \mathbf{d} is called a *real Schur root* of Q .

Theorem 6. ([1]) *If the dimension vector \mathbf{d} is a real Schur root of a Dynkin quiver Q , then the singular set (or discriminant) in $\text{Rep}(Q, \mathbf{d})$ (the complement of the open orbit) is a linear free divisor with group $\text{Gl}_{Q, \mathbf{d}}/Z_0$.*

For the quivers of type E_6, E_7 and E_8 the isomorphism class of the linear free divisor depends on the orientation of the arrows. The significance of the condition that Q be a Dynkin quiver is that by a theorem of Gabriel, $\text{Rep}(Q, \mathbf{d})$ then contains

finitely many orbits. In fact all that is needed for the proof of Theorem 6 is that there be only finitely many orbits of codimension 0 and 1. Michel Brion has informed us of a much simpler proof than the one given in [1].

In all of these examples, the group G_{X_0} ($= \text{Gl}_{Q,\mathbf{d}}/Z_0$) is reductive.

If we take *any* quiver whose underlying graph is a tree, the dimension vector with 1 at each node is a real Schur root, and the complement of the open $\text{Gl}_{Q,\mathbf{d}}$ -orbit is a normal crossing divisor.

1.2. Some non-reductive examples. Take a star quiver Q with $n + 1$ sources. The dimension vector \mathbf{d} assigning 1 at each of the sources and n at the central sink is a real Schur root, and the discriminant in $\text{Rep}(Q, \mathbf{d})$ is a linear free divisor. We can identify $\text{Rep}(Q, \mathbf{d})$ with the space of matrices of size $n \times (n + 1)$ - each column representing a map from one of the sources to the sink. The group $\text{Gl}_{Q,\mathbf{d}}$ consists of $\text{Gl}_n(\mathbb{C})$ acting by left multiplication, and $(\text{Gl}_1(\mathbb{C}))^{n+1}$ = diagonal matrices of size $(n + 1) \times (n + 1)$ acting by right-multiplication. Here we are still in the reductive situation. Now if we reduce the dimension at the sink below n , the difference (1) drops below 1, and \mathbf{d} is no longer a real Schur root. We can remedy the situation by increasing the size of the group acting on the right, departing from a purely quiver-theoretic framework. In [2] we do this by grafting a quotient of the path algebra of certain auxiliary quivers onto $\text{Gl}_{Q,\mathbf{d}}$. By using auxiliary quivers such as these it is possible to generate an unlimited number of different linear free divisors. The group in these examples is not in general reductive.

2. SINGULARITIES OF FUNCTIONS ON LINEAR FREE DIVISORS

Let X_0 be a linear free divisor with group $G = G_{X_0}$ and equation h . Then h is a semi-invariant of the action of G on $\mathbb{C}[\text{Rep}(Q, \mathbf{d})]$, corresponding to a character χ : for $g \in G$, $h \circ g = \chi(g)h$. Let $SG = \text{Ker}(\chi)$. It is easy to see that $X_t := \{h = t\}$ is a single orbit of SG .

If $f : \mathbb{C}^n \rightarrow \mathbb{C}$ is a function, the global T^1 (for right-equivalence) of $f|X_t$ is equal to $\mathbb{C}[X_t]/J(f|X_t)$, and thus to $\mathbb{C}[x_1, \dots, x_n]/df(\text{Der}(-\log h)) + (h-t)$, where $\text{Der}(-\log h)$ is the (global) module of vector fields annihilating h . The family of these quotients has as organising centre the quotient, $\mathbb{C}[x_1, \dots, x_n]/df(\text{Der}(-\log h) + (h))$, the T^1 for \mathcal{K}_h -equivalence of $f|X_0$, which we will denote by $T^1_{\mathcal{K}_h} f$.

Proposition 7. *Let X_0 be a free divisor. If $\dim T^1_{\mathcal{K}_h} f = \mu < \infty$ then for each fixed value of $t \neq 0$, $\mu = \sum_{x \in X_t} \mu(f|X_t; x)$.*

If f is a linear form defining a hyperplane H then the critical points of $f|X_t$ are the points where $T_x X_t = H$, so μ , if finite, is the degree of the Gauss map $X_t \rightarrow (\mathbb{C}\mathbb{P}^{n-1})^\vee$.

Proposition 8. *Let f be a linear form. If $\mu < \infty$ then all of the critical points of $f|X_t$ are non-degenerate, and the Gauss map is an unramified cover of its image, which is equal to the projective quotient of the open orbit of the dual representation of G .*

There are not always linear forms for which $\mu < \infty$. Their existence depends upon the nature of the dual (contragredient) representation of G on $(\mathbb{C}^n)^\vee$: they make up its open orbit.

Proposition 9. ([4], page 71) *If the group G is reductive, the dual representation of G has an open orbit whose complement has equation \bar{h} .*

In all of the examples coming from real Schur roots of quivers, $\bar{h} = h$. Where G is non-reductive, there may exist no good forms, or there may be a dual linear free divisor quite different to X_0 .

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Local geometry of singular spaces in the non archimedean setting

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This is a report on some work in progress with Raf Cluckers and Georges Comte on the local geometry of singular spaces in the non archimedean setting. In the talk we presented a construction of local densities in the p -adic and motivic settings and discussed their relations with normal cones and projections. In this abstract we shall only consider the p -adic case.

Let X be a possibly singular subvariety of dimension d of \mathbb{C}^n . Let x be a point of X . Recall that the algebraic multiplicity of X at x may be expressed in terms of the tangent cone to X at x (taking multiplicities in account). It is also equal to the local density of X at x , that is, the limit for $\varepsilon \mapsto 0$ of the volume of the intersection of X with a ball of radius ε around x . It can also be expressed as the local degree at x of a generic linear projection to a d -dimensional plane.

If one replaces \mathbb{C} by \mathbb{R} , and X is a semialgebraic (or subanalytic) subset of dimension d of \mathbb{R}^n , Kurdyka and Raby [5] defined local densities at a point x and showed how it can be computed on the tangent cone to X at x . Comte proved a local Crofton formula in [1] that expresses the local density at x as an average over the grassmannian of d -dimensional linear spaces of local densities of the local projections (with multiplicities) of X .

The category of definable subsets of \mathbf{Q}_p^n is the p -adic analogue of the category of semialgebraic sets of \mathbb{R}^n and shares many similar properties, cf. [2][3][4]. The naive analogue of the definition of local densities does not work, but it is possible to circumvent that difficulty by using a suitable regularization device. For any finite index subgroup Λ of the multiplicative group \mathbf{Q}_p^\times there is a natural notion of Λ -cone and one can define a tangent Λ -cone for a definable subset of \mathbf{Q}_p^n at a point. It is also possible to define suitable multiplicities for tangent Λ -cone.

Our main results are:

- 1) The following analogue of the result of Kurdyka and Raby: if Λ is small enough, the local density can be computed on the Λ -cone with multiplicities.
- 2) A p -adic analogue of Comte's local Crofton formula.

