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Singularities

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Abstract. Singularity theory is concerned with the local and global structure of maps and spaces that occur in algebraic, analytic or differential geometric context. It uses methods from algebra, topology, algebraic geometry and complex analysis.

Mathematics Subject Classification (2000): 14Bxx, 32Sxx,58Kxx.

Introduction by the Organisers

The workshop *Singularity Theory* that was held in September 2012 was the continuation of a long sequence of workshops on the subject that over the years took place at Oberwolfach. It was organized by A. Némethi (Budapest), D. van Straten (Mainz) and V. A. Vassiliev (Moscow). It was attended by 53 participants with a broad geographic representation. Funding from the Marie Curie Program of the EU provided complementary support for young researchers and PhD students.
The schedule of the meeting followed more of less the standard format of three morning and two afternoon talks of one hour each. An exception was the first thursday morning slot, which was used for three shorter presentations by younger participants. On three evenings additional presentations and forum-discussions took place, so that, taking the traditional wednesday afternoon hike into account, a total of 28 talks were given. From the abstracts it is clearly visible that a broad spectrum of topics in singularity theory was covered, showing that the field is vibrant as ever.

New questions are keeping the theory of singularities very much alive. N. A'Campo gave a new approach to the description of the monodromy of plane curve singularities in terms of flips of triangulated surfaces. This takes up the ideas around cluster algebras and Fock-Goncharov coordinates. The conjectured relationship between Hilbert-schemes of curve singularities, the compactified Jacobian and Severi-strata in the versal base on the one hand, and knot invariants of the link on the other hand, were subject of talks by A. Oblomkov, E. Gorsky and V. Shende. These exciting new developments hold much promise for the future and underline how much more there is to be learned about the simplest class of plane curve singularities.
The theory of normal surface singularities lost one famous conjecture, but aquired an exciting new one: J. de Bobadilla (joint with M. Pe Pereira) presented their recent proof of the Nash-conjecture for surfaces, and J. Stevens gave a conjectural characterisation of all simple normal surface singularities.
Several talks were related to mirror symmetry and categorical structures related to singularities. R. Buchweitz gave an overview of non-commutative singularity theory, where spaces and resolutions are described by appropriate categories. A. Ishii reported on crepant resolutions of cones over lattice polytopes determined by dimer-models, K. Ueda spoke about mirror symmetry and categorifications around Arnol'ds strange duality, and in the talk of W. Ebeling strange duality was generalised to the orbifold setting. C. Sevenheck reported on work (joint with T. Reichelt) that aims at giving a more precise description of the now classical cases of mirror symmetry (as isomorphism of A- and B- model Frobenius manifolds) on the level of D-modules and GKZ-systems. The talk by A. Varchenko explained how the axiomatics of master functions give rise to Frobenius-like structures associated to arrangements. B. Pike and M. Schulze presented some new results around free divisors.
In the realm of symplectic singularity theory, M. Garay explained how the perspective of singularity theory can be used in the analysis of hamiltonian systems, which results in a proof of the Herman conjecture. Y. Namikawa presented his proof of the classification of symplectic homogeneous complete intersections.
There were also a number of talks that represented beginnings of new theory. E. Faber described a new notion of transversality of singular varieties, H. D. Nguyen described the first steps in the study of right equivalence in characteristic p and D. Kerner described an attempt to define equisingularity discriminants. The talk of K. Saito about $A_{\frac{1}{2}\infty}$ and $D_{\frac{1}{2}\infty}$, the simplest transcendental curves, opened up a whole new field of exploration. A. Libgober reported on the recently discovered link between the fundamental group of cuspidal curve complements and Mordell-Weil groups of elliptic curves over function fields.
Furthermore, there were talks of a general nature: H. Hauser gave an overview of approximation theorems and how to prove them, J. Christophersen formulated a new general comparison theorem in deformation theory and B. Teissier explored the connections between toric geometry and resolutions.
In the global theory of singularities and Thom-polynomials there were talks by M. Kazarian, describing a new topological recursion for Hurwitz numbers and A. Szűcs

presented a striking new result on the impossibility to describe homology classes by manifolds with mild singularities. The meeting was closed by J. Schürmann, who showed how his formalism of homological Chern-classes can effectively be used to study characteristic numbers of Hilbert-schemes.

To summarize, we think the meeting was a great succes: old and new conjectures were presented by older and younger participants. Old and new friendships were celebrated, old and new collaborations were started or continued. The organisers thank the Oberwolfach staff for their efficient handling of the boundary conditions, which helped to create the unique Oberwolfach atmosphere.

Workshop: Singularities

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Abstracts

Monodromy and Flips

Norbert A'Campo

Introduction.

The geometric monodromy of an isolated complex hypersurface singularity is a mapping class of a relatif diffeomorphism of the local nearby fiber. A representatif is constructed as follows. Let the hypersurface $X \subset \mathbf{C}^{n+1}, n > 0$, be the zero level of the polynomial mapping $f : \mathbf{C}^{n+1} \to \mathbf{C}$ and let $0 \in X$ be an isolated singularity of f. For $0 < \delta << \epsilon << 1$ the differentiable manifold with boundary $(F, \partial F) := \{p \in \mathbf{C}^{n+1} \mid ||p|| \leqslant \epsilon \& f(p) = \delta\}$ does not depend up to diffeomorphism on δ and ϵ. With a partition of unity one constructs a vector field V tangent to the tube $X_\delta := \{||f|| = \delta\}$ such that one has $(Df)_p(V) = 2\pi i f(p), p \in X_\delta$. Moreover, one asks that V is tangent to the boundary of X_δ and that all flow lines of V in the boundary of X_δ close at time 1. The flow of V with stopping time 1 defines a representatif of the geometric monodromy. A basic fact is that no representatif of the geometric monodromy preserves a complex structure on F, except if there is no singularity at 0, i.e. $(Df)_0 \neq 0$. Our far away aim is the study of the action of the geometric monodromy on the Teichmueller space T_F of marked complex structures on F. This aim is indeed far away since no workable Teichmueller theory nor computation of geometric monodromy in complex dimensions greater 1 is available.

In dimension $n = 1$, i.e. for plane curve singularities, tools are available.

We will work with the enhanced Teichmueller theory developped by Rinat Kashaev, Vladimir Fock, Alexander Goncharov and Bob Penner, see [4], [5], [6], [7]. It turns out that the discription of the nearby fiber and of the geometric monodromy by real morsification gives a method for reaching the above aim.

Teichmueller Theory.

First we explain briefly enhanced Teichmueller Theory. Let $(F, \partial F)$ be an oriented, connected surface with non empty boundary and non positif Euler characteristic. A marking is a labelled system of embedded, and pairwise disjoint relative arcs $a_1, a_2, \cdots, a_N$ that cut the surface F into hexagons. A relatif arc in F is an embedded copy of the interval $(I, \partial I, I = [0, 1]$, in F with $\partial I \subset \partial F$. Using a system of $N = 6g - 6 + 3r$ arcs one can cut the surface $S_{g,r}$ in $4g - 4 + 2r$ hexagons. Each hexagon H has three sides that belong to the boundary of F, the remaining three sides consist of arcs of the cutting system. The boundary ∂H of a hexagon H is defined to be the union of its three boundary sides. Observe, that $H \setminus \partial H$ is homeomorphic to an ideal hyperbolic triangle. Let $(F, \partial F, \sigma)$ be a triple, such that σ is a marking for $(F, \partial F)$. A σ-marked hyperbolic structure on $(F, \partial F)$ is a hyperbolic structure on $F \setminus \partial F$ such that all arcs of σ are geodesics and such that for each hexagon H the induced hyperbolic structure on $H \setminus \partial H$ is isometric to an ideal hyperbolic triangle. It is important to notice, that the hyperbolic structure on $F \setminus \partial F$ is not required to be complete. Let $T_\sigma(F)$ be the space of σ-marked hyperbolic structures on $(F, \partial F)$. The topology of this space can be defined by using the Gromov-Hausdorff distance between metrical completions. Here an important result of this Teichmueller Theory.

Theorem 1. *Let F be the surface $S_{g,r}, r > 0, 2g-2-r < 0$. Let σ be a marking. Then the space $T_\sigma(F)$ is homeomorphic to $\mathbf{R}^{6g-6+3r}$.*

More precisely, for each arc a of the system σ one defines a coordinate function $f_a : T_\sigma(F) \to \mathbf{R}$ on $T_\sigma(F)$ as follows: Let (Δ, Δ') be a pair of ideal hyperbolic triangles that are glued along the arc a. Let O_Δ be the centrum of the incercle of Δ, let M_Δ be the point of intersection of the incercle of Δ with the arc a, and finally let $M_{\Delta'}$ be the the point of intersection of the incercle of Δ' with the arc a. Consider the broken geodesic $O_\Delta, M_\Delta, M_{\Delta'}$ which can be turning left or right at the point M_Δ and which depends on the given σ-marked hyperbolic stucture $t \in T_\sigma(F)$. One defines the value $f_a(t) = \pm|M_\Delta M_{\Delta'}| \in \mathbf{R}$ where the sign $\pm$ is $+$ if the broken geodesic turns right at M_Δ. Here, $|M_\Delta M_{\Delta'}| = |f_a(t)|$ denotes the hyperbolic length of the segment $M_\Delta M_{\Delta'}$ on a. It is important to observe that the value $f_a(t)$ does not depend on the ordering of the pair triangles (Δ, Δ') that meet along a. Putting all coordinate function f_a together one obtains a map $f_\sigma : T_\sigma(F) \to \mathbf{R}^{6g-6+3r}$. The Theorem states that this map is a homeomorphism.

One can also use as coordinate map $c_\sigma : T_\sigma(F) \to \mathbf{R}^{6g-6+3r}_{>0}$ given by putting $c_a(t) = e^{2f_a(t)}$. The value $c_a(t)$ is a crossratio of the 4 points at infinity of a lift of the union of the two triangles (Δ, Δ').

A flip is an elementary change of marking: an arc a_i belonging to the marking σ on F defines an ideal quadrilateral with diagonal a_i. The flip (about a_i) changes the marking σ to the marking σ' by replacing a_i with the other diagonal b_i, again labelled by i, of the ideal quadrilateral. The tautological map $\tau_{\sigma,\sigma'} : T_\sigma(F) \to T_{\sigma'}(F)$ induces a coordinate change map $c_{\sigma'} \times c_\sigma^{-1} : \mathbf{R}^{6g-6+3r}_{>0} \to \mathbf{R}^{6g-6+3r}_{>0}$ which is of cluster type.

Two markings are related by a sequence of flips. By composing the above coordinate changes we get for two markings σ, σ' a coordinate change $c_{\sigma',\sigma} : \mathbf{R}^{6g-6+3r}_{>0} \to \mathbf{R}^{6g-6+3r}_{>0}$.

Topology of isolated plane curve singularities by divides.
Now we explain how to cut by an arc system the local nearby fiber of an isolated plane curve singularity in hexagons as above, see [1], [2], [3]. Here the singularity A_1 is an exception since the Euler caracteristic of the fiber is 0. Without making any topological restriction, we may assume that all local branches admit a real parametrization. We perturb the parametrizations in order to get a divide P for the singularity. The divide is a system of generic relative embeddings of the union

of r copies of the interval I in the euclidean unit disk D. We consider this divide as a planar 4-valent graph, which we modify as follows:

– at each double point, we replace the double point by a cercle with four points of valency 3, exactly as modifying a street crossing by a turnabout. The result is a 3-valent planar graph.

– we remove all edges that run to the boundary of the D, but keep the endpoint that is in the interior of D as a 2-valent vertex. The result is a $2,3$-valent graph with $2r$ 2-valent vertices.

– thicken the graph with a framing that does each along edge NOT coincides with the planar framing. We obtain a 3-valent ribbon surface $F = F_P$. The surface F_P is orientable since every edge cycle of the graph has an even number of edges. The divide link L_P is naturally oriented. We orient its (Seifert-)surface F_P consistently. Observe that the number of ribbons is $2r$ less then the number of edges in the previous $2,3$-valent graph. In fact, the number of ribbons is given by $N = 6g - 6 + 3r$ where g is the genus of the ribbon surface.

– label the ribbons from 1 to N and cut the i-th ribbon with an arc a_i. The system $\sigma_P = a_1, a_2, \cdots, a_N$ is a marking for the ribbon surface F_P.

The oriented ribbon surface F_P with marking σ_P is a topological model for the nearby fiber of the plan curve singularity. The monodromy mapping class is obtained as follows. Each double point of P contributes in F_P with an annulus. Let $\delta_.$ be system of core curves of these annuli. The complementairy region in D of P are signed. Each $+$ or $-$-region also contributes in F_P with an annulus. Let δ_+ and δ_+ be the systems of corresponding core curves. The geometric monodromy is represented by the mapping class T_P obtained by composing the right Dehn twists about these curves: first do the twists about the curves in δ_+ next about the curves in $\delta_.$ and finally about the curves in δ_-. The monodromy T_P is the composition of three multi-twists $T_- \circ T_. \circ T_+$.