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Limits of gradient and conormal geometry

ADAM PARUSIŃSKI

Let $f : (\mathbb{K}^{n+1}, 0) \rightarrow (\mathbb{K}, 0)$, where \mathbb{K} denotes either \mathbb{R} or \mathbb{C} , be an analytic function germ. Consider the set of limits of secant lines and gradient directions:

$$\mathcal{E}_f := \{(l, \xi) \in \mathbb{P}^n \times \mathbb{P}^n; \exists x_i \rightarrow 0, df(x_i) \neq 0, \text{ s.t. } [x_i] \rightarrow l \text{ and } [\nabla f(x_i)] \rightarrow \xi\},$$

where ∇f denotes the gradient of f and $\mathbb{P}^n = \mathbb{P}^n(\mathbb{K})$.

In the complex analytic case the geometric structure of \mathcal{E}_f is described in [5]. We shall recall this description below. In this case it is more convenient to replace the limits of gradient directions by the limits of tangent hyperplanes

$$\mathcal{E}_f := \{(l, H) \in \mathbb{P}^n \times \check{\mathbb{P}}^n; \exists x_i \rightarrow 0, df(x_i) \neq 0, \text{ s.t. } [x_i] \rightarrow l \text{ and } T_{x_i} f \rightarrow H\},$$

where $T_x f$ denotes the tangent space at x to the level f through x . Note that in this case \mathcal{E}_f equals the (reduced) exceptional divisor of the blowing up of the product of the maximal ideal m_0 of 0 times the jacobian ideal $J_f = (\partial f / \partial z_i)_{i=1, \dots, n+1}$. Therefore \mathcal{E}_f has pure dimension n .

Complex absolute case. Let $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ be the germ of a complex analytic set. The geometric structure of the set of limits of secant lines and tangent hyperplanes to the regular part X_{reg} of X :

$\mathcal{E}_X := \{(l, H) \in \mathbb{P}^n \times \check{\mathbb{P}}^n; \exists x_i \rightarrow 0, x_i \in X_{reg}, \exists H_i \supset T_{x_i}X, \text{ s.t. } [x_i] \rightarrow l \text{ and } H_i \rightarrow H\}$ is described in [4], Corollaire 2.1.3, and in [1], as a finite union of dual correspondences

$$(1) \quad \mathcal{E}_X = \bigcup_{\alpha} D_{\alpha}.$$

Recall that a subset of D of $\mathbb{P}^n \times \check{\mathbb{P}}^n$ is called a *dual correspondence* if it is of the form

$$D = D_V = \text{Closure}\{(x, H) \in V_{reg} \times \check{\mathbb{P}}^n; H \supset T_x V\},$$

where V_{reg} is the set of regular points of an irreducible subvariety V of \mathbb{P}^n . Then D_V establishes the classical duality between V and its dual $\check{V} \subset \check{\mathbb{P}}^n$ and $D_V \subset Inc$ where by Inc we denote the incidence subvariety of $\mathbb{P}^n \times \check{\mathbb{P}}^n$

$$Inc = \{(l, H) \in \mathbb{P}^n \times \check{\mathbb{P}}^n; l \subset H\}.$$

The proof of (1) is based on the Lagrangian specialization applied to the deformation $\mathcal{X} \rightarrow \mathbb{C}$ of X to its normal cone at the origin.

Complex relative case. Assume that $\mathbb{C}^{n+1} \subset \mathbb{C}^{n+2}$. Let $l_0 \in \mathbb{P}^{n+1}$ and $H_0 \in \check{\mathbb{P}}^{n+1}$ be given by $l_0 = (0 : \dots : 0 : 1)$, $H_0 = (0 : \dots : 0 : 1)$ respectively. Then the standard projections $\pi_{l_0} : \mathbb{P}^{n+1} \setminus l_0 \rightarrow \mathbb{P}^n$ and $\pi_{H_0} : \check{\mathbb{P}}^{n+1} \setminus H_0 \rightarrow \check{\mathbb{P}}^n$ are given in the homogeneous coordinates by $(x_1 : \dots : x_{n+1} : y) \rightarrow (x_1 : \dots : x_{n+1})$ and $(\eta_1 : \dots : \eta_{n+1} : \xi) \rightarrow (\eta_1 : \dots : \eta_{n+1})$ respectively. Thus the cartesian product $\pi_0 = \pi_{l_0} \times \pi_{H_0}$ defines a rational map

$$\pi_0 : \mathbb{P}^{n+1} \times \check{\mathbb{P}}^{n+1} \dashrightarrow \mathbb{P}^n \times \check{\mathbb{P}}^n$$

that is defined in the complement of $\Sigma_0 = (\{l_0\} \times \check{\mathbb{P}}^{n+1}) \cup (\mathbb{P}^{n+1} \times \{H_0\})$.

Theorem 12. (c.f. [5])

For each irreducible component \mathcal{E}_{γ} of \mathcal{E}_f there is a dual correspondence $\tilde{\mathcal{E}}_{\gamma} \subset \mathbb{P}^{n+1} \times \check{\mathbb{P}}^{n+1}$ such that $\tilde{\mathcal{E}}_{\gamma} \not\subset \Sigma_0$ and $\mathcal{E}_{\gamma} = \text{Closure}\{\pi_0(\tilde{\mathcal{E}}_{\gamma} \setminus \Sigma_0)\}$.

The proof of this proposition is by reduction to the absolute case. Here are the main points. Following [1] consider the following diagram

$$\begin{array}{ccccccc} \overline{E_Y C_f} & \xrightarrow{\bar{\pi}} & E_0 C_f & \xrightarrow{\tilde{e}_Y} & C_f & \xrightarrow{\gamma} & \check{\mathbb{P}}^n \\ & & \tilde{\tau}_f \downarrow & & \tau_f \downarrow & & \\ & & E_Y U & \xrightarrow{e_0} & U & & \\ & & \delta \downarrow & & & & \\ & & \mathbb{P}^n & & & & \end{array}$$

where $\tau_f : C_f \rightarrow \mathbb{C}$ is the relative conormal space (equal in this case to the blowing up of J_f), e_0 is the blowing-up of 0, \tilde{e}_0 is the blowing-up of $\tau_f^{-1}(0)$, γ and δ are

the induced projections, \bar{n} is the normalization map. Let $\zeta_f: E_Y C_f \rightarrow U$ be given by

$$\zeta_f = \tau_f \circ \tilde{e}_Y = \tilde{\tau}_f \circ e_Y.$$

Then $\mathcal{E}_f = \zeta_f^{-1}(0)$.

Fix $\bar{\mathcal{E}}_\gamma$ an irreducible component of $\bar{n}^{-1}(\mathcal{E}_\gamma)$ and define

$$\begin{aligned} a_\gamma &= \text{mult}_{\bar{\mathcal{E}}_\gamma}(z_i)_{i=1, \dots, n+1} \\ b_\gamma &= \text{mult}_{\bar{\mathcal{E}}_\gamma}(\partial f / \partial z_i)_{i=1, \dots, n+1} \\ c_\gamma &= \text{mult}_{\bar{\mathcal{E}}_\gamma} f. \end{aligned}$$

(These multiplicities are well-defined since $\bar{\mathcal{E}}_\gamma$ is a divisor in a normal variety). By the transversality of relative polar varieties, cf. [2] 8.7. Lemme de Transversalité, $c_\gamma = a_\gamma + b_\gamma$. It is also easy to see that $c_\gamma = a_\gamma + b_\gamma$ is equivalent to $\mathcal{E}_\beta \not\subset \text{Inc}$. A more precise argument given in [5] shows that

$$\mathcal{E}_f \cap \text{Inc} = \mathcal{E}_f \cap Z_f,$$

where $Z_f = \overline{\zeta_f^{-1}(f^{-1}(0) \setminus \{0\})}$. In particular, $\dim \mathcal{E}_f \cap Z_f = n - 1$.

Define an analytic subset $X \subset \mathbb{C}^{n+1} \times \mathbb{C}$ by the equation $(f(z))^{a_\gamma} = y^{c_\gamma}$ and consider $\mathcal{E}_X \subset \mathbb{P}^{n+1} \times \check{\mathbb{P}}^{n+1}$. Then an elementary computation. cf. [5], shows that π_0 sends the family of those components of \mathcal{E}_X which are not entirely contained in Σ_0 onto the family of those components $\bar{n}(\bar{\mathcal{E}}_\beta)$ of \mathcal{E}_f that satisfy $a_\beta/c_\beta = a_\gamma/c_\gamma$.

Real case. A similar, though less exact, description of \mathcal{E}_f can be obtained in the real analytic or even the o-minimal case. The techniques of conormal geometry such as Lagrange specialization were extended to the real analytic set-up by Kashiwara and Fu. The reader may consult [7], [3] and the bibliography quoted therein. The o-minimal case is treated in [6].

In what follows the word *definable* will mean definable in an o-minimal extension of \mathbb{R} , in particular subanalytic if we work in the real analytic category.

Let $X \subset \mathbb{R}^n$ and $f: X \rightarrow \mathbb{R}$ be definable, $0 \in X$. Consider

$$\begin{aligned} \mathcal{E}_X &:= \{(l, \xi) \in \mathbb{P}^n \times \mathbb{P}^n; \exists X_{reg} \ni x_i \rightarrow 0, \exists \mathbb{P}^n \ni \xi_i \rightarrow \xi \text{ s.t. } [x_i] \rightarrow l \text{ and } \xi_i \perp T_{x_i} X\} \\ \mathcal{E}_f &:= \{(l, \xi) \in \mathbb{P}^n \times \mathbb{P}^n; \exists f_{reg} \ni x_i \rightarrow 0, \exists \mathbb{P}^n \ni \xi_i \rightarrow \xi \text{ s.t. } [x_i] \rightarrow l \text{ and } \xi_i \perp T_{x_i} f\}. \end{aligned}$$

Here by X_{reg} and f_{reg} we mean the set of C^1 -regular points of X and f respectively, $T_x f$ denotes the tangent space to the level of f through x . We have the following results.

Theorem 13. *Let $X \subset \mathbb{R}^n$ be definable, $0 \in X$. Then there exists a finite family of definable \mathbb{R}^+ -cones $V_i \subset \mathbb{R}^n$ such that*

$$\mathcal{E}_X \subset \bigcup \mathcal{E}_{V_i}.$$

In particular $\mathcal{E}_X \subset \text{Inc} = \{(l, \xi) \in \mathbb{P}^n \times \mathbb{P}^n; l \perp \xi\}$

Theorem 14. *Let $f : (\mathbb{R}^{n+1}, 0) \rightarrow (\mathbb{R}, 0)$ be real analytic. Then there is a finite family of subanalytic \mathbb{R}^+ -cones V_i and \mathbb{R}^+ homogeneous subanalytic functions $\varphi_i : V_i \rightarrow \mathbb{R}$ such that*

$$\mathcal{E}_f \subset \bigcup \mathcal{E}_{\varphi_i}.$$

Moreover, $Inc \cap \mathcal{E}_f$ is a finite union of sets of the form \mathcal{E}_V .

Theorem 14 holds also for continuous functions definable in a polynomially bounded o-minimal extension of \mathbb{R} . For an arbitrary o-minimal extension we only get the following inclusion

$$\mathcal{E}_f \setminus Inc \subset \bigcup \mathcal{E}_{\varphi_i}.$$

Example 1. Let $\alpha(r) = (-\ln r)^{-1}$ for $r > 0$ and $\alpha(0) = 0$. The function $\alpha : [0, 1) \rightarrow [0, \infty)$ is continuous and is definable in any o-minimal extension of \mathbb{R} that contains the real exponential function (and hence by the dihotomy theorem in any non-polynomially bounded o-minimal structure). Then, $\alpha(r)$ satisfies

$$(2) \quad r\alpha'(r) = \alpha^2(r).$$

In particular,

$$\frac{r\alpha'(r)}{\alpha(r)} \rightarrow 0, \quad \text{as } r \rightarrow 0.$$

(A continuous function vanishing at 0 and satisfying the above property does not exist if the structure is polynomially bounded.) Consider the function $f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0)$ defined in polar coordinates by

$$f(r, \theta) = \alpha(r) \cos \theta.$$

Split the gradient ∇f of f into its radial component $\partial_r f = \frac{\partial f}{\partial r}$ and the spherical one $\nabla' f = \nabla f - \partial_r f$. Then the limit of the secants lines and the gradient directions along a curve is contained in Inc iff along this curve the spherical part $\nabla' f$ dominates the radial one $\partial_r f$: $\frac{\partial_r f}{\|\nabla' f\|} \rightarrow 0$. In our case

$$\nabla' f = -r^{-1}\alpha(r)\sin\theta, \quad \partial_r f = \alpha'(r) \cos \theta$$

Hence for almost all secant directions (that is if $\sin\theta \neq 0$), $\nabla' f$ dominates $\partial_r f$ and therefore $Inc \subset \mathcal{E}_f$. Therefore, for dimensional reasons, $Inc \cap \mathcal{E}_f$ is not a finite union of sets of the form \mathcal{E}_V .

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Deformations of a Stanley-Reisner abelian surface

JAN CHRISTOPHERSEN

This project originated from a wish to find applications for general results of Klaus Altmann and the author on deformations of Stanley-Reisner schemes. See [1] and [2]. It is work in progress. We study deformations of a degenerate abelian surface and discover a Calabi-Yau 3-fold with Euler number 6.

We start with a torus, i.e. a smooth topological surface with genus 1. It follows from Euler's formula that the number of vertices in a triangulation must be greater than or equal to 7. There exists a minimal triangulation with 7 vertices, 21 edges and 14 faces. We denote the corresponding simplicial complex by Γ . The edge graph is the complete graph K_7 so each vertex has valency 6. The automorphism group of Γ is the Frobenius group F_{42} .

The Stanley-Reisner ideal of Γ in $\mathbb{C}[x_0, \dots, x_6]$ is generated by 21 cubic monomials. Let X be the complex projective space defined by this ideal in \mathbb{P}^6 . In general, if A_Γ is the Stanley-Reisner ring over \mathbb{C} of a simplicial complex Γ and $\mathbb{P}(\Gamma) = \text{Proj}(A_\Gamma)$, then $H^i(\mathbb{P}(\Gamma), \mathcal{O}_{\mathbb{P}(\Gamma)}) \simeq H^i(\Gamma, \mathbb{C})$. Moreover if Γ is an oriented combinatorial manifold then $\omega_{\mathbb{P}(\Gamma)}$ is trivial. Thus, if our X can be deformed to a smooth projective scheme X_t , then X_t must be an abelian surface.