Two core curves δ, δ' of the same type $+, .$ or $-$ are disjoint and also disjoint in the following stronger sense: No arc a of the system σ_P intersects both δ and δ'. Moreover, a core curve δ and an arc a intersect transversely in at most one point. It follows that we can compute rather easily the system $T_P(P)$ by applying a sequence of flips to the system P.

Our main result is:

Theorem 2. *The triple (F_P, σ_P, T_P) describes the action of the geometric monodromy on the Teichmueller space $T_{\sigma_P}(F_P)$. More precisely, a structure $t \in T_{\sigma_P}(F_P)$ with coordinates $c_\sigma(t)$ is mapped by T_P to the structure $s \in T_{\sigma_P}(F_P)$ with coordinates $c_\sigma(s) = c_{T_P(\sigma),\sigma}(c_\sigma(t))$.*

A more conformal information can be obtained by the following trick. Let σ_P be a marking of F_P as above. We double F_P by making a boundary connected sum of two copies of F_P. The boundary connected sum is along open intervals in each boundary component of F_P. These open intervals are chosen to have closures that are disjoint from the arcs of the marking σ_P. The resulting surface G_P is again with non empty boundary and marked by the arcs of the markings σ_P in the copies together with the gluing intervals. On both sides of each gluing interval appear two quadrilaterals, which we triangulate by adding diagonals. We denote this marking again by σ_P. We let on one copy act the monodromy T_P and on the other its inverse T_P^{-1}. We denote by S_P this mapping class of G_P. Let $t \in T_{\sigma_P}(G_P)$ by a structure such that the length of the boundary components with respect to metric completion of t is zero. Hence we may think the boundary components as punctures of G_P and the surface G_P as a complete hyperbolic surface and hence also as a conformal surface. Important is to notice that the image structure of t by G_P is also such a punctured surface. The mapping class S_P acts on usual Teichmueller space of G_P and preserves the boundary connected sum decomposition of G_P. Also this action can be computed as a composition of flips.

Our future project is to compute asymptotics. For instance, let c be an isotopy class of a closed curve in $F_P \subset G_P$ and let t be a structure on G_P. Compute the growth rate of the length with respect to t of $T_P^n(c)$ as $n \to \infty$. We speculate, that growth rate gives a filtration on the linear space of multi-curves in F_P , and hence also on the spaces of regular functions of the representation spaces of the fundamental group of F_P into $SL(2, \mathbf{C})$ by using the theorem of Josef Przytycki and Adam Sikora [8]. From this filtration we speculate to get new insight in non abelian Hodge Theory of plane curve singularities.

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Noncommutative Singularity Theory - A Survey

Ragnar-O. Buchweitz

In two talks we explained recently obtained extensions of the classical McKay-correspondence in the context of representations of algebras (over fields of characteristic 0). First we reported on work of Amiot, Iyama, and Reiten [1]:

Theorem 1. *Let A be a graded bimodule-d-Calabi–Yau algebra of Gorenstein invariant $a \in \mathbb{Z}$, in that A is of finite projective dimension over its enveloping algebra $A^{\mathrm{op}} \otimes A$ and further*

$$\mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}} \otimes A}(A, A^{\mathrm{op}} \otimes A) \cong A[-d](a)$$

in the derived category of A–bimodules. Assume $e \in A$ is an idempotent such that $\overline{A} = A/AeA$ is finite-dimensional and $eA(1-e) = 0$.

If A is noetherian, then $R = eAe$ is (Iwanaga–)Gorenstein *and the stable category of graded maximal Cohen–Macaulay R–modules is equivalent to the bounded derived category $D^b(\overline{A})$.*

Forgetting the grading, the stable category $\underline{MCM}(R)$ of maximal Cohen–Macaulay R–modules becomes equivalent to $\mathcal{C}_{d-1}(\overline{A})$, the $(d-1)$ cluster category of the artinian algebra $\overline{A}$.

The classical McKay correspondence is a rather special case of this result when $d = 2$: If $\tilde{\Delta}$ is an *extended Coxeter-Dynkin diagram*, then its preprojective algebra $A = \Pi\tilde{\Delta}$, Morita equivalent to the twisted group algebra $S{*}G$; see [5]; satisfies the hypotheses.

Here $G \leqslant SL(2, \mathbb{C})$ is the finite group corresponding to $\tilde{\Delta}$, and $S = \mathbb{C}[u, v]$ with the induced G-action. Taking for $e \in \Pi\tilde{\Delta}$ the idempotent corresponding to the trivial representation of G, one finds $R = S^G$, the ring of the corresponding Kleinian surface singularity, and $\overline{A} = \Pi\tilde{\Delta}/(e) \cong \Pi\Delta$, the preprojective algebra of the Coxeter-Dynkin diagram itself.

One knows that the derived category of the minimal resolution of singularities of $\operatorname{Spec} R$ is equivalent to that of $S{*}G$ or $\Pi\tilde{\Delta}$ by [12]; see also [15] for the case of three dimensional quotient singularities with crepant resolutions of singularities.

The statement on graded maximal Cohen-Macaulay R-modules then recovers results by Kajiura–Saito–Takahashi [8, 9] and Lenzing–de la Peña [13], as well as Ueda [17] in the surface case. In [1], Amiot, Iyama, and Reiten extend these results to some three dimensional cyclic quotient singularities.

It is interesting to note that the preprojective algebra of an extended Coxeter-Dynkin diagram made its first entrance into singularity theory through the differential geometric work of Kronheimer; see [4] for a survey of that point of view and how one obtains both semi–universal deformation and simultaneous resolution of Kleinian singularities as moduli spaces of representations of that algebra.

A further application of the above theorem to singularity theory arises from Bridgeland's [2] "*rolled-up helix algebras*" for Fano varieties with tilting object (ongoing joint work with L. Hille). Here R is the homogenous coordinate ring of the anti-canonical embedding of the Fano variety, while A is the endomorphism algebra of a tilting object pulled back to the canonical bundle.

In the second talk, we used Kalck's recent presentation at ICRA 2012 in Bielefeld, available at [10], to explain work by Iyama–Kalck–Wemyss–Yang [7], based on earlier work of these authors, e.g. [11], as well as Burban–Kalck [3], De Thanhoffer de Völcsey–Van den Bergh [16], and others, on the "*relative singularity category*", the triangulated category obtained as the Verdier quotient $D^b(\tilde{X})/\pi^* \operatorname{perf}(X)$, where $\operatorname{perf}(X)$ is the category of perfect complexes on the singular space X and $\pi : \tilde{X} \to X$ is a resolution of singularities.

If $D^b(\tilde{X}) \cong D^b(A)$ for an algebra A, then A represents a *noncommutative desingularization* of X; see [6, 14, 18] for surveys of that theory. Similar to the above, one finds an exact equivalence of triangulated categories

$$\underline{MCM}(R) \cong \frac{D^b(A)/\operatorname{perf}(R)}{D^b(\overline{A})}$$

in case the local ring $R = \mathcal{O}_{X,x}$ of the isolated singularity is Gorenstein. Here $A = \operatorname{End}_R(R \oplus M)$ is the endomorphism ring of the direct sum of the ring with a suitable maximal Cohen–Macaulay R–module M, the idempotent e is given by the projection onto the direct summand R, and $\overline{A} = A/AeA$ identifies with the stable endomorphism ring of M over R.

These results extend to "special" maximal Cohen-Macaulay modules on (rings of) rational surface singularities, such as quotient singularities by finite subgroups $G \leqslant GL(2, \mathbb{C})$, and the so obtained stable categories correspond to partial desingularizations.

Last, but not least, I want to thank Susanne Müller for her excellent and much appreciated support in preparing this extended abstract!

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Hilbert schemes of points on planar curves and knot homology of their link

Alexei Oblomkov
(joint work with J. Rasmussen and V. Shende)

Hilbert scheme of points on planar singular curves and knot invariants. An algebraic knot is constructed from a plane curve singularity by intersecting the curve with a small three sphere surrounding the singularity (see for example [2] for an introduction). The new formula for the HOMFLY-PT invariant is written in terms of the Euler characteristics of the Hilbert scheme of points on the singular curve. These Hilbert schemes appear naturally in the recent studies of the BPS states [12]. The geometric (or more precisely gauge theoretic) interpretation of the knot invariants was a starting point for the topological vertex theory which is an ancestor of the GW/DT correspondence conjecture. We hope that while exploring this simple case of algebraic knots we will achieve a better understanding of the recent physical conjectures on quantum invariants of knots [17].

Let $C = \{E(x,y) = 0\} \subset \mathbb{C}^2$ be a planar curve. Then $C^{[n]}$ stands for the Hilbert scheme of n points on C, that is, the set of ideals $I \subset \mathbb{C}[x,y]$ that contain E and have codimension n. If C is smooth, the Hilbert scheme is the n-th symmetric power of the curve; for the singular curve it is a partial resolution of the symmetric power. If we assume that $E(0,0) = 0$, then $C^{[n]}_{(0,0)}$ is the punctual Hilbert scheme (i.e. the moduli space of ideals defining a fat point supported at $(0,0)$): algebraically, it is the set of ideals from $C^{[n]}$ that contains x^N, y^N for some N. Motivated by the construction of Nakajima and Yoshioka [9], we introduce the following nested Hilbert scheme:

$$C^{[l]}_{(0,0)} \times C^{[l+m]}_{(0,0)} \supset C^{[l,\,l+m]}_{(0,0)} := \{(I,J) | I \supset J \supset I \cdot (x,y)\}$$

When $m = 0$ we get back the Hilbert scheme: in general, $C^{[l+m]}_{(0,0)}$ maps to $C^{[l]}_{(0,0)}$, with smooth fibers that are constant over the locus of ideals, with a fixed minimal number of generators. It is therefore possible to restate our conjecture in terms of Euler characteristics of loci with a fixed minimal number of generators [11]. Let us fix notations for the knot invariants. We use the normalization for the HOMFLY-PT polynomials of the link L from the paper [4]:

$$a\,\bar{P}(\nearrow\!\!\!\!\nwarrow) - a^{-1}\,\bar{P}(\nwarrow\!\!\!\!\nearrow) = (q - q^{-1})\,\bar{P}()(), \quad a - a^{-1} = (q - q^{-1})\bar{P}(\text{unknot})$$

The links $L_{C,(0,0)}$ that constitute the intersection of the curve C with the small 3-sphere around $(0,0)$ are called *algebraic*. When $E = x^n - y^m$, the link is called a torus link $T_{m,n}$.

Conjecture 1. [11] *Let* $\mu = \dim \mathbb{C}[[x,y]]/(\frac{\partial E}{\partial x}, \frac{\partial E}{\partial y})$ *be the Milnor number of the singularity at* $(0,0)$. *Then,*

$$\bar{P}(L_{C,(0,0)}) = (a/q)^{\mu-1} \sum_{l,m} q^{2l}(-a^2)^m \chi(C^{[l,\,l+m]}_{(0,0)})$$

Theorem 2. [11] *The conjecture holds in the following cases:*

- $a = -1$ *and* C *is any planar curve*
- $C = \{x^n = y^m\}$, *and* $C = \{(t^4, t^6 + t^7)\}$.

The first case of the theorem is closely related to the main result of [1]. Let us discuss possible strategies for the proof of the conjecture. It is known that an algebraic link is obtained by iterative application of cabling to the unknot: (s,r)-cabling K^r_s of a knot K is a knot that travels r times along K and s times along the meridian of the torus surrounding K. For example, the link of the curve $(t^4, t^6 + t^7)$ is obtained by $(2,13)$ cabling of the trefoil, and $T_{m,n}$ is the (m,n)-cabling of the unknot. The HOMFLY-PT invariant of K^r_s can be expressed in terms of colored invariants of the knot K (see below). Interestingly, the cabling procedure is in many regards similar to the procedure of 'thickening' of an algebraic curve C. Furthermore, the Hilbert schemes of points on the thickened curve are conjecturally related to the colored HOMFLY-PT invariants. These observations present a clear path to the proof of the conjecture, as discussed below.