We recall some particularly nice formulae from [2]. If Γ is a 2-dimensional combinatorial manifold, let f_0 be the number of vertices, f_1 the number of edges and $f_0^{(k)}$ be number of vertices with valency k . Then

$$\begin{aligned} \dim T_{\mathbb{P}(\Gamma)}^0 &= f_0 - 1 + h^1(\Gamma) \\ \dim T_{\mathbb{P}(\Gamma)}^1 &= 4f_0^{(3)} + 2f_0^{(4)} + f_1 + h^2(\Gamma) \\ &= f_0 + 9\chi(\Gamma) + h^2(\Gamma) + \sum_{k \geq 6} 2(k-5)f_0^{(k)} \\ \dim T_{\mathbb{P}(\Gamma)}^2 &= \sum_{k \geq 6} \frac{1}{2}k(k-5)f_0^{(k)}. \end{aligned}$$

On our case we get $\dim T_X^0 = 8$, $\dim T_X^1 = 22$ and $\dim T_X^2 = 21$.

Let $\mathcal{D}ef_X$ be the versal deformation of X as a compact complex space. There is a 1-dimensional contribution to T^1 from $H^1(\Theta_X)$ which we will not discuss. Instead, consider the subspace $\mathcal{D}ef_X^a \subset \mathcal{D}ef_X$ of algebraic deformations, or in this case not locally trivial deformations. The tangent space is the 21-dimensional $H^0(\mathcal{T}_X^1)$.

Since the vertex valencies are as small as 6 we may describe the obstruction equations as in [2]. We get 21 binomial quadratic equations, easily constructed

from Γ but still too complicated to describe here. The space $\mathcal{D}ef_X^a$ has dimension 9, is reduced and is the union of 29 components of which one is a unique smoothing component.

The 28 other components are interesting in their own right as the generic fibers will be non-smoothable degenerate abelian surfaces. In fact one can describe these surfaces exactly.

Let S be the unique smoothing component. It is the germ of a 9 dimensional affine toric variety. One can get the equations by saturating the ideal of the 21 binomial quadratic equations. Thus it corresponds to a cone σ and we break with convention by setting S to be the germ of of $\text{Spec}\mathbb{C}[\sigma \cap M]$ where M is a rank 9 lattice.

We use now the definitions and constructions originally used by Batyrev and Borisov to study mirror symmetry of complete intersections in toric varieties; see [3]. It turns out that σ is a completely split reflexive Gorenstein cone of index 3. It is the Cayley cone of 3 different lattice 6-simplices in \mathbb{R}^6 . It has the wonderful property that the cone generators are also the semi-group generators.

Following the nef-partition Batyrev-Borisov construction we take the Minkowski sum of the 3 simplices in \mathbb{R}^6 and get a reflexive polytope and so a Fano (highly singular) toric 6-fold. The corresponding complete intesection is a Calabi-Yau 3-fold. Thanks to Maximilian Kreuzer and the computer program PALP the stringy Hodge numbers are computed to be $h^{1,1} = 15, h^{1,2} = 12$. In particular the Euler number is 6.

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The topology of corank one singularities and its applications to the contact geometry of space curves

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Let M^m and N^n be real C^∞ -smooth closed (compact without boundary) manifolds of dimensions m and n , respectively, where $l = n - m \geq 0$. Consider a stable smooth mapping $f : M \rightarrow N$. We will assume that f is a mapping of corank ≤ 1 . This means that the dimension of the kernel of the derivative $f_{*x} : T_x M \rightarrow T_{f(x)} N$ does not exceed 1 for any $x \in M$.

Germes of f are classified with respect to the left-right equivalence (smooth transformations of local coordinates in N and M). The mapping f can have only singularities of types A_μ , where $0 \leq \mu \leq m/(l+1)$. We recall that f has a singularity of type A_μ at a given point $x \in M$ if its local algebra at x is isomorphic

to the \mathbb{R} -algebra $\mathbb{R}[[t]]/(t^{\mu+1})$ of truncated polynomials in one variable of degree at most μ .

The multi-singularity of f at a point $y \in N$ is the unordered set of singularities of f at points from $f^{-1}(y)$. Multi-singularities of the mapping f are classified by elements $\mathcal{A} = A_{\mu_1} + \cdots + A_{\mu_p}$ of the free additive Abelian semigroup \mathbb{A} generated by the symbols A_0, A_1, A_2, \dots . The number $\text{codim}_l \mathcal{A} = (l+1) \sum_{i=1}^p \mu_i + pl$ is called the codimension of a multi-singularity of type \mathcal{A} .

The mapping f can have only multi-singularities of codimension at most n . The set \mathcal{A}_f of points $y \in N$, where f has a multi-singularity of type $\mathcal{A} \in \mathbb{A}$, is a smooth submanifold of codimension $\text{codim}_l \mathcal{A}$ in N . The Euler characteristic $\chi(\mathcal{A}_f)$ of the manifold \mathcal{A}_f is the alternating sum of its Betti numbers (the ranks of the homology groups with compact supports).

We find a complete system of universal linear relations between the Euler characteristics of the manifolds of multi-singularities of mappings under consideration. Namely, we prove that for any $\mathcal{A} \in \mathbb{A}$ such that $\text{codim}_l \mathcal{A} \equiv n-1 \pmod{2}$, the Euler characteristic $\chi(\mathcal{A}_f)$ is a linear combination

$$(1) \quad \chi(\mathcal{A}_f) = \sum_X K_{\mathcal{A}}^{(l)}(X) \chi(X_f)$$

of the Euler characteristics $\chi(X_f)$, where the summation is carried over all $X \in \mathbb{A}$ such that $\text{codim}_l X \equiv n \pmod{2}$ and $\text{codim}_l X > \text{codim}_l \mathcal{A}$.

The universality of the relation (1) means that all its coefficients do not depend on f and on the topology of the manifolds M, N . We show that every coefficient $K_{\mathcal{A}}^{(l)}(X)$ is a rational number depending only on \mathcal{A}, X , and on the parity of the number l . Moreover, we produce a simple combinatorial algorithm for the calculation of the numbers $K_{\mathcal{A}}^{(l)}(X)$.

The completeness of the system of relations (1) in the simplest case $m < n$ means the following. Let $W_{m,n}$ be the class of all stable smooth corank ≤ 1 mappings of smooth closed m -dimensional manifolds into smooth closed manifolds of dimension n . Then any universal linear relation with real coefficients between the Euler characteristics of manifolds of multi-singularities of mappings $f \in W_{m,n}$ is a linear combination of the relations of the form (1) over all $\mathcal{A} \in \mathbb{A}$ such that $\text{codim}_l \mathcal{A} \equiv n-1 \pmod{2}$ and $\text{codim}_l \mathcal{A} < n$.

Similar results are valid for singularities of the front of a stable Legendre mapping of corank ≤ 1 , for singularities of the Maxwell set of global minima of a generic family of smooth functions depending on $k \leq 6$ parameters, and for singularities in some other problems. We apply these results to the contact geometry of space curves.

Multidimensional generalizations of the Bose theorem on supporting circles of a plane curve. A supporting circle of a plane curve is a tangent circle such that the curve lies on one side of it. The Bose theorem claims that, for any smooth closed convex generic curve on Euclidean plane, the difference between the number of supporting curvature circles and the number of supporting circles touching the curve at three points is equal to 2 both for externally supporting

circles (such that the curve lies outside the disks bounded by these circles) and for internally supporting circles (such that the curve lies inside the disks).

Consider, for example, a smooth closed generic curve in Euclidean space \mathbb{R}^{10} . Suppose that it is convex (this means that any hyperplane in the ambient space intersects the curve at at most 10 points, counting multiplicities). Take an element $\mathcal{A} = A_{\mu_1} + \cdots + A_{\mu_p} \in \mathbb{A}$ such that $\mu_1 > 0, \dots, \mu_p > 0, \mu_1 + \cdots + \mu_p = 11$. Let $\chi(\mathcal{A})$ denote the number of externally supporting hyperspheres that are tangent to the curve at exactly p geometrically distinct points with multiplicities μ_1, \dots, μ_p . Then

$$\begin{aligned} &42\chi(A_{11}) - 14\chi(A_9 + 2A_1) - 5\chi(A_7 + A_3 + A_1) + 5\chi(A_7 + 4A_1) \\ &\quad - 4\chi(2A_5 + A_1) - 2\chi(A_5 + 2A_3) + 2\chi(A_5 + A_3 + 3A_1) \\ &\quad - 2\chi(A_5 + 6A_1) + \chi(3A_3 + 2A_1) - \chi(2A_3 + 5A_1) \\ &\quad + \chi(A_3 + 8A_1) - \chi(11A_1) = 252. \end{aligned}$$

A similar relation is valid for internally supporting hyperspheres.

Multidimensional generalizations of the Freedman theorem on the number of triple tangent planes of a curve in 3-space. A triple tangent plane of a smooth curve in \mathbb{R}^3 is a plane tangent to the curve at three distinct points. The Freedman theorem claims that the number T of triple tangent planes of any smooth closed connected generic curve without flattening (zero-torsion) points in \mathbb{R}^3 is even. Banchoff, Gaffney, and McCrory generalized this statement to the case of a curve having flattening points. Namely, $T \equiv N/2 \pmod{2}$ for any smooth closed connected generic curve in \mathbb{R}^3 , where N is the total number of points of transversal intersections of the curve with its osculating planes at flattening points. We generalize these results to curves in $\mathbb{R}P^n$.

Consider a smooth closed generic curve γ (not necessarily connected) in $\mathbb{R}P^n$. By θ we denote the number of noncontractible connected components of γ . Take an element $\mathcal{A} = A_{\mu_1} + \cdots + A_{\mu_p} \in \mathbb{A}$ such that $\mu_1 > 0, \dots, \mu_p > 0$, and $\mu_1 + \cdots + \mu_p = n$. A hyperplane π in $\mathbb{R}P^n$ is called a tangent \mathcal{A} -hyperplane of γ if it is tangent to γ at exactly p geometrically distinct points with multiplicities μ_1, \dots, μ_p . Let $\chi(\mathcal{A})$ be the number of all tangent \mathcal{A} -hyperplanes of γ and let $\tilde{\chi}(\mathcal{A})$ be the number of tangent \mathcal{A} -hyperplanes π such that the number of points where γ intersects π transversally is congruent to $n + p + \theta$ modulo 4. Then, for any odd $n \geq 3$,

$$(2) \quad \chi(nA_1) \equiv \tilde{\chi}(A_3 + (n-3)A_1) \pmod{2},$$

$$(3) \quad \chi(2A_{(n-1)/2} + A_1) \equiv \tilde{\chi}(A_n) \pmod{2}.$$

In particular, if $\chi(A_3 + (n-3)A_1) = 0$, then the number $\chi(nA_1)$ of n -tangent hyperplanes of γ is even; if $\chi(A_n) = 0$ (that is γ has no flattening points), then the number $\chi(2A_{(n-1)/2} + A_1)$ is even.

Notice that if $n = 3$, then (2) is the same as (3). Moreover, the congruence (2) for this case leads to the formula

$$T \equiv \frac{N + \theta \cdot C}{2} \pmod{2},$$

where C is the number of flattening points of the curve γ .

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The space of arcs: their local rings at the generically stable points

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Let X be a singular variety over a perfect field. We show a finiteness result in the space of arcs X_∞ of X , which is an extension of the stability result of Denef and Loeser. From this it follows a Curve selection lemma for generically stable subsets of X_∞ . As a consequence, we extend to all dimensions the problem of wedges, proposed by M. Lejeune-Jalabert, and we obtain that an affirmative answer to this problem is equivalent to the surjectivity of the Nash map. More precisely, if $h \in X_\infty$ is the generic point of a generically stable subset of X_∞ , our result asserts that the completion $\widehat{\mathcal{O}_{X_\infty, h}}$ of the local ring $\mathcal{O}_{X_\infty, h}$ is a Noetherian ring. We will also show that the ring $\mathcal{O}_{X_\infty, h}$ is not in general a Noetherian ring, and that $\mathcal{O}_{(X_\infty)_{\text{red}}, h}$ is not in general an excellent ring, although it may be Noetherian.

Singularities, Finite Dimensional Algebras and Mirror Symmetry

ATSUSHI TAKAHASHI

I talked on the mirror symmetry for isolated weighted homogeneous hypersurface singularities R of dimension 2 ($R := A/(f)$, $A := \mathbb{C}[x, y, z]$). For these singularities, we can construct some equivalent triangulated categories: $D_{Sg}^{gr}(R) := D^b(\text{gr} - R)/D^b(\text{grproj} - R)$ the category of the singularity, $\underline{\text{CM}}^{gr}(R)$ the (stable) category of graded maximal Cohen-Macaulay modules and $\text{HMF}_A(f)$ the (homotopy) category of graded matrix factorizations of f .

By the mirror symmetry for Landau-Ginzburg orbifold theory in physics, we can conjecture that the following properties hold for the above categories.

Conjecture. ([T])

Let \mathcal{T} be one of equivalent categories $D_{Sg}^{gr}(R)$, $\underline{\text{CM}}^{gr}(R)$ and $\text{HMF}_A(f)$.

- (i) \mathcal{T} has a full strongly exceptional collection. In other words, \mathcal{T} is equivalent to a bounded derived category of finitely generated (right) modules over a finite dimensional algebra defined by the path algebra with relations for a quiver.
- (ii) \mathcal{T} has a Serre functor S such that $S^h \simeq [d]$ for some positive integers h and d where $[1]$ is the translation functor on \mathcal{T} . In fancy words, \mathcal{T} is a "fractional non-commutative Calabi-Yau of dimension d/h ".
- (iii) Assume that the singularity f has a "dual" singularity f^* . The Grothendieck group $K_0(\mathcal{T})$ with a symmetrized Euler pairing $\chi + {}^t\chi$ is isomorphic as a lattice to the second homology group of the Milnor fiber of the dual singularity f^* with a (minus of) intersection form.

The following are our main results:

Theorem.

The above conjecture is true for ADE singularities ([KST1]), and the Arnold's 14 exceptional singularities, as a result, we have a categorification of the Arnold's strange duality ([KST2]).

Moreover, the part (ii) is true for all graded isolated singularities ([KST2]) and the weak version of (i), that \mathcal{T} has a full exceptional collection, is true ([KST3]).

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Division theorems for the rational cohomology of discriminant complements and applications to automorphism groups

ALEXEI GORINOV

Let n and k be integers satisfying $1 \leq k \leq n + 1$. Set $\underline{d} = (d_1, \dots, d_k)$ to be a collection of integers such that $2 \leq d_1 \leq \dots \leq d_k$. Denote by $\Pi_{\underline{d},n}$ the \mathbb{C} -vector space of all k -tuples (f_1, \dots, f_k) , where $f_i, i = 1, \dots, k$, is a homogeneous polynomial in $n + 1$ variables of degree d_i with coefficients in \mathbb{C} . For every $(f_1, \dots, f_k) \in \Pi_{\underline{d},n}$ denote by $\text{Sing}(f_1, \dots, f_k)$ the projectivisation of the set of all $x \in \mathbb{C}^{n+1} \setminus \{0\}$ such that

- $f_i(x) = 0, i = 1, \dots, k$,
- the gradients of $f_i, i = 1, \dots, k$ at x are linearly dependent.