Cabling for knot invariants. The specialization to $a = q^n$ of the colored HOMFLY-PT knot invariant $P_\lambda(L) \in \mathbb{Q}(q,a)$ is constructed by means of R-matrix $R \in End(V_\lambda, V_\lambda)$ [15] where V_λ is an irreducible finite-dimensional representation of $U_q(sl(n))$. Its value on the unknot is fixed to be the q-dimension of V_λ, and the usual HOMFLY-PT knot invariant corresponds to the case $\lambda = (1)$. The arguments from the paper [16], where the case $a = q^2$ was treated, can be extended to the general case:

Theorem 3. *Let* $(r,s) = 1$. *The HOMFLY-PT invariant of* (r,s)*-cabling of a knot* K *can be expressed as follows:*

$$\bar{P}_\nu(K^r_s)(q,a) = \sum_{\mu} q^{\frac{r}{s}c(\mu) - rsc(\nu)} a^{-r(s-1)|\nu|} C^{s;\nu}_{\mu} \bar{P}_\mu(K)(q,a)$$

Here $c(\lambda)=\sum_{(i,j)\in\lambda} i-j$ *and* $C^{s;\nu}_\mu$ *are given by the Schur function expansion:*

$$s_\nu(x_1^s,x_2^s,\dots)=\sum_\mu C^{s;\nu}_\mu s_\mu(x_1,x_2\dots)$$

Colored invariants and PT-spaces. When one searches for a moduli space that would match with the colored knot $\bar{P}_\lambda$ invariant, the first guess would be the moduli space of the ideals on the thick curve in $\mathbb{C}^3$

$$C_\lambda=\{E^{\lambda_1}(x,y)=0, zE^{\lambda_2}(x,y)=0,\dots\}$$

that has a fat point of shape λ as a generic cross-section. As it turns out, this moduli space doesn't quite do the job, but the following close cousin passes numerical tests. The moduli space $PT_n^\lambda(mC)_{(0,0)}$ consists of pairs of a pure sheaf F with support on C and map s surjective outside $(0,0)$ such that:

$$[\mathcal{O}_{\mathbb{C}^3}\xrightarrow{s}F]\in PT_n^\lambda(mC)_{(0,0)} \text{ iff } Ker(s)=(E^{\lambda_1},zE^{\lambda_2},\dots).$$

This moduli space appears naturally in the study of moduli spaces of pairs [12]: when one counts curves on a Calabi-Yau threefold that are homologous to $\beta\in H_2(Y)$, it is generally expected (and shown in some cases [13]) that the count is given in terms of so-called BPS states, which mathematically manifest themselves as topological invariants of the moduli spaces of sheaves on the singular curves that are homologous to β.

In the case when $\lambda=(m)$ we deal with sheaves on the fat but still planar curve mC. Thus Appendix B to [12] contains the proof of $PT_n^{(m)}(C)_{(0,0)}=(C^{[n]}_{(m)})_{(0,0)}$ where $C_{(m)}$ is a planar curve $E^m=0$, i.e. m-fattening of C. On the other hand, if $\lambda=(1^m)$, then one immediately sees the match with the m-step nested Hilbert scheme; and the case of general partition is a hybrid of these cases.

From the cabling formula, we see that for algebraic knot K there are unique powers $f(\lambda,K)$, $g(\lambda,K)$ such that $q^{f(\lambda,K)}a^{g(\lambda,K)}\bar{P}_\lambda(K)|_{a=q=0}=1$. We define the $sl(\infty)$ invariant by

$$\bar{P}^\infty_\lambda(K):=q^{f(\lambda,K)}a^{g(\lambda,K)}\bar{P}_\lambda(K)|_{a=0}.$$

Conjecture 4 (Oblomkov, Shende)**.** *If* $L_{C,(0,0)}$ *be a link of singularity of* C *at* $(0,0)$*, then*

$$\bar{P}^\infty_\nu(K)=\sum_n \chi(PT_n^\nu(C)_{(0,0)})q^{2n}.$$

One- and two-leg PT-vertex theory [14, 8] implies the conjecture for the unknot case and $T_{2,2}$ (Hopf link). In the case when C is given by $x^m=y^n$ we have $\mathbb{C}^*$ action on $P_n^\nu(C)_{(0,0)}$. I can show that the $\mathbb{C}^*$-fixed locus is a union of linear spaces: thus computation of the $\chi(P_n^\nu(C))$ is purely combinatorial. Meanwhile, we have an explicit formula for the colored invariants of torus knots and it should be possible to relate these combinatorial procedures. To prove the cabling formula for PT moduli spaces we need to understand how the topology of the moduli spaces changes when we vary curve in the family with the central element of the family being non-reduced curve.

Homological version of the conjecture. In this section we discuss the Poincaré polynomial of the triply graded HOMFLY homology $\overline{\mathrm{H}}^{i,j,k}(K)$ of Khovanov and Rozansky [7] of links of singularities of planar curves. We write their graded dimension as

$$\mathcal{P}(K) = \sum_{i,j,k} a^i q^j t^k \overline{\mathrm{H}}^{i,j,k}(K).$$

The specialization to the generating series of the Euler characteristics reproduces HOMFLY-PT invariant $\overline{P} = \mathcal{P}|_{t=-1}$. In other words the Khovanov-Rozansky homology is a much stronger isotopy invariant of the knot than the HOMFLY-PT invariant, in the same way as the homology of a topological space is a much richer invariant than the Euler characteristics. Notice, in particular, the very impressive recent result of Mrowka and Kronheimer [5] stating that the Khovanov homology [6] distinguishes the unknot. Besides being a stronger isotopy invariant, the knot homology satisfies various functorial properties.

Using the same convention as in our HOMFLY-PT conjecture, we have

Conjecture 5. [10]

$$(a/q)^{\mu-1} \sum_{l,m} q^{2l} a^{2m} t^{m^2} P^{vir}(C^{[l,l+m]}_{C,(0,0)}) = \mathcal{P}(L_{C,(0,0)}),$$

where P^{vir} is a virtual Poincaré polynomial.[1]

In the case when the curve admits a $\mathbb{C}^*$-action, we have derived a combinatorial formula for the algebro-geometric side of the conjecture [10]. The combinatorics of the Hilbert scheme is much easier then the combinatorics of the homological algebra underlying the definition of the Khovanov-Rozansky homology. In particular, programming an algorithm for the computation of the Khovanov-Rozansky homology of torus knots appears to be problematic and no guess for the $\mathcal{P}(T_{m,n})$ for $n > 3$ was available at the moment of appearance of our conjecture. On the other hand, we produce an explicit combinatorial formula for the knot invariant. Thus we were able to check our conjecture for knots $T_{2,n}$, $T_{3,n}$, and a few other torus knots $T_{m,n}$ where m, n are small and knot invariant computations are available. The conjecture above leads us to a conjecture relating representation theory of rational DAHA to the theory of the Khovanov-Rozansky homology of torus links [10, 3].

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[1]The virtual Poincaré polynomial P_t^{vir} is related to the usual Poincaré polynomial P_t as follows: $P_t^{vir}(X) = P_t(X)$ if X is smooth and projective; $P_t^{vir}(X \sqcup Y) = P_t^{vir}(X) + P_t^{vir}(Y)$.

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Compactified Jacobians and q, t-Catalan numbers

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(joint work with Mikhail Mazin)

Consider the plane curve singularity with one Puiseux pair (m, n). Its compactified Jacobian JC is defined (e.g. [1],[2]) as the moduli space of rank 1 degree 0 torsion free sheaves on a complete rational curve with this unique singularity. It has been shown in [8] that JC admits a pavement by the affine cells, and the dimensions of these cells were computed.

The combinatorics of this cell decomposition was studied in [6] and [7]. The cells $\Sigma(D)$ can be naturally labelled by the Young diagrams D in the $m \times n$ rectangle located below the diagonal, and the dimension of the $\Sigma(D)$ can be written combinatorially in terms of D. In particular, the Euler characteristic of JC equals to the number of such diagrams, which is known to be equal to the generalized Catalan number $\frac{(m+n-1)!}{m!n!}$.

Let

$$c_{m,n}(q,t) = \sum_{D} q^{\delta-|D|} t^{\delta-\dim\Sigma(D)},$$

where $\delta = (m-1)(n-1)/2$. It is proved in [6] that for $m = n+1$, the polynomial $c_{n,n+1}(q,t)$ coincides with the q,t-Catalan number $c_n(q,t)$ introduced by A. Garsia and M. Haiman in [3] (see also [4],[5]). In particular, it is symmetric in q and t. In [7] we conjecture that the symmetry

$$c_{m,n}(q,t) = c_{m,n}(t,q) \tag{1}$$

holds for all coprime m and n. We also study the weaker form of this identity:

$$c_{m,n}(q,1) = c_{m,n}(1,q). \tag{2}$$

We prove that (1) holds for $\min(m,n) \leqslant 3$ and in the case $m = n+1$ described above, and (2) holds for $m = kn \pm 1$. As a corollary from (2), for $m = kn \pm 1$ we prove a surprisingly easy formula for the Poincaré polynomial of the compactified Jacobian:

$$P_{JC}(t) = \sum_D t^{2|D|},$$

where the summation is made over the Young diagrams D in $m \times n$ rectangle located below the diagonal.

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Some aspects of the connection between toric geometry and resolution of singularities

Bernard Teissier

We know from [2] that normal toric varieties over a field admit (non embedded) resolutions of singularities described by the regular refinements of their fan. The toric *embedded* resolution of singularities for affine toric varieties over an algebraically closed field k was proved in [3] and [5]. The combinatorics works as follows: an affine toric variety $X_0 \subset \mathbf{A}^N(k)$ over k is defined by a prime binomial ideal $I_0 = (u^{m^\ell} - \lambda_\ell u^{n^\ell})_{\ell \in L}$ in $k[u_1, \ldots, u_N]$. The monomial u^m corresponds to a point m in the lattice $M \simeq \mathbf{Z}^N$, and $\lambda_\ell \in k^*$. The vectors $m^\ell - n^\ell \in M$ determine dual hyperplanes H_ℓ in the real vector space $N_{\mathbf{R}}$ generated by the dual lattice $N \simeq \check{\mathbf{Z}}^N$

of M. The intersections with the first quadrant of these hyperplanes determine a fan Σ_0 subdividing the fan whose maximal cone is the first quadrant. The strict transform of X_0 by the corresponding birational map $\pi(\Sigma_0)\colon Z(\Sigma_0)\to \mathbf{A}^N(k)$ of normal toric varieties is the normalization of X_0. The strict transform of X_0 by a birational toric map $\pi(\Sigma)\colon Z(\Sigma)\to \mathbf{A}^N(k)$ corresponding to a regular fan Σ subdividing Σ_0 is non singular and transversal to the toric boundary. Such subdivisions provide embedded pseudo[1] resolutions of X_0. The fan Σ can be chosen so as to contains the regular faces of the *weight cone* $\beta=\mathbf{R}^N_{\geqslant 0}\cap(\bigcap_\ell H_\ell)$, and then $\pi(\Sigma)$ is an embedded resolution.

One may wonder whether such toric maps also (pseudo) resolve the spaces obtained by suitable deformations of the binomial equations. This question comes from the basic observation of [5]: Given a local integral domain R with maximal ideal m and a rational valuation of R corresponding to an inclusion $R\subset R_\nu$ of R in a valuation ring R_ν of its field of fractions, such that $m_\nu\cap R=m$ and $R/m\to R_\nu/m_\nu$ is an isomorphism, we have a faithfully flat specialization of Spec$R$ to the affine toric variety (which may be of infinite embedding dimension) corresponding to the associated graded ring $\mathrm{gr}_\nu R=\bigoplus_{\varphi\in\Phi}\mathcal{P}_\varphi/\mathcal{P}^+_\varphi$ of R with respect to the filtration associated to ν, where $\mathcal{P}_\varphi=\{x\in R|\nu(x)\geqslant\varphi\}$, $\mathcal{P}^+_\varphi=\{x\in R|\nu(x)>\varphi\}$. The fact that ν is a rational valuation implies that $\mathrm{gr}_\nu R$ is a k-algebra and each homogeneous component is a vector space of dimension 1 over k. There is therefore a presentation $\mathrm{gr}_\nu R=k[(U_i)_{i\in I}]/(U^{m^\ell}-\lambda_\ell U^{n^\ell})_{\ell\in L}$ where U^m denotes a monomial, $\lambda_\ell\in k^*$, the sets I and L may be infinite, but countable.

We note that the degrees which actually appear in the graded algebra are the valuations of the elements of R, which form a subsemigroup of the semigroup $\Phi_+\cup\{0\}=(R_\nu\backslash\{0\})^{mult.}/\{units\}$ of non negative elements of the (totally ordered) value group Φ of ν. In fact $\mathrm{gr}_\nu R$ is isomorphic to the semigroup algebra over k of the semigroup $\Gamma=\nu(R\setminus\{0\})$. If R is noetherian the semigroup Γ is well ordered and therefore has a unique minimal system of generators, indexed by an ordinal, which is at most ω^h where h is the (archimedian, or real) rank of the value group. By transfinite induction one defines γ_{i+1} as the smallest non zero element of Γ which is not in the semigroup generated by the previous ones.

Let us concentrate on the case where the semigroup Γ is finitely generated and R is a local equicharacteristic and complete noetherian domain with an algebraically closed residue field k. Pick and fix a field of representatives $k\subset R$. Then R appears as an *overweight* deformation of its associated graded ring, in the sense of [6]: there is a continuous and surjective map of k-algebras

$$k[[u_1,\dots,u_N]]\xrightarrow{\pi}R,\ \text{determined by}\ \ u_i\mapsto\xi_i,$$

for any choice of elements $\xi_i\in R$ whose valuations are the minimal generators of the semigroup Γ or equivalently are such that their initial forms minimally generate the k-algebra $\mathrm{gr}_\nu R$. Giving to u_i the weight $\gamma_i=\nu(\xi_i)\in\Gamma\subset\Phi_+\cup\{0\}$ determines a weight w on $k[[u_1,\dots,u_N]]$, with its filtration by weight and the associated graded ring $\mathrm{gr}_w k[[u_1,\dots,u_N]]\simeq k[U_1,\dots,U_N]$, now graded by the weight: $\deg U_i=\gamma_i$.

[1] This means that the restriction over the non singular part is not necessarily an isomorphism.

Moreover the valuation ideals of R are the images by π of the weight ideals of $k[[u_1,\ldots,u_N]]$ and so the map π induces a surjection of graded k-algebras

$$k[U_1,\ldots,U_N] \xrightarrow{\mathrm{gr}_w\pi} \mathrm{gr}_\nu R, \text{ determined by } \; U_i \mapsto \mathrm{in}_\nu \xi_i,$$

whose kernel is a binomial ideal $(u^{m^\ell}-\lambda_\ell u^{n^\ell})_{\ell\in L}$; it is essentially the presentation of the semigroup algebra of Γ over k which corresponds to an affine toric variety X_0. By flatness the kernel of π is generated by series $F_\ell = u^{m^\ell}-\lambda_\ell u^{n^\ell}+\sum_p c_p^{(\ell)}u^p$ with $c_p^{(\ell)}\in k,\ w(u^p)>w(u^{m^\ell})=w(u^{n^\ell})$, for $\ell\in L$, a finite set. Let us call X the formal subspace of $\mathbf{A}^N(k)$ defined by the ideal $I=(F_\ell)_{\ell\in L}$; it is an *overweight deformation* of the affine toric variety X_0.

For a regular fan Σ with support the first quadrant of $\check{\mathbf{R}}^N$, the corresponding birational toric map $Z(\Sigma)\to\mathbf{A}^N(k)$ is described in each chart $Z(\sigma)$ corresponding to a maximal cone $\sigma=\langle a^1,\ldots,a^N\rangle$ of Σ, where $a^j\in N$, by

$$\begin{array}{l} u_1 = y_1^{a_1^1}\ldots y_N^{a_1^N} \\ \cdot \qquad \cdot \\ \cdot \qquad \cdot \\ \cdot \qquad \cdot \\ u_N = y_1^{a_N^1}\ldots y_N^{a_N^N} \end{array}$$

and the valuation ν of R picks a point in the strict transform of X. A combinatorial argument explained in [8] shows that one can find regular fans Σ subdividing the fan Σ_0 corresponding to the initial binomials of the F_ℓ, and such that for appropriate $\sigma\in\Sigma$ the transforms of the F_ℓ can be written

$$F_\ell\circ\pi(\sigma)=$$
$$y_1^{\langle a^1,n^\ell\rangle}\ldots y_N^{\langle a^N,n^\ell\rangle}(y_1^{\langle a^1,m^\ell-n^\ell\rangle}\ldots y_N^{\langle a^N,m^\ell-n^\ell\rangle}-\lambda_\ell+\textstyle\sum_p c_p^{(\ell)}y_1^{\langle a^1,p-n^\ell\rangle}\ldots y_N^{\langle a^N,p-n^\ell\rangle}).$$

The point is to find fans for which the inequalities $w(u^p)>w(u^{n^\ell})$ induce inequalities $\langle a^i,p-n^\ell\rangle>0$. The largest torus-invariant charts of $Z(\Sigma)$ in which the strict transform meets the toric boundary correspond to cones σ of Σ whose intersection with the weight cone β is of maximal dimension $r=\mathrm{dim}R$. The variables $y_{i_j},\ 1\leqslant j\leqslant r$ corresponding to the vectors $a^{j_i}\in\beta$ do not appear in the transformed binomials $y_1^{\langle a^1,m^\ell-n^\ell\rangle}\ldots y_N^{\langle a^N,m^\ell-n^\ell\rangle}-\lambda_\ell$ and can be taken as local coordinates on the strict transform of X. In fact, *at the point picked by the valuation*, this strict transform is a deformation of the strict transform of X_0 and hence non singular. In summary:

Theorem 1. *Given a rational valuation ν on a complete equicharacteristic local domain R with an algebraically closed residue field k, if the semigroup of values $\nu(R\setminus\{0\})$ is finitely generated, say by N generators, there is a continuous surjection $k[[u_1,\ldots,u_N]]\xrightarrow{\pi} R$ such that some of the toric modifications of $\mathbf{A}^N(k)$ in the coordinates u_i which resolve the singularities of the toric variety corresponding to $\mathrm{gr}_\nu R$ also produce an embedded local uniformization of the valuation ν on the space $X\subset\mathbf{A}^N(k)$ corresponding to R.*

In the situation of the theorem, by flatness of the deformation, the valuation ν is Abhyankar, which means in this case that the Abhyankar inequality $\dim \mathrm{gr}_\nu R \leqslant \dim R$ (see [5]) is an equality. Since local uniformization for Abhyankar valuations of algebraic function fields has been proved by Knaf and Kuhlmann in [4], it is natural to ask whether in general the Abhyankar condition implies that the semigroup Γ is finitely generated. An attempt to prove this is in progress. Combined with the theorem above it would have as consequence that the Abhyankar valuations are exactly the quasi-monomial ones, a fact proved by Cutkosky for valuations of rank one using embedded resolution of singularities (see [1], Prop. 2.8).

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The Nash problem for surfaces

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(joint work with María Pe Pereira)

1. Introduction

Nash problem [10] was formulated in the sixties (but published later) in the attempt to understand the relation between the structure of resolution of singularities of an algebraic variety X over a field of characteristic 0 and the space of arcs (germs of parametrized curves) in the variety. He proved that the space of arcs centred at the singular locus (endowed with an infinite-dimensional algebraic variety structure) has finitely many irreducible components and proposed to study the relation of these components with the essential irreducible components of the exceptional set of a resolution of singularities.

An irreducible component E_i of the exceptional divisor of a resolution of singularities is called *essential*, if given any other resolution the birational transform of E_i to the second resolution is an irreducible component of the exceptional divisor. Nash defined a mapping from the set of irreducible components of the space of arcs centred at the singular locus to the set of essential components of a resolution as follows: he assigns to each component W of the space of arcs centred at the singular locus the unique component of the exceptional set which meets the lifting of a generic arc of W to the resolution. Nash established the injectivity of this mapping. For the case of surfaces it seemed plausible for him that the mapping is also surjective, and posed the problem as an open question. He also proposed to study the mapping in the higher dimensional case. Nash resolved the question positively for the surface singularities of type A_k. As a general reference for Nash problem the reader may look at [10] and [6].

Ishii and Kollar showed in [6] a 4-dimensional example with non-bijective Nash mapping. Very recently there have appeared 3-dimensional counterexamples as well. The first ones are due to T. de Fernex [1]. Later J. Kollar showed even simpler counterexamples [7]: even the A_4-threefold singularity, defined by the equation $x^2 + y^2 + z^2 + w^5 = 0$ is a counterexample. In the same paper he proposes a revised higher dimensional conjecture.

On the positive side, recently, the author of this report and M. Pe Pereira have resolved affirmatively Nash question for surfaces [4]:

Main Theorem. *Nash mapping is bijective for any surface defined over an algebraically closed field of characteristic* 0*.*

The core of the result is the case of normal surface singularities. After settling this case it is not so difficult to deduce from it the general surface case.

The proof is based on the use of convergent wedges and topological methods. A wedge is a uniparametric family of arcs. The use of wedges in connection to Nash problem was proposed by M. Lejeune-Jalabert [8].

2. Sketch of the proof

The idea of our proof is as follows: let (X, O) be a normal surface singularity and

$$\pi : \tilde{X} \to (X, O)$$

be the minimal resolution of singularities. Let $E = \cup_i E_i$ its decomposition in irreducible components.

Given any irreducible component E_i we define by N_{E_i} the Zariski closure in the arc space of X of the set of non-constant arcs whose lifting to the resolution is centered at E_i. We say that there is an adjacency from E_j to E_i in $N_{E_i} \subset N_{E_j}$. Nash conjecture consists in proving that there are no non-trivial adjacencies.

A wedge is a morphism

$$\alpha : Spec(\mathbb{C}[[t, s]]) \to (X, O).$$

Its special arc is

$$\alpha(t,0) : Spec(\mathbb{C}[[t]]) \to (X,O),$$

and its generic one is alpha itself, but viewed as

$$\alpha : Spec(\mathbb{C}((S))[[t]]) \to (X,O).$$

A wedge realises an adjacency $N_{E_i} \subset N_{E_j}$ if its generic arc belongs to E_j and its special one lifts to the resolution trasversely to E_i at a non singular point of E_i.

The starting point of the proof of Nash conjecture for surfaces is the following Theorem, which is the implication " (1) $\Rightarrow$ (a) " of Corollary B of [3]:

Theorem 1 ([3]). *An essential divisor E_i is in the image of the Nash mapping if there is no other essential divisor $E_j \neq E_i$ such that there exists a* convergent *wedge realizing an adjacency from E_j to E_i.*

As in [11], taking a suitable representative we may view α as a uniparametric family of mappings

$$\alpha_s : \mathcal{U}_s \to (X,O)$$

from a family of domains $\mathcal{U}_s$ to X with the property that each $\mathcal{U}_s$ is diffeomorphic to a disk. For any s we consider the lifting

$$\tilde{\alpha}_s : \mathcal{U}_s \to \tilde{X}$$

to the resolution. Notice that $\tilde{\alpha}_s$ is the normalization mapping of the image curve.

On the other hand, if we denote by Y_s the image of $\tilde{\alpha}_s$ for $s \neq 0$, then we may consider the limit divisor Y_0 in $\tilde{X}$ when s approaches 0. This limit divisor consists of the union of the image of $\tilde{\alpha}_0$ and certain components of the exceptional divisor of the resolution whose multiplicities are easy to be computed. We prove an upper bound for the Euler characteristic of the normalization of any reduced deformation of Y_0 in terms of the following data: the topology of Y_0, the multiplicities of its components and the set of intersection points of Y_0 with the generic member Y_s of the deformation. Using this bound we show that the Euler characteristic of the normalization of Y_s is strictly smaller than one. This contradicts the fact that the normalization is a disk.

The proof of Theorem 1 has two parts. The first consists of proving that if there is an adjacency then there exists a *formal wedge*

$$\alpha : Spec(\mathbb{C}[[t,s]]) \to (X,O)$$

realising the adjacency. For that, firstly it is used a Theorem of A. Reguera [12] which produces wedges defined over large fields. Then a specialisation argument is performed to produce a wedge defined over the base field $\mathbb{C}$. This was done independently in [9]. The second part is an argument based on D. Popescu's Approximation Theorem, which produces the convergent wedge from the formal one. In [5] the author of the report and M. Pe Pereira paper give an alternative proof of the first part giving in one step a formal wedge defined over $\mathbb{C}$.

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Dimer models and crepant resolutions

Akira Ishii

(joint work with Kazushi Ueda)

1. Dimer models and moduli spaces

Let $T = \mathbb{R}^2/\mathbb{Z}^2$ be a real two-torus equipped with an orientation. A *dimer model* on T consists of

- a finite set $B \subset T$ of black nodes,
- a finite set $W \subset T$ of white nodes, and
- a finite set E of edges, consisting of embedded closed intervals e on T

such that

- one boundary of an edge belongs to B, and the other boundary belongs to W,
- two edges intersect only at the boundaries,
- every node is contained in at least two edges, and
- every connected component of $T \setminus \cup_{e \in E} e$ is simply connected.

1.1. **Lattice polygon from a dimer model.** Suppose a dimer model (B, W, E) is given. There is a lattice polygon associated with (B, W, E) constructed as follows.

Definition 1. *A* perfect matching *is a subset $D \subset E$ such that for every node, there is a unique edge in D containing it.*

We can measure the distance of two perfect matchings as follows. Consider the orientation of an edge which goes from black to white. If we regard a perfect matching as a 1-chain on T, then the difference of two perfect matchings becomes a 1-cycle. The height change of two perfect matchings D, D' is defined as

$$h(D, D') := [D - D'] \in H_1(T, \mathbb{Z}) \cong H^1(T, \mathbb{Z}) \cong \mathbb{Z}^2.$$

We fix a reference perfect matching D_0 and let Δ be the convex hull of the set

$$\{h(D, D_0) \mid D \text{ is a perfect matching}\} \subset \mathbb{R}^2.$$

Put Δ in $\mathbb{R}^2 \times \{1\} \subset \mathbb{R}^3$ and take the cone $C(\Delta)$ over Δ. Then we can consider the Gorenstein affine toric variety $X_{C(\Delta)}$ associated with $C(\Delta)$.