Set $\Sigma_{\underline{d},n}$ to be the subset of $\Pi_{\underline{d},n}$ consisting of all (f_1, \dots, f_k) such that $\text{Sing}(f_1, \dots, f_k) \neq \emptyset$. If $(f_1, \dots, f_k) \in \Pi_{\underline{d},n} \setminus \Sigma_{\underline{d},n}$, then the subvariety X of $\mathbb{C}P^n$ defined by $f_1 = \dots = f_k = 0$ is smooth, and f_1, \dots, f_k generate the homogeneous ideal of X . For this reason the space $\Pi_{\underline{d},n} \setminus \Sigma_{\underline{d},n}$ can be viewed as the space of equations for some smooth complete intersections of multidegree \underline{d} in $\mathbb{C}P^n$. The case $k = n + 1$ does not quite agree with this interpretation (it would correspond to “empty complete intersections”), but we include it nonetheless.

The group $\text{GL}_{n+1}(\mathbb{C})$ acts on $\Pi_{\underline{d},n}$ in an obvious way:

$$(0.1) \quad \text{GL}_{n+1}(\mathbb{C}) \times \Pi_{\underline{d},n} \ni (A, (f_1, \dots, f_k)) \mapsto (f_1 \circ A, \dots, f_k \circ A);$$

this action preserves $\Sigma_{\underline{d},n}$ (and hence, $\Pi_{\underline{d},n} \setminus \Sigma_{\underline{d},n}$).

Theorem 10. *Suppose $\underline{d} \neq (2)$. Then the geometric quotient of $\Pi_{\underline{d},n} \setminus \Sigma_{\underline{d},n}$ by $\text{GL}_{n+1}(\mathbb{C})$ exists, and the Leray spectral sequence of the corresponding quotient map degenerates over \mathbb{Q} (or modulo a sufficiently large prime) at the second term.*

This theorem generalises a recent result of J. Steenbrink and C. Peters for the case $k = 1$ [7]. Notice that since the quotient mapping is not proper, the degeneration of the Leray sequence can not be proven using the standard weight argument.

Corollary 11.

$$H^*(\Pi_{\underline{d},n} \setminus \Sigma_{\underline{d},n}, \mathbb{Q}) \cong H^*((\Pi_{\underline{d},n} \setminus \Sigma_{\underline{d},n})/\text{GL}_{n+1}(\mathbb{C}), \mathbb{Q}) \otimes H^*(\text{GL}_{n+1}(\mathbb{C}), \mathbb{Q}).$$

This isomorphism holds on the algebra level and respects the mixed Hodge structures.

By the Leray-Hirsch principle, in order to prove theorem 10, it suffices to construct global cohomology classes on $\Pi_{\underline{d},n} \setminus \Sigma_{\underline{d},n}$ (over \mathbb{Q} or modulo a prime $p, p \gg 0$) such that their pullbacks under any orbit map generate the cohomology of the group $\text{GL}_{n+1}(\mathbb{C})$ (as a topological space). Such classes can be realised as linking numbers with some natural subvarieties of $\Sigma_{\underline{d},n}$.

It turns out that in our situation working with integer coefficients is just a little bit more difficult than with the rationals. However, taking this little extra effort pays off, since it enables one to determine explicitly which multiple of the generator

of the highest cohomology group of $GL_{n+1}(\mathbb{C})$ comes from the cohomology of $\Pi_{\underline{d},n} \setminus \Sigma_{\underline{d},n}$ via an orbit map. This (together with some simple computations) implies the following results:

Theorem 12. *Let d be an integer > 2 . Then the order of the subgroup of $GL_{n+1}(\mathbb{C})$ consisting of the transformations that fix $f \in \Pi_{(d),n} \setminus \Sigma_{(d),n}$ divides*

$$\prod_{i=0}^n ((-1)^{n-i} + (d-1)^{n-i+1})(d-1)^i.$$

Actually, an analogous statement can be proven for arbitrary \underline{d} , but the resulting formula is a bit messy (and was therefore banned from the report).

Theorem 13. *The order of the subgroup $PGL_{n+1}(\mathbb{C}), n \geq 1$, consisting of the transformations that preserve a smooth hypersurface of degree $d > 2$ divides*

$$(0.2) \quad \frac{1}{n+1} \prod_{i=0}^{n-1} \frac{1}{C_{n+1}^i} ((-1)^{n-i} + (d-1)^{n-i+1}) \text{LCM}(C_{n+1}^i (d-1)^i, (n+1)(d-1)^n).$$

(Here LCM stands for the least common multiple.)

Theorem 14. *Let $f : \mathbb{C}P^n \rightarrow \mathbb{C}P^n$ be a ramified covering of degree d^n . Then the order of the group formed by the automorphisms $g : \mathbb{C}P^n \rightarrow \mathbb{C}P^n$ such that $f \circ g = f$ divides*

$$d^{n^2-1} \prod_{i=2}^{n+1} \frac{1}{C_{n+1}^i} \text{LCM}(C_{n+1}^i, (n+1)d^{i-1}).$$



By the Lefschetz principle, the statements of theorems 12, 13 and 14 are in fact true over any algebraically closed field of characteristic 0.

Here we list the values of (0.2) for $n = 2, 3, 4$ and $3 \leq d \leq 10$.

Table 1: Values of (0.2) for small n and d

$d \backslash n$	2	3	4
3	$432 = 2^4 \cdot 3^3$	$414720 = 2^{10} \cdot 3^4 \cdot 5$	$218972160 = 2^{14} \cdot 3^5 \cdot 5 \cdot 11$
4	$18144 = 2^5 \cdot 3^4 \cdot 7$	$2^{10} \cdot 3^8 \cdot 5 \cdot 7$	$2^{11} \cdot 3^{16} \cdot 5 \cdot 7 \cdot 61$
5	$2^8 \cdot 3 \cdot 5^2 \cdot 13$	$2^{19} \cdot 3^2 \cdot 5^3 \cdot 13 \cdot 17$	$2^{30} \cdot 3^2 \cdot 5^5 \cdot 13 \cdot 17 \cdot 41$
6	$2^4 \cdot 3^3 \cdot 5^4 \cdot 7$	$2^9 \cdot 3^4 \cdot 5^9 \cdot 7 \cdot 13$	$2^9 \cdot 3^5 \cdot 5^{16} \cdot 7 \cdot 13 \cdot 521$
7	$2^4 \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 31$	$2^{10} \cdot 3^8 \cdot 5^2 \cdot 7^3 \cdot 31 \cdot 37$	$2^{14} \cdot 3^{16} \cdot 5^2 \cdot 7^4 \cdot 11 \cdot 31 \cdot 37 \cdot 101$
8	$2^7 \cdot 3 \cdot 7^4 \cdot 43$	$2^{13} \cdot 3^2 \cdot 5^2 \cdot 7^9 \cdot 43$	$2^{15} \cdot 3^2 \cdot 5^2 \cdot 7^{16} \cdot 11 \cdot 43 \cdot 191$
9	$2^{12} \cdot 3^5 \cdot 7 \cdot 19$	$2^{28} \cdot 3^7 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19$	$2^{46} \cdot 3^9 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 19 \cdot 331$
10	$2^5 \cdot 3^8 \cdot 5^2 \cdot 73$	$2^{11} \cdot 3^{17} \cdot 5^3 \cdot 41 \cdot 73$	$2^{11} \cdot 3^{32} \cdot 5^5 \cdot 41 \cdot 73 \cdot 1181$

An explicit description of all automorphism groups of smooth projective hypersurfaces of given degree > 2 is known in very few cases; in fact, to the author's

knowledge, there are three such cases: plane cubics and quartics and cubic surfaces.