1.2. **Quiver with relations from a dimer model.** We can also construct a quiver with relations from (B, W, E). This is done by taking the dual: The set V of vertices is the set of connected components of $T \setminus \cup_{e \in E} e$. The set A of arrows is the set E of edges of the dimer model, where the oriention of an arrow is determined so that the white node is on the right of the arrow.

The relations of the quiver are described as follows: For an arrow $a \in A$, there exist two paths $p_+(a)$ and $p_-(a)$ from $t(a)$ to $s(a)$, the former going around the white node connected to $a \in E = A$ clockwise and the latter going around the black node connected to a counterclockwise. Then the ideal of the path algebra is generated by $p_+(a) - p_-(a)$ for all $a \in A$.

We consider the moduli space of representations of this quiver with relations with respect to the dimension vector $(1, 1, \dots, 1)$. The moduli space depends on a stability parameter θ and is denoted by $\mathcal{M}_\theta$.

1.3. **Non-degenerate dimer models.**

Definition 2. *A dimer model is* non-degenerate *if for every edge $e \in E$, there exists a perfect matching D which contains e.*

A stability parameter θ is *generic* if θ-stability coincides with θ-semistability.

Theorem 3 ([7])**.** *If a dimer model is consistent and θ is generic, then $\mathcal{M}_\theta$ is a crepant resolution of $X_{C(\Delta)}$.*

Example 4. *Suppose that a dimer model is given by a tesselation of T by regular hexagons of the same size. Then the associated polygon Δ is a triangle and we have $X_{C(\Delta)} \cong \mathbb{C}^3/G$, where G is a finite abelian subgroup of* $\mathrm{SL}(3, \mathbb{C})$*. The associated quiver is the McKay quiver for G, whose path algebra modulo relations is Morita equivalent to $G\#\mathbb{C}[x, y, z]$. The moduli space $\mathcal{M}_\theta$ coindides with the Hilbert scheme of G-orbits for a suitable choice of the stability parameter θ. The above theorem* *is a generalization of Nakamura's theorem which states that G-Hilb is a crepant resolution of* $\mathbb{C}^3/G$ *for a finite abelian subgroup of* $\mathrm{SL}(3,\mathbb{C})$ [12].

2. Derived equivalences

2.1. **Consistent dimer models.** In the McKay case, we have a derived equivalence $D^b(\operatorname{coh}\mathcal{M}_\theta) \cong D^b(G\#\operatorname{mod}\mathbb{C}[x,y,z])$ established by Bridgeland, King and Reid[2]. For a dimer model, non-degeneracy is not enogh to generalize this result and we need the notion of consistency.

We need some space to state the definition of consistency ([9], [1]) and we omit the precise definition here. We note that it is equivalent to the non-degeneracy plus the cancellation property of the path algebra modulo the relations.

Theorem 5 ([8])**.** *If a dimer model is consistent, it is non-degenerate and the universal representation induces an equivalence*

$$D^b(\operatorname{coh}\mathcal{M}_\theta) \cong D^b(\operatorname{mod}\mathbb{C}\Gamma) \tag{1}$$

where $\mathbb{C}\Gamma$ *is the path algebra of the quiver modulo the relations.*

See also [11], [3] and [5]. Note that Gulotta[6] constructs a consistent dimer model for an arbitrary convex lattice polygon Δ.

2.2. **Induction on Δ.** The basic strategy in [8] uses an induction on the lattice polygon. If Δ is a basic triangle, then we can see that $\mathcal{M}_\theta \cong \mathbb{C}^3$ and $\mathbb{C}\Gamma \cong \mathbb{C}[x,y,z]$, where (1) is trivial. This is the first step of the induction. Suppose that Δ is not a basic triangle. Take a vertex D of the polygon Δ (which we call a corner) and consider the convex hull of $\Delta \cap \mathbb{Z}^2 \setminus D$. We can regard D as a perfect matching of G and we can choose a subset S of D such that $G' = (B, W, E \setminus S)$ is a consistent dimer model which determines Δ'. Here, we use the special McKay correspondence of Wunram-Riemenschneider [13] to choose the subset $S \subset D$. Moreover, we can show

Proposition 6. *We can choose generic stability parameters* θ *for* G *and* θ' *for* G' *so that the equivalence* (1) *holds for* G *and* θ *if and only if it holds for* G' *and* θ'.

2.3. **Variation of moduli spaces.** To make Proposition 6 work as the induction step, we show

Theorem 7 ([10])**.** *Let* G *be a consistent dimer model. If* (1) *holds for one generic stability parameter, it holds for any generic stability parameter.*

This follows from arguments of [2] but we can directly prove this by looking at the chamber structure for the parameter space. As a corollary of the arguments, we can also prove the following generalization of [4].

Theorem 8 ([10])**.** *Let* G *be a consistent dimer model and* Δ *the associated lattice polygon. Then for any projective crepant resolution* Y *of* $X_{C(\Delta)}$*, there is a genreric stability parameter* θ *such that* $Y \cong \mathcal{M}_\theta$.

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Topological recursion for the genus zero descendant Hurwitz potential

Maxim Kazarian

(joint work with Sergey Lando, Dmitry Zvonkine)

The Hurwitz numbers enumerate the number of possible ways to represent a given permutation as the product of a given number of transpositions. In topological terms, they describe the number of topologically distinct meromorphic functions on a Riemann surface of given genus with prescribed critical values and prescribed behavior at poles. In the case when the surface has genus zero, a closed formula for these numbers was proposed by Hurwitz a century ago. His arguments were algebraic and based on the study of combinatorics of the permutation group.

We propose a new recursion for Hurwitz numbers which has topological origin: it is derived form the cohomological information contained in the stratification of the Hurwitz space by the multisingularity types possessed by the functions. It appeared as a result of our research project developed in [1, 2, 3]. We expect that variations of this approach could be adopted to other families of Hurwitz numbers for which closed formulas are not known at the moment.

The compactification of the space of genus zero meromorphic functions is smooth (in contrast with the case of higher genera). The local singularities of functions provide a stratification of the space of functions which can be studied by the methods of singularity theory: the classification of possible singularities has its own adjacencies, normal forms, versal deformations etc. The new feature comparing with the classical classifications in singularity theory is the appearance of nonisolated singularities that have to be included into the classification of possible local degenerations — these singularities are attained on singular curves if the function is constant on one of its components.

The information about ajacencies of singularity strata is converted to the cohomological relations between the classes represented by these strata which, in turn, can be reformulated as a relation between the corresponding Hurwitz numbers.

All the obtained relations between the studied numbers are written as a partial differential equation on the generating function for these numbers. The generating function is denoted by $\mathcal{Y}$. It is an infinite power series in an infinite number of variables q and $t_{\lambda,\nu}$, $\lambda \geqslant 0$, $\nu \geqslant 0$. The Taylor coefficient of the monomial $q^n t_{\lambda_1,\nu_1} \dots t_{\lambda_\ell,\nu_\ell}$ is a certain cohomological invariant (the so called degree) associated to the space of degree n rational functions with n simple marked poles having zeroes of order $\lambda_1, \dots, \lambda_\ell$ at another ℓ marked points, and ν_i's are the powers of the so called ψ-classes attached to these points (the adjective 'descendant' in the name of the potential refers to the presence of these ψ-classes).

Theorem. *The series $\mathcal{Y}$ obeys the following differential equations valid for any $m \geqslant 0$ and $s \geqslant 0$:*

$$\frac{\partial \mathcal{Y}}{\partial t_{m,s+1}} = \frac{\partial \mathcal{Y}}{\partial t_{m,s}} + s\,\frac{\partial \mathcal{Y}}{\partial t_{m+1,s}} - \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \sum_{\sigma_1,\dots,\sigma_\ell} \frac{\partial \Psi_{\ell,|\sigma|-s}}{\partial t_{m,0}} \prod_{i=1}^{\ell} \sigma_i \frac{\partial \mathcal{Y}}{\partial t_{0,\sigma_i}},$$

where $\Psi_{\ell,a}$ is the explicitly given series

$$\Psi_{\ell,a} = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\substack{\nu_1+\dots+\nu_k=\ell+k-3 \\ \lambda_1+\dots+\lambda_k=a}} \binom{|\nu|}{\nu_1,\dots,\nu_k} \prod_{i=1}^{k} t_{\nu_i,\lambda_i}.$$

As long as we can see, these kind of equations has never appeared before. It would be an interesting problem to relate it to some known equations of integrable hierarchies appearing in modern mathematical physics.

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Comparison theorems for deformation functors

Jan Arthur Christophersen
(joint work with Jan Kleppe)

This is work in progress regarding comparisons of deformations of algebras to deformations of schemes via invariant theory. We generalize comparison theorems of Kleppe and Schlessinger for projective schemes. We consider deformation functors for a scheme X which is a good quotient of a quasi-affine scheme X' by a linearly reductive group G and compare them to invariant deformations of any affine G-scheme containing X' as an open invariant subset.

Given a projective scheme X defined by equations $f_1, \dots, f_m \in k[x_0, \dots, x_n]$, perturbing the equations in a flat manner so that they remain homogeneous induce deformations of X. In practice this is often the only way to construct examples of deformations. In more stringent terms we have a map between the degree 0 embedded deformations of the affine cone $C(X)$ and deformations of X in $\mathbb{P}^n$. If we take into account trivial deformations we get a map to the deformations of X as scheme. The question is when do we get all deformations this way.

In terms of deformation functors on Artin rings, if $R = k[x_0, \dots, x_n]$ and $S = R/(f_1, \dots, f_m)$ then the above describes maps $\mathrm{Def}^0_{S/R} \to \mathrm{Hilb}_{X/\mathbb{P}^n}$ where $\mathrm{Def}^0_{S/R}$ is the functor of degree 0 deformations of S as R-algebra and $\mathrm{Def}^0_S \to \mathrm{Def}_X$ where Def^0_S is the functor of degree 0 deformations of S as k-algebra. In [1] the second author gave exact conditions for when these maps are isomorphisms. The goal of our research is to generalize these to other situations where one can compare deformations of algebras to deformations of schemes.

The comparison map for projective schemes factors through deformations of the open subset of $C(X)$ where the vertex $\{0\}$ is removed. Thereafter one compares deformations to $X = (C(X) \setminus \{0\})/k^*$ via the quotient map. We generalize this to schemes X which are good quotients of a quasi-affine scheme X' by a linearly reductive group G.

We assume that $X' \subseteq \operatorname{Spec} S$ and that G acts on S inducing the action on X'. We can then compare Def^G_S to Def_X where Def^G_S is the functor of invariant deformations of S. If this situation is embedded in another one we can compare with the local Hilbert functor as well. The main examples are closed subschemes of toric varieties corresponding to ideals in the Cox ring, but the group need not be by a quasi-torus. Thus many moduli constructions serve as examples. This generality allows us also to say something about affine schemes like quotient singularities as well.

Linearly reductive groups have many properties coming from the Reynolds operator which make it possible to prove things, e.g. taking invariants is exact. Another reason to work with them is that the functor of invariant deformations is well defined and has the usual nice properties of a good deformation theory. This was proven by Rim in [2].

Our main result on the local Hilbert functor is too technical to state here but we introduce depth conditions along the complement of X' in $\operatorname{Spec} S$ and along the locus where the quotient map is not a G-bundle that imply that the above comparison maps are isomorphisms. As corollaries we have precise statements for subschemes of toric varieties and weighted projective space.

For the abstract deformation functor Def_X the results are not as exact due to the presence of infinitesimal automorphisms. It is not clear what the correct assumptions should be but we found it useful to use results of Altmann regarding rigidity of $\mathbb{Q}$-Gorenstein toric singularities as a guide. We get depth conditions as above but also along the locus where the isotropy groups are not finite. An important ingredient are what we call a set of Euler derivations coming from the Lie algebra of G. This was explained to us by Dmitry Timashev and made it possible to work with general groups and not just tori which we had originally studied.

We can apply these results to rigidity questions for toric varieties. All though we are at the moment only able to reprove known results of Altmann and Totaro we believe the techniques will lead to new applications.

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Towards transversality of singular varieties: splayed divisors

Eleonore Faber

In this talk we present a natural generalization of transversally intersecting smooth hypersurfaces in a complex manifold: hypersurfaces, whose components intersect in a transversal way but may be themselves singular. We call these hypersurfaces "splayed" divisors. Splayed divisors are characterized by certain properties of their Jacobian ideals. They can also be characterized in terms of K. Saito's logarithmic derivations. As applications we consider the relation of splayed divisors with free and normal crossing divisors and consider certain properties of their Chern classes. This talk contains joint work with Paolo Aluffi (Florida State University).

The geometric idea for the generalization is that two singular hypersurfaces D_1 and D_2 in a complex manifold S intersect "transversally" at a point p if their "tangent spaces" fill out the whole space and the ideal of their intersection is reduced. The notion of tangent space for singular hypersurfaces can be made precise by means of K. Saito's logarithmic derivations [11]: if a divisor D in a complex manifold S of dimension n is locally at a point $p = (x_1, \ldots, x_n)$ given by $D = \{f(x) = 0\}$, then the $\mathcal{O}_{S,p}$-module of logarithmic derivations (along D) is defined as

$$\mathrm{Der}_{S,p}(\log D) = \{\delta \in \mathrm{Der}_{S,p} : \delta(f) \in (f)\mathcal{O}_{S,p}\}.$$

We say that two divisors D_1 and D_2 are *splayed* at p if their equations may be written in terms of disjoint sets of analytic coordinates at that point. Denote by

$D = D_1 \cup D_2$ their union. Then D is called a *splayed divisor* if D_1 and D_2 are splayed at any point of their intersection. For an example of a splayed divisor D in a threedimensional S, see fig. 1.

FIGURE 1. $D = \{x(y^2 - z^3) = 0\}$ (left) is splayed and $D' = \{x(x + y^2 - z^3) = 0\}$ (right) is not splayed.

We show that being splayed is equivalent to the fact that the logarithmic derivations along D_1 and D_2 satisfy the equation

$$\mathrm{Der}_{S,p}(\log D_1) + \mathrm{Der}_{S,p}(\log D_2) = \mathrm{Der}_{S,p},$$

which corresponds to the definition of transversal intersection of two submanifolds of S (see [7, Prop. 15]).

Several other characterizations of splayed divisors are discussed, see [7, 4]: consider the Jacobian ideals (the ideals generated by the partial derivatives of the defining equations) of $D_1 = \{g(x) = 0\}, D_2 = \{h(x) = 0\}$ and $D = \{gh(x) = 0\}$, which are denoted by J_g, J_h and J_{gh}, respectively. It is clear that for a splayed D, the Jacobian ideal satisfies

$$(gh, J_{gh}) = g(h, J_h) + h(g, J_g),$$

when the defining equations g and h are chosen in separated variables. We show that this *Leibniz property* already characterizes splayed divisors.

Further, given two divisors D_1, D_2 meeting at a point p and without common components, there is a natural monomorphism

$$\frac{\mathrm{Der}_{S,p}}{\mathrm{Der}_{S,p}(-\log(D_1 \cup D_2))} \hookrightarrow \frac{\mathrm{Der}_{S,p}}{\mathrm{Der}_{S,p}(-\log(D_1))} \oplus \frac{\mathrm{Der}_{S,p}}{\mathrm{Der}_{S,p}(-\log(D_2))}$$

involving quotients of modules of logarithmic derivations. The divisors D_1 and D_2 are splayed at p if and only if this monomorphism is an isomorphism (Theorem 2.4 of [4]). We also mention an analogous statement involving sheaves of *logarithmic differentials* (Theorem 2.12 of [4]) giving a partial answer to a question raised in [7], but only subject to the vanishing of an Ext module: D_1 and D_2 are splayed at a point p if the natural inclusion

(1) $$\Omega^1_{S,p}(\log D_1) + \Omega^1_{S,p}(\log D_2) \subseteq \Omega^1_{S,p}(\log D)$$

is an equality and $\mathrm{Ext}^1_{\mathcal{O}}(\Omega^1_{S,p}(\log D), \mathcal{O}) = 0$. Thus, if D is free at p, then D_1 and D_2 are splayed at p if and only if the two modules in (1) are equal. In general this condition alone does not imply splayedness, as the example of the union of a cone and a plane in three-space shows.
We also mention an intrinsic characterization of splayedness by M. Schulze (see [13] or Remark 2.17 in [4]) in terms of logarithmic residues.

Finally we give some applications of splayed divisors (see [7, 8]): first we comment on the relationship between splayed divisors and free divisors. Free divisors are a generalization of normal crossing divisors and appear frequently in different areas of mathematics, e.g., in deformation theory as discriminants or in combinatorics as free hyperplane arrangements, see e.g. [1, 10, 5, 9, 6, 12] for more examples. Then we give a partial answer to a question of H. Hauser about the characterization of normal crossing divisors by their Jacobian ideals: it is shown that if $D = \bigcup_{i=1}^n D_i$ is locally the union of smooth irreducible components then D has normal crossings if and only if D is locally free and its Jacobian ideal is radical. We briefly sketch that the Hilbert–Samuel polynomial $\chi_{D,p}$ of a splayed divisor $(D,p) = (D_1,p) \cup (D_2,p)$ satisfies the natural additivity condition

$$\chi_{D,p}(t) = \chi_{D_1,p}(t) + \chi_{D_2,p}(t) - \chi_{D_1 \cap D_2,p}(t).$$

As another application in the direction of transversal intersection, one can consider implications for different notions of *Chern classes* associated with divisors: the characterizing conditions for splayed divisors in terms of their logarithmic derivations globalize nicely, and give conditions on morphisms of *sheaves* of logarithmic derivations and differentials characterizing splayedness at all points of intersection of two divisors. These conditions imply identities involving Chern classes for these sheaves (Corollary 2.20 of [4]). For curves on surfaces these identities actually characterize splayedness. Also, there is a different notion of 'Chern class' that can be associated with a divisor D in a nonsingular variety V, namely the Chern-Schwartz-MacPherson (c_{SM}) class of the complement $V \smallsetminus D$. In previous work, P. Aluffi has determined several situations where this c_{SM} class *equals* the Chern class $c(\mathrm{Der}_V(-\log D))$ of the sheaf of logarithmic differentials, see [2, 3]. It is then natural to expect that c_{SM} classes of complements of splayed divisors, and more general subvarieties, should satisfy a similar type of relations as the one obtained for ordinary Chern classes of sheaves of derivations. One can show that for subvarieties defined by pullbacks from the factors of a product, joins of projective varieties and in the case of curves, the corresponding expected relation of c_{SM} classes does hold, see [4]. We hope to prove the validity of this relation for arbitrary splayed subvarieties in the future.

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Simple surface singularities

Jan Stevens

Simple hypersurface singularities were classified by Arnol'd in the famous ADE list. In the surface case these are exactly the rational double points. In Giusti's list of simple isolated complete intersection singularities no surface singularities occur. In the classification of simple determinantal codimension two singularities by Frühbis-Krüger and Neumer [2] the surface singularities are the rational triple points.

Here we address the question:

Question. *What are the simple normal surface singularities?*

As there is no obvious group action in the problem of classifying singularities of arbitrary embedding dimension, we take simple to mean that there occur only finitely many isomorphism classes in the versal deformation.

We conjecture the following answer:

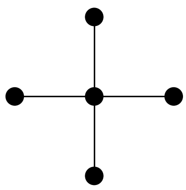
Conjecture. *Simple normal surface singularities are exactly the rational singularities, whose resolution graph can be obtained from the graphs of rational double points and rational triple points by making (some) vertex weights more negative.*

Without normality there are more simple singularities. The standard example of a nonnormal isolated singularity, two planes in 4-space meeting transversally in one point, has no nontrivial deformations at all, so is certainly simple. It is an old unsolved question whether rigid normal surface singularities (or rigid reduced curve singularities) exist. If they do, they are rather special. Our conjecture includes the statement that there are no rigid normal surface singularities, and even more, that there are no singularities for which infinitesimal deformations exist, but they all are obstructed.

The singularities in the conjecture make up the parts I, II and III in Laufer's list of taut singularities [3]: the graphs with at most one vertex of valency three and no higher valencies. As the graphs in question are star-shaped, all these singularities are quasi-homogeneous.

The problem in studying rational singularities of multiplicity at least four is that their deformation space has (in general) many components, and for only one, the Artin component, one has good methods to study adjacencies: it suffices to look at deformations of the resolution; in the case of almost reduced fundamental cycle there is even a complete description of the adjacencies [4]. Using deformations on the Artin component we can show:

Proposition. *A rational singularity, whose graph is not obtainable from a double or triple point graph by making vertex weights more negative, is not simple. It deforms into a singularity with a modulus in the exceptional divisor, with (unweighted) graph of the form*



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It follows that every non-simple rational singularity is adjacent to such a singularity.

For the following classes of rational singularities it is known or we can prove that they are simple:

- quotient singularities [1],
- singularities with reduced fundamental cycle, occurring in the conjecture,
- rational quadruple points, occurring in the conjecture.

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Mirror symmetry for Calabi-Yau manifolds and mirror symmetry for singularities

Kazushi Ueda

(joint work with Masahiro Futaki, Masanori Kobayashi, Makiko Mase, and Yuichi Nohara)

The lattice of vanishing cycles equipped with the intersection form is called the *Milnor lattice*, which is one of the central objects in singularity theory. The Milnor lattice admits a categorification called the *Fukaya-Seidel category*, which is an A_∞-category whose objects are vanishing cycles and whose spaces of morphisms are Lagrangian intersection Floer complexes.

Fukaya-Seidel categories appear in homological mirror symmetry for Fano manifolds. If we take the projective space $\mathbb{P}^n$ as an example, then the mirror is given by the Laurent polynomial

$$W = x_1 + \cdots + x_n + \frac{1}{x_1 \cdots x_n}$$

defining a regular map $W : (\mathbb{C}^\times)^n \to \mathbb{C}$, and one has an equivalence

$$D^b \operatorname{coh} \mathbb{P}^n \cong D^b \mathfrak{Fuk}\, W \tag{1}$$

of triangulated categories [7, 2].

Fukaya-Seidel categories also appear in homological mirror symmetry for singularities. If we take a Brieskorn-Pham polynomial

$$f = x_1^{p_1} + \cdots + x_n^{p_n}$$

as an example, then the mirror is the Brieskorn-Pham singularity

$$R = \mathbb{C}[x_1, \dots, x_n]/(f)$$

equipped with a grading by the abelian group

$$L = \mathbb{Z}\vec{x}_1 \oplus \cdots \oplus \mathbb{Z}\vec{x}_n \oplus \mathbb{Z}\vec{c}/(p_1\vec{x}_1 - \vec{c}, \dots, p_n\vec{x}_n - \vec{c})$$

of rank one, and one has an equivalence

$$D^b \mathfrak{Fuk}\, f \cong D^b_{\mathrm{sing}}(\operatorname{gr} R) \tag{2}$$

of triangulated categories [3]. Here, the category on the right hand side is the *stable derived category*, defined as the quotient category $D^b(\operatorname{gr} R)/D^{\mathrm{perf}}(\operatorname{gr} R)$ of the bounded derived category of finitely-generated L-graded R-modules by the full subcategory consisting of bounded complexes of projective modules. Similar result has been proved also for arbitrary Sebastiani-Thom sum of singularities of types A and D [1].

Now assume that the Milnor fiber of f can be compactified to a Calabi-Yau manifold Y. A typical example is the case when $n = 3$ and $f = x^2 + y^3 + z^7$, which defines one of Arnold's 14 exceptional unimodal singularities called the E_{12}-singularity. The mirror Calabi-Yau manifold $\check{Y}$ of Y is obtained as (a crepant resolution of) the quotient of Y by a suitable abelian group, and one expects an equivalence

$$D^b \mathfrak{Fuk}\, Y \cong D^b \operatorname{coh} \check{Y} \tag{3}$$

of triangulated categories [5]. The Fukaya-Seidel category $\mathfrak{Fuk}\, f$ is a *directed subcategory* of $\mathfrak{Fuk}\, Y$, and the stable derived category $D^b_{\mathrm{sing}}(\operatorname{gr} R)$ is a directed subcategory of $D^b \operatorname{coh} \check{Y}$ [11], so that it is natural to expect the existence of a commutative diagram

$$\begin{array}{ccc} D^b \mathfrak{Fuk}\, f & \hookrightarrow & D^b \mathfrak{Fuk}\, Y \\ \downarrow \wr & & \downarrow \wr \\ D^b_{\mathrm{sing}}(\operatorname{gr} R) & \hookrightarrow & D^b \operatorname{coh} \check{Y} \end{array}$$

where horizontal arrows are embeddings of directed subcategories and vertical arrows are homological mirror symmetry. This helps, for instance, to understand strange duality for Arnold's 14 exceptional unimodal singularities in the context of mirror symmetry for K3 surfaces [4].

On the other hand, the compatibility

$$\begin{array}{ccc} D^b \mathfrak{Fuk}\, W & \hookrightarrow & D^b \mathfrak{Fuk}\, Y \\ \downarrow \wr & & \downarrow \wr \\ D^b \operatorname{coh} \mathbb{P}^n & \hookrightarrow & D^b \operatorname{coh} \check{Y} \end{array}$$

of homological mirror symmetry for the projective space and that for its Calabi-Yau hypersurface is known by [8, 9, 10, 6], and it is an interesting problem to generalize this to, say, complete intersections in toric stacks.