A smooth cubic curve in $\mathbb{C}P^2$ can have 18, 36 or 54 projective automorphisms, depending on the value of the j -invariant. The least common multiple of these numbers is $2^2 \cdot 3^3$, which is 4 times smaller than the corresponding item of table 1.

For $n = 2, d = 4$ the value of (0.2) is 18144, which is 9 times the LCM of the orders of the projective automorphism groups of smooth plane quartics (the list of these groups is given e.g. in [1, section 6.5.2]).

The list of automorphism groups of smooth cubics in $\mathbb{C}P^3$ is given in [4] by T. Hosoh who corrected an earlier classification by B. Segre [8]; the least common multiple of the orders of those groups is $3240 = 2^3 \cdot 3^4 \cdot 5$; compare this with the $n = 3, d = 3$ item of table 1.

If $n \geq 3, d \geq 3$ and $(d, n) \neq (4, 3)$, then any automorphism of a smooth hypersurface of degree d in $\mathbb{C}P^n$ is known [6, theorem 2] to be the restriction of a projective transformation, so in these cases theorem 13 implies that the order of the full automorphism group divides (0.2).

The expression (0.2) is majorated by $d^{\frac{3}{2}n(n+1)}(n+1)^{n-1}$; since (0.2) is divisible by the order of the projective automorphism group of any smooth hypersurface of degree d in $\mathbb{C}P^n$, it can hardly be expected to be a sharp bound. Indeed, smaller bounds are known; the best one known to the author is

$$(0.3) \quad J(n+1)d^n$$

given by A. Howard and A. J. Sommese [5] (here J is the Jordan function, i.e., $J(m)$ is the minimal integer such that any finite subgroup of $\mathrm{GL}_m(\mathbb{C})$ contains a normal Abelian subgroup of index $\leq J(m)$; B. Weisfeiler proved [12] that $J(m) \leq (m+1)!m^{a \ln m + b}$ for some $a, b \in \mathbb{R}$). However, theorem 13 gives additional information on the orders of automorphism groups; in a sense, asymptotically as $d \rightarrow \infty$, it provides much more restrictions than (0.3), since the number of divisors of $x \in \mathbb{Z}$ grows more slowly than any positive power of x as $x \rightarrow \infty$ (see, e.g., [3, theorem 317]).

The idea of the proof of theorem 10 is based on the following remark. The first columns of the Vassiliev spectral sequences that compute the Borel-Moore homology of the determinant varieties [10] (i.e., the spaces of degenerate $(n+1) \times (n+1)$ -matrices) and of the discriminant varieties $\Sigma_{d,n}$ (see [11, 2, 9]) coincide up to a dimension shift. This work may be viewed as an attempt to understand the relationship between the corresponding cohomology classes.

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Cohomology of the moduli space of stable curves of genus four

ORSOLA TOMMASI

(joint work with Jonas Bergström [2])

Let us denote by $\mathcal{M}_{g,n}$ the moduli space of irreducible non-reduced smooth projective curves of genus g with n marked points ($2g + n - 2 > 0$), and by $\overline{\mathcal{M}}_{g,n}$ its Deligne-Mumford compactification, the moduli space of stable curves. Starting from Mumford’s paper [10], which deals with the cohomology of $\overline{\mathcal{M}}_2$, the cohomology of moduli spaces of stable curves has been determined in several cases. Most results were deduced from a study of the cohomology of moduli spaces of smooth curves. On the moduli spaces of stable curves there is a stratification by the topological type of the curves, and the strata can be explicitly described in terms of moduli spaces of the form $\mathcal{M}_{g,n}$. From this fact, it follows that there is a relationship between the rational cohomology of the moduli spaces of smooth curves and of stable curves. This was made precise by Getzler and Kapranov in [7] by using the formalism of modular operads.

In the case of stable curves of genus four, the application of Getzler-Kapranov’s formula requires the knowledge of the rational cohomology of the moduli spaces of smooth curves of genus g with n marked points for all g, n satisfying $0 \leq g \leq 4$, $\max\{0, 3 - 2g\} \leq n \leq 8 - 2g$.

The cohomology of \mathcal{M}_4 was computed in [12]. The space \mathcal{M}_4 can be written as the union of locally closed strata, in such a way that each stratum is a geometric quotient of the complement of a discriminant in a certain vector space. In view of the generalized Leray-Hirsch theorem of [11], the rational cohomology of these geometric quotients can be easily read off the cohomology of the corresponding

discriminant complement. Hence, it is enough to calculate the cohomology of the complements of discriminants occurring in the stratification. This is achieved by using a topological method due to Vassiliev and Gorinov (see [13], [9]).

In the case $g = 3, n = 2$ and $g = 2, n = 4$, information on the rational cohomology of $\mathcal{M}_{g,n}$ is obtained by a different method. In [1], Jonas Bergström proves that for these moduli spaces, the number of points over any finite field is a polynomial in the number of elements of the field. This ensures that the rational cohomology of $\mathcal{M}_{2,4}$ and $\mathcal{M}_{3,2}$ (over \mathbf{C}) is of Tate Hodge type. Moreover, it is straightforward to obtain from the number of points over finite fields the Euler characteristics of the rational cohomology of these spaces in the Grothendieck group of rational Hodge structures. These Euler characteristics contain enough information on the cohomology of $\mathcal{M}_{2,4}$ and $\mathcal{M}_{3,2}$, to be used in Getzler-Kapranov's formula. Note that the relationship between the number of points and the cohomology of the smooth stack $\mathcal{M}_{g,n}$ follows from an adaptation of the results of [3]. The contributions of all other moduli spaces $\mathcal{M}_{g,n}$ with $g \leq 3$ to the cohomology of $\overline{\mathcal{M}}_4$ is known from previous works of Getzler and Looijenga ([4], [5], [6], [8]).

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On rational cuspidal plane curves, open surfaces and local singularities

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(joint work with J. Fernández de Bobadilla, I. Luengo and A. Némethi)

In the last Conference on Singularity Theory in Oberwolfach there were several lectures devoted to the theory of normal surface singularities and their topological/analytical invariants. One of the fundamental questions is what kind of analytic invariants of an analytic complex normal surface singularity can be determined from the topology (i.e. from the link) of the singularity. To have a chance to answer these type of questions, one has to assume two types of restrictions: a topological one – e.g. that the link is a rational homology sphere – and an analytic one – e.g. that the singularity is \mathbb{Q} -Gorenstein. For such class of singularities several conjectures were presented and discussed in Oberwolfach 2003: the so called “Seiberg-Witten invariant conjecture” (of Nicolaescu and Némethi), the “Universal abelian cover conjecture” (of Neumann and Wahl) and the “Geometric genus conjecture” of Némethi.

In [6] we found counter-examples to these conjectures using hypersurface superisolated singularities. This class of singularities “contains” in a canonical way the theory of complex projective plane curves. They were introduced in by Luengo in order to show that the μ -constant stratum, in general, is not smooth.

The Seiberg-Witten Conjecture of Nicolaescu and Némethi is a generalization of the “Casson invariant conjecture” of Neumann and Wahl [10].

If the link of a normal surface singularity $(X, 0)$ is a rational homology sphere then the geometric genus p_g of $(X, 0)$ has an “optimal” topological upper bound. Namely,

$$(SWC) \quad p_g \leq \mathbf{sw}(M) - (K2 + s)/8.$$

Moreover, if $(X, 0)$ is a \mathbb{Q} -Gorenstein singularity then in (SWC) the equality holds.

Here, $\mathbf{sw}(M)$ is the Seiberg-Witten invariant of the link M of $(X, 0)$ associated with its canonical $spin^c$ structure, K is the canonical cycle associated with a fixed resolution graph G of $(X, 0)$, and s is the number of vertices of G (see [9]).

A hypersurface singularity $f : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}, 0)$, $f = f_d + f_{d+1} + \dots$ (where f_j is homogeneous of degree j) is called *superisolated* if the projective plane curve $C := \{f_d = 0\} \subset \mathbb{P}^2$ is reduced with isolated singularities $\{p_i\}_{i=1}^\nu$, and these points are not situated on the projective curve $\{f_{d+1} = 0\}$. The link of f is a rational homology sphere if the curve C is rational and cuspidal (i.e. if all the germs (C, p_i) are locally irreducible).

In [6] some superisolated singularities with $\nu = \#Sing(C) \geq 2$ which do not satisfy (SWC) were showed. Moreover, in all the counterexamples $p_g > \mathbf{sw}(M) - (K2 + s)/8$ (contrary to the inequality predicted by the general conjecture !). On

the other hand, even after an intense search of the existing cases, the authors were not able to find any counterexample with $\nu = 1$.

To understand the relationship between (SWC) and the pair (\mathbb{P}^2, C) , (C being the rational cuspidal curve which is the tangent cone of the corresponding superisolated surface singularity) leads the authors of [3, 4, 5] to the classification problem of the rational cuspidal projective plane curves. That is, to determine, for a given d , whether there exists a projective plane curve of degree d having a fixed number of unibranch singularities of given topological type. One of the integers which help in the classification problem is the logarithmic Kodaira dimension $\bar{\kappa}$ of open surface $\mathbb{P}^2 \setminus C$. The classification of curves with $\bar{\kappa}(\mathbb{P}^2 \setminus C) < 2$ has been recently finished by Miyanishi and Sugie [7], Tsunoda [14]. and Tono [12].

This remarkable problem of classification is not only important for its own sake, but it is also connected with crucial properties, problems and conjectures in the theory of open surfaces, and in the classical algebraic geometry:

- **Coolidge and Nagata problem**, see [1, 8]. It predicts that every rational cuspidal curve can be transformed by a Cremona transformation into a line.

- **Orevkov's conjecture** [11] which formulates an inequality involving the degree d and numerical invariants of local singularities.

- **Rigidity conjecture** of Flenner and Zaidenberg, [2]. Fix one of 'minimal logarithmic compactifications' (V, D) of $\mathbb{P}^2 \setminus C$, that is V is a smooth projective surface with a normal crossing divisor D , such that $\mathbb{P}^2 \setminus C = V \setminus D$, and (V, D) is minimal with these properties. The sheaf of the logarithmic tangent vectors $\Theta_V \langle D \rangle$ controls the deformation theory of the pair (V, D) . The *rigidity conjecture* asserts that every \mathbb{Q} -acyclic affine surfaces $\mathbb{P}^2 \setminus C$ with $\bar{\kappa}(\mathbb{P}^2 \setminus C) = 2$ is rigid and has unobstructed deformations. That is,

$$h^1(\Theta_V \langle D \rangle) = 0 \quad \text{and} \quad h^2(\Theta_V \langle D \rangle) = 0.$$

In fact, the Euler characteristic $\chi(\Theta_V \langle D \rangle) = h^2(\Theta_V \langle D \rangle) - h^1(\Theta_V \langle D \rangle)$ must vanish because $h^0(\Theta_V \langle D \rangle) = 0$. Note that the open surface $\mathbb{P}^2 \setminus C$ is \mathbb{Q} -acyclic if and only if C is a rational cuspidal curve.

The aim of the propose talk is to present some of these conjectures and related problems, and to complete them with some results and new conjectures from the recent work of the authors in [3].

1. Following Tono in [13] can be proved that $\chi(\Theta_V \langle D \rangle) \geq 0$ and

Theorem 1 *Let C be an irreducible, cuspidal, rational projective plane curve with $\bar{\kappa}(\mathbb{P}^2 \setminus C) = 2$. The following conditions are equivalent:*

(i) *Orevkov's conjecture is true.*

(ii) $\chi(\Theta_V \langle D \rangle) \leq 0$.

In such a case, the curve C can be transformed by a Cremona transformation of \mathbb{P}^2 into a straight line (i.e., the Coolidge-Nagata problem has a positive answer).

2. Author's 'compatibility property' is a sequence of inequalities, conjecturally satisfied by the degree and local invariants of the singularities of a rational cuspidal curve.

Consider a collection $(C, p_i)_{i=1}^{\nu}$ of locally irreducible plane curve singularities, let $\Delta_i(t)$ be the characteristic polynomial of the monodromy action associated with (C, p_i) , and $\Delta(t) := \prod_i \Delta_i(t)$, with $\deg \Delta(t) = 2 \sum \delta(C, p_i)$. Then $\Delta(t)$ can be written as $1 + (t-1)\delta + (t-1)^2 Q(t)$ for some polynomial $Q(t)$. Let c_l be the coefficient of $t^{(d-3-l)d}$ in $Q(t)$ for any $l = 0, \dots, d-3$.

Conjecture CP *Let $(C, p_i)_{i=1}^{\nu}$ be a collection of local plane curve singularities, all of them locally irreducible, such that $2\delta = (d-1)(d-2)$ for some integer d . If $(C, p_i)_{i=1}^{\nu}$ can be realized as the local singularities of a degree d (automatically rational and cuspidal) projective plane curve then*

$$c_l \leq (l+1)(l+2)/2 \text{ for all } l = 0, \dots, d-3.$$

If $\nu = 1$, we may replace Δ by the semigroup $\Gamma_{(C,p)} \subset \mathbb{N}$ of the local germ (C, p) . and reformulate **CP** in terms of $\Gamma_{(C,p)}$ and the degree d . It turns out that the inequalities of c_l are replaced by a very precise and mysterious distribution of the elements of the semigroup with respect to the intervals $I_l := ((l-1)d, ld]$:

Conjecture CP Assume that $\nu = 1$. Then for any $l > 0$, the interval I_l contains exactly $\min\{l+1, d\}$ elements from the semigroup $\Gamma_{(C,p)}$.

The main result of [3] is:

Theorem 2 If $\bar{\kappa}(\mathbb{P}^2 \setminus C) \leq 1$, then Conjecture CP is true (in fact with $n_l = 0$). Moreover, (SWC) holds for the corresponding superisolated singularity.

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