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The vanishing cycles of types $\mathbf{A}_{\frac{1}{2}\infty}$ and $\mathbf{D}_{\frac{1}{2}\infty}$

Kyoji Saito

We introduce two real entire functions $f_{\mathrm{A}_{\frac{1}{2}\infty}}$ and $f_{\mathrm{D}_{\frac{1}{2}\infty}}$ in two variables, having only two critical values 0 and 1. Associated maps $\mathbf{C}^2 \to \mathbf{C}$ define topologically locally trivial fibrations over $\mathbf{C}\backslash\{0,1\}$. The critical points over 0 and 1 are ordinary double points, whose associated vanishing cycles in the generic fiber span its middle homology group and their intersection diagram forms the bi-partite decomposition of quivers of type $\mathrm{A}_{\frac{1}{2}\infty}$ and $\mathrm{D}_{\frac{1}{2}\infty}$, respectively (see the diagram below). Coxeter element of type $\mathrm{A}_{\frac{1}{2}\infty}$ and $\mathrm{D}_{\frac{1}{2}\infty}$ are introduced as the product of the monodromies of the fibrations around 0 and 1, which acts also on the Hilbert space obtained by completing the middle homology group. Then the spectra ("logarithm" of the eigenvalues of the) Coxeter element is absolutely continuous on the interval $(-\frac{1}{2}, \frac{1}{2})$ (except at 0 for the type $\mathrm{D}_{\frac{1}{2}\infty}$). This should give the datum of the good section for the construction of the primitive form associated with the function.

$$\Gamma_{\mathrm{A}_{\frac{1}{2}\infty}}: \quad \gamma_{A,1}^{(1)} \longrightarrow \gamma_{A,0}^{(1)} \longleftarrow \gamma_{A,1}^{(2)} \longrightarrow \gamma_{A,0}^{(2)} \longleftarrow \gamma_{A,1}^{(3)} \longrightarrow \gamma_{A,0}^{(3)} \longleftarrow \quad \cdots$$

$$\Gamma_{\mathrm{D}_{\frac{1}{2}\infty}}: \quad \begin{array}{l} \gamma_{D,0}^{+} \\ \quad \nwarrow \\ \qquad \gamma_{D,1}^{(1)} \longrightarrow \gamma_{D,0}^{(1)} \longleftarrow \gamma_{D,1}^{(2)} \longrightarrow \gamma_{D,0}^{(2)} \longleftarrow \gamma_{D,1}^{(3)} \longrightarrow \cdots \\ \quad \swarrow \\ \gamma_{D,0}^{-} \end{array}$$

On the Artin Approximation Theorem

Herwig Hauser

Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ be variables and let $f \in \mathbb{C}\{x, y\}^p$ be a vector of convergent power series in x and y. Let c be a natural integer. Artin's Approximation Theorem [1] asserts that whenever $y(x)$ is a formal power series solution of $f = 0$, say

$$f(x, y(x)) = 0,$$

there exists a convergent power series solution $\widetilde{y}(x)$, say

$$f(x, \widetilde{y}(x)) = 0,$$

which approximates $y(x)$ up to degree c,

$$\widetilde{y}(x) \equiv y(x) \ \text{ modulo } \ (x)^{c+1}.$$

This result has many variations and extensions, e.g. the passage from approximate solutions to formal exact solutions, or the difficult nested subring case. In the talk, which represents joint work with Guillaume Rond from Marseille, we address the more general question of how to describe the entire solution set of $f(x, y) = 0$ inside the infinite dimensional spaces of formal or convergent power series.

Defining a natural partition of this set by locally open sets, one tries to construct isomorphisms of formal power series spaces (to be defined suitably) which map each stratum of the solution set to a cartesian product of a finite dimensional singular variety with an infinite dimensional smooth variety. Such a product decomposition, if it can be proven to exist, would globalize the theorem of Grinberg-Kazhdan and Drinfeld [2], [3] on the local factorization of arc spaces and extend it to the multivariate case. Similarly, it would generalize Denef-Loeser's fibration theorem [4]. At the same time, it would yield a quite conceptual understanding of the proof of the approximation theorem.

In the talk we indicate some of the key steps and problems in carrying out this program.

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Right simple singularities in positive characteristic

Hong Duc Nguyen
(joint work with Gert-Martin Greuel)

We classify isolated hypersurface singularities $f \in \mathfrak{m}^2 \subset K[[x_1, \ldots, x_n]]$, K an algebraically closed field of characteristic $p > 0$, which have no moduli (modality 0) w.r.t. right equivalence, meaning that there are only finitely many right equivalence classes. These singularities are called right simple, following Arnol'd, who classified right simple singularities for $K = \mathbb{R}$ and $\mathbb{C}$ (cf. [1]). He showed that the simple singularities are exactly the ADE-singularities, i.e. the two infinite series $A_k, k \geqslant 1$, $D_k, k \geqslant 4$, and the three exceptional singularities E_6, E_7, E_8. It turned out later that the ADE-singularities of Arnol'd are also exactly those of modality 0 for contact equivalence. In the late eighties, Greuel and Kröning showed in [2] that the contact simple singularities over a field of positive characteristic are again exactly the ADE-singularities but with a few more normal forms in small characteristic.

A classification w.r.t. right equivalence in positive characteristic however, was never considered so far. A surprising fact of our classification is that for any fixed $p > 0$ there exist only finitely many right simple singularities. For example, if $p = 2$ and n is even, there is just one right simple hypersurface,

$$x_1x_2 + x_3x_4 + \ldots + x_{n-1}x_n,$$

while for n odd no right simple singularity exists. A table with normal forms for any $n \geqslant 1$ and any $p > 0$ is given (see also [4]):

Let $p = \mathrm{char}(K) > 2$.

(i) A plane curve singularity $f \in \mathfrak{m}^2 \subset K[[x,y]]$ is right simple if and only if it is right equivalent to one of the following forms

Name	Normal form	
A_k	$x^2 + y^{k+1}$	$1 \leqslant k \leqslant p-2$
D_k	$x^2y + y^{k-1}$	$4 \leqslant k < p$
E_6	$x^3 + y^4$	$3 < p$
E_7	$x^3 + xy^3$	$3 < p$
E_8	$x^3 + y^5$	$5 < p$

Table (a)

(ii) A hypersurface singularity $f \in \mathfrak{m}^2 \subset K[[\mathbf{x}]] = K[[x_1, \ldots, x_n]], n \geqslant 3$, is right simple if and only if it is right equivalent to one of the following forms

Normal form	
$g(x_1,x_2) + x_3^2 + \ldots + x_n^2$	g is one of the singularities in Table (a)

Table (b)

Let $p = \mathrm{char}(K) = 2$. A hypersurface singularity $f \in \mathfrak{m}^2 \subset K[[\mathbf{x}]] = K[[x_1, \ldots, x_n]]$ with $n \geqslant 2$, is right simple if and only if n is even and if it is right equivalent to

$$A_1 : x_1x_2 + x_3x_4 + \ldots + x_{n-1}x_n.$$

The problem is even interesting for univariate power series ($n = 1$) where we give a complete classification (see [3], [4]). Moreover we show that:

If $f(x) \in K[[x]]$ is a univariate power series such that its Milnor number $\mu := \mu(f)$ is finite. Then

$$\mathcal{R}\text{-mod}(f) = [\mu/p], \text{the integer part of } \mu/p.$$

A major point of this paper is the clarification of the notion of modality and its relations to formal deformation theory. We give a precise definition of the number of moduli (modality) for families of power series parametrized by an algebraic variety. In fact, we give two definitions of G-modality, both related to the action of an algebraic group G on a variety X and show that they coincide, a result which is valid in any characteristic.

Moreover, we prove that the G-modality is upper semicontinuous for G the right resp. the contact group.

We introduce the notion of G-completeness which suffices to determine the modality and show that the usual semiuniversal deformation with section of an isolated hypersurface singularity is complete. In contrast to the complex analytic case the semiuniversal deformation is not sufficient to determine the modality; we have to consider versal deformations with section.

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The number of components of a linear free divisor

Brian Pike

A *free divisor* is a germ of a complex hypersurface with the property that its module of logarithmic vector fields is a free module. These hypersurfaces are of significant interest because, for example, the discriminants of versal unfoldings of isolated complete intersection singularities are always free divisors. Though the definition dates from 1980 ([10]), these objects remain mysterious. For instance, it is not completely understood which hyperplane arrangements are free divisors.

In the last few years many have studied *linear* free divisors, free divisors where the module of logarithmic vector fields is generated by 'linear' (homogeneous of degree 0) vector fields. Every linear free divisor arises from a rational representation $\rho : G \to \mathrm{GL}(V)$ of a connected complex linear algebraic group G on a complex vector space V which has a Zariski open orbit Ω. Moreover, $\dim(G) = \dim(V)$, the linear free divisor $(X, 0)$ is simply $V \setminus \Omega$, and $(X, 0)$ is defined by a reduced homogeneous polynomial of degree $\dim(G)$. This overlap of singularity theory and representation theory has proven to be very fertile. Connections have been found between these linear free divisors and representations of quivers ([2]), F-manifolds ([3]), versions of Grothendieck's comparison theorem (e.g., [5]), Bernstein-Sato polynomials ([7]), etc. The linear free divisors in dimensions $\leqslant 4$ have been classified ([5]), although it seems that the difficulty of classification increases dramatically with dimension. Nontrivial infinite families of linear free divisors (each in a different ambient space) have been exhibited ([4]), along with several nontrivial ways of constructing linear free divisors from existing linear free divisors ([9, 6, 1]).

Representations having open orbits have been studied before under a different name (see e.g., [11]). A *prehomogeneous vector space* $\rho : G \to \mathrm{GL}(V)$ is a rational representation of a connected complex linear algebraic group G on a finite-dimensional vector space V, having a (Zariski) open orbit Ω. When $\dim(G) > \dim(V)$, the complement X of Ω may not be of pure dimension. Even so, the hypersurface components of X are closely related to the group $X_1(G)$ of multiplicative characters $\chi : K \to \mathbb{G}_m \simeq \mathbb{C}^*$, where G_{v_0} is the isotropy subgroup at some $v_0 \in \Omega$ and $K = G/([G, G] \cdot G_{v_0})$; for instance, $X_1(G)$ is a free abelian group with $\mathrm{rank}(X_1(G))$ equal to the number of irreducible hypersurface components of X. Since K is a abelian connected complex linear algebraic group,

$K \simeq (\mathbb{G}_m)^k \times (\mathbb{G}_a)^\ell$, where $\mathbb{G}_a \simeq (\mathbb{C}, +)$, and k is the number of irreducible hypersurface components of X. The number ℓ may be detected as the dimension of the vector space $A(G)$ of *additive functions*, rational homomorphisms $K \to \mathbb{G}_a$.

In ongoing work ([8]), we investigate the additive functions of prehomogeneous vector spaces. In the special case where $\rho : G \to \mathrm{GL}(V)$, $\dim(G) = \dim(V)$, produces a linear free divisor $X = V \setminus \Omega$, then we prove that there are no nontrivial additive functions; hence $\ell = 0$, and the number of irreducible components of X is exactly

$$\dim_{\mathbb{C}}(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]),$$

where $\mathfrak{g}$ may be interpreted as either the Lie algebra of G or as the 'linear' logarithmic vector fields of X. A key step in the proof is the use of a criterion, due to Michel Brion (published in [6]), for X to be a linear free divisor. This result simplifies and unifies many previous results.

A natural question for further research is whether any similar results hold for arbitrary free divisors.

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Discriminant of transversal singularity type and further stratification of the singular locus

Dmitry Kerner
(joint work with Maxim Kazarian, András Némethi)

Let X be an analytic space with non-isolated singularity, let Z be a connected component of the singular locus. Assume Z is locally complete intersection, while X is a strict locally complete intersection (i.e. its tangent cone at each point is a complete intersection).

The (topological) transversal type of X along Z is generically constant but at some points of Z it degenerates. We introduce the discriminant of the transversal type, the subscheme of Z that reflects these degenerations. The scheme structure is imposed by various compatibility properties and is often non-reduced.

In the global case, Z being compact, we compute the class of the discriminant in the Picard group $Pic(Z)$. Further, we define the natural stratification of the singular locus and compute the classes of the simplest strata.

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On the structure of homogeneous symplectic varieties of complete intersection

Yoshinori Namikawa

A normal complex algebraic variety X is a *symplectic variety* if there is a holomorphic symplectic 2-form ω on the regular part X_{reg} of X and ω extends to a (possibly degenerate) holomorphic 2-form on a resolution $f : \tilde{X} \to X$.

Example: Let $\mathfrak{g}$ be a semisimple complex Lie algebra and let G be its adjoint group. Let us consider the *adjoint quotient map* $\chi : \mathfrak{g} \to \mathfrak{g}//G$. If $\mathrm{rank}(\mathfrak{g}) = r$, then $\mathfrak{g}//G$ is isomorphic to the r-dimensional affine space $\cong \mathbf{C}^r$. The nilpotent variety N is, by definition, the set of all nilpotent elements of $\mathfrak{g}$ and we have $N = \chi^{-1}(0)$. The nilpotent variety decomposes into the disjoint union of (finite number of) nilpotent orbits. There is a unique nilpotent orbit O_{reg} that is open dense in N, which we call the *regular nilpotent orbit*. Then $N = \overline{O_{reg}}$. The regular nilpotent orbit O_{reg} coincides with the regular part of N and it admits a holomorphic symplectic form ω_{KK} so called the *Kostant-Kirillov* 2-form. Then (N, ω_{KK}) is a symplectic variety. Moreover $N \subset \mathfrak{g}$ is defined as a complete intersection of r homogeneous polynomials (with respect to the standard $\mathbf{C}^*$-action on $\mathfrak{g}$).

In this talk I characterize the nilpotent varieties of semisimple Lie algebras among affine symplectic varieties.

Let (X, ω) be a singular affine symplectic variety of dimension $2n$ embedded in an affine space $\mathbf{C}^{2n+r}$ as a complete intersection of r homogeneous polynomials. Assume that ω is also homogeneous, i.e. there is an integer l such that $t^*\omega = t^l \cdot \omega$ for $t \in \mathbf{C}^*$.

Main Theorem: *One has $(X, \omega) \cong (N, \omega_{KK})$, where N is the nilpotent variety of a semisimple Lie algebra $\mathfrak{g}$ together with the Kostant-Kirillov 2-form ω_{KK}.*

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The Herman conjecture

Mauricio Garay

Consider the space $\mathbb{R}^{2n}$ with coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ and let $U \subset \mathbb{R}^{2n}$ be an open subset. The interior product with the symplectic form $\sum_{i=1}^n dq_i \wedge dp_i$ induces an isomorphism of sheaves between vector fields and differential one-forms. Given an analytic function $H : U \to \mathbb{R}$, , called the *hamiltonian*, the vector field associated to the one-form dH is called the *hamiltonian vector field* of H.

In simple examples, hamiltonian dynamical systems are easily integrated. This is the case for instance for the Kepler problem but already for the next case in difficulty, where two heavy bodies attract a smaller one, classically known as the *problem of the moon* or in a less poetic way as the *restricted three body problem*, the situation turns out to be incredibly complicated. For this reason Poincaré turned towards the qualitative theory of differential equations [13].

In the fifties, Kolmogorov discovered the existence of hamiltonian systems carrying invariant n-dimensional tori which persist under small perturbations. These n-dimensional tori are the closure of dense trajectories of the dynamical system, but only those which fill the tori sufficiently fast, define such robust tori. This rate can be given explicitly in the following technical terms.

Linear trajectories on a torus $(\mathbb{R}/\mathbb{Z})^n$ are defined by a single vector $\alpha \in \mathbb{R}^n$, called the *frequency*. The frequency α is called (C, τ)-*diophantine* if one has the estimate $|(\alpha, i)| \geqslant \frac{C}{\|i\|^\tau}$ for any vector $i \in \mathbb{Z}^n \setminus \{0\}$. The set of such vectors is denoted by $\Omega_{C,\tau}$.

For any constants C, τ, the diophantine trajectories define persistent tori under perturbation provided that a second condition called *Kolmogorov's non-degeneracy* is satisfied. This condition says that, in first approximation, the tori are smoothly parametrised by the frequency of the hamiltonian motion [9].

In the sixties, Arnold proved, under Kolmogorov's conditions, the existence of a positive measure set of invariant tori for perturbed integrable systems and Moser extended the theory to the differentiable case [2, 12]. The new-born KAM – acronym for Kolmogorov-Arnold-Moser– theory was used by Arnold in the problem of the moon but he rapidly discovered that, due to the symmetries of the problem, Kolmogorov's non-degeneracy condition is not fulfilled. Nevertheless, by a real "tour de force", he proved the existence of a positive measure of robust tori, and eventually pointed out that his proof could be adapted to the N-body problem [3].

Then together with Piartly, he proposed to study diophantine approximation in the more general context of manifolds in euclidean space and this was later developped as a subject in itself by Margulis and his school.

In the nineties, Herman started his investigation on the N-body problem and discovered that Arnold's claim was incorrect : new difficulties appear in the N-body problem [1]. The computations for non-degeneracy conditions turned out to be so difficult that Herman proposed an acronym BLC meaning "Bonjour les calculs" in order to point out each time there were awful computations. In 1998, during his ICM lecture, he made the following striking conjecture for discrete time hamiltonian systems [8] :

In the neighbourhood of a diophantine elliptic fixed point, a real analytic symplectomorphism has a positive measure set of invariant tori.

The conjecture seemed odd, since it was known, after the work of Katok, that the absence of non-degeneracy condition is responsible for chaotic motions [10]. Nevertheless :

Theorem 1 ([7])**.** *The Herman conjecture is true.*

Katok's theorem seems to contradict the possibility of more general conjectures, it does not : the subtle point is that the neighbourhood in which KAM theory applies is prescribed by the perturbation, so that one may approach any such system by a chaotic one. In some sense, there is an unexpected problem of quantifier between the size of the perturbation and the existence of chaotic motions, and this was a major source of our misunderstanding of KAM theory. In fact, there exists a KAM theorem without any kind of non-degeneracy condition [7].

The proof of the Herman conjecture is based on group actions in infinite dimensional spaces, an idea which goes back to Moser. Symplectomorphisms act by change of variables on the space of hamiltonians and KAM theory can be interpreted as the relation between this action and its linearisation. Indeed, the invariant tori form the fibres of some map $\pi : U \to \Omega_{C,\tau}$. The graph of this map defines a family of lagrangian manifolds which ideal sheaf we denote by I. As the family is lagrangian, its conormal sheaf I/I^2 gets identified with its tangent sheaf. Thus, in this abstract form, we study the orbit of our hamiltonian function H under the action of the group of symplectomorphism modulo I^2.

So the situation is very similar to that of versal deformation theory and finite determinacy theorems but... there are important differences. The action involves differential operators, thus the usual methods of commutative algebra do not apply in this context. Moreover, solving infinitesimal condition introduces small denominators which turn out to be responsible for the presence of unbounded operators. Fortunately, Kolmogorov and Arnold overcame this difficulty and their method can be put in abstract form, in a way similar to what Sergeraert, Hamilton and Zehnder did for Moser's proof. In fact, due to the analytic context, one can go much further than implicit function theorems and begin to construct a whole theory of versal deformation in direct limits of Banach spaces. Here is a simple example. We denote by $\mathcal{O}_{\mathbb{C}^n,0}$ the algebra of germs of holomorphic functions at the origin in $\mathbb{C}^n$ and by $\mathcal{M}_{\mathbb{C}^n,0}$ its maximal ideal.

Theorem 2 ([5]). *Let $f \in \mathcal{M}_{\mathbb{C}^n,0}$ be a map germ, G a closed subgroup of automorphism of the algebra $\mathcal{O}_{\mathbb{C}^n,0}$ and $\mathfrak{g} \subset Der(\mathcal{O}_{\mathbb{C}^n,0}, \mathcal{M}_{\mathbb{C}^n,0})$ a closed vector space such that $e^{\mathfrak{g}} \subset G$. Assume that the "infinitesimal action" $\rho : \mathfrak{g} \to M,\ v \mapsto v(f)$ admits a bounded right inverse then, in the neighbourhood of f, the space $f + \mathcal{M}^2_{\mathbb{C}^n,0}$ is locally a G-homogeneous space.*

For the definition of a bounded map in this context see [4, 5]. Then a second difficulty arises : the base $\Omega_{C,\tau}$ which parametrises the invariant tori is not at all a smooth manifold. After base change, this set might behave wildly and one might expect that the pre-image of a set of positive measure consists of a single point. This is confirmed by the recent discovery of Eliasson, Fayad and Krikorian who constructed examples of curves which pass through only one point of some $\Omega_{C,\tau}$, so the situation seems lost !

In fact, using Dani-Kleinbock-Margulis techniques (see [11]), one can prove a base change property for which such pathologies do not occur, namely :

Theorem 3 ([6]). *For any l-curved mapping $f : \mathbb{R}^d \supset U \to \mathbb{R}^n,\ f(0) = \alpha \in \Omega_{C,\tau}$ the density of the set $f^{-1}(\Omega_{C,\tau'})$ at the origin is equal to 1 where*

$$\tau' := kn + kl(n+1) + kdl(n+1)^2 + (n+1)\tau.$$

Here l-curved means that the image is contained in an affine space whose associated vector space is spanned by the partial derivatives of f of order at most l, at each point. So, we can indeed integrate base changes in KAM theory and formulate a generalisation of the KAM theorem as an abstract theorem on group actions ; then repeating Martinet's original proof of the versal deformation theorem, we prove the Herman conjecture.

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On realizing homology classes by maps of restricted complexity

András Szűcs
(joint work with Mark Grant)

We show that in every codimension greater than one there exists a mod 2 homology class in some closed manifold (of sufficiently high dimension) which cannot be realized by an immersion of closed manifolds. The proof gives explicit obstructions (in terms of cohomology operations) for realizability of mod 2 homology classes by immersions. We also prove the corresponding result in which the word 'immersion' is replaced by 'map with some restricted set of multi-singularities'.

Let $f\colon M^{n-k} \to N^n$ be a continuous map of codimension k between closed manifolds (all manifolds and maps between them are assumed smooth, unless stated otherwise). Then f is said to *realize* both the mod 2 singular homology class $z = f_*[M] \in H_{n-k}(N;\mathbb{Z}_2)$ (where $[M] \in H_{n-k}(M;\mathbb{Z}_2)$ is the fundamental class of the domain manifold) and its Poincaré dual cohomology class $x \in H^k(N;\mathbb{Z}_2)$. We address the following questions. When can a (co)homology class be realized by an immersion? When can a (co)homology class be realized by a map whose complexity is restricted by prescribing some finite set of allowed multi-singularity types?

Theorem 1. *For any $k > 1$ there exists a closed manifold N_k and cohomology class $x_k \in H^k(N_k;\mathbb{Z}_2)$ which cannot be realized by an immersion. The manifold N_k can be chosen to have dimension $4k+3$ if k is even, and $4k+15$ if k is odd.*

The proof of Theorem 1 makes use of the following explicit obstructions to realizability by immersions, in terms of stable cohomology operations.

Theorem 2. *Let $k > 1$ and let I be an admissible sequence of excess $e(I) = k$, i.e. $I = (i_1, \dots, i_r)$, where $i_1, \dots, i_r$ are natural numbers such that $i_j \geqslant 2i_{j+1}$ and $e(I) = \sum(i_j - 2i_{j+1}) = k$. Let $Sq^I = Sq^{i_1} \dots Sq^{i_r}$ be the corresponding monomial* *in the Steenrod algebra. If the cohomology class $x \in H^k(N;\mathbb{Z}_2)$ is realizable by an immersion, then $Sq^I(x)$ is the reduction* mod 2 *of an integral class.*

In particular, if k is even and $\beta(x^2)$ is nonzero (where β is the Bockstein associated to reduction mod 2) then x cannot be realized by an immersion.

The obstruction $\beta(x^2)$ in the case k even is very natural:

Claim 3. *$\beta(x^2)$ is the integer cohomology class realized by the singular set of any generic map realizing x.*

Now we turn to non-realizability of homology classes by singular maps. Let τ be a finite set of codimension k multi-singularities. A *multi-singularity* is a finite multiset of stable local singularities. Recall [2] that a stable map $f\colon M^{n-k} \to N^n$ is called a *τ-map* if at each point $y \in N$ the pre-image $f^{-1}(y) \subseteq M$ is finite and the local singularities of f at the pre-image points, counted with multiplicity, form an element of τ.

Theorem 4. *Let $k > 1$, and let τ be any finite set of multi-singularities in codimension k. Then there exists a closed manifold N_k and cohomology class $x_k \in H^k(N_k;\mathbb{Z}_2)$ which cannot be realized by a τ-map.*

Theorems 1 and 4 should be contrasted with the well known fact that any one-dimensional cohomology class $x \in H^1(N;\mathbb{Z}_2)$ in a closed manifold is realizable by an embedding of a closed manifold.

History of the question realizing homology classes by manifolds. It is somewhat surprising that a result such as Theorem 1 has not found its way into the literature before now. Ever since Poincaré and the birth of homology, basic questions concerning realization of homology classes by maps from closed manifolds have had a profound effect on the development of Algebraic Topology. Thom showed in his landmark paper [6] that every mod 2 homology class in a finite polyhedron can be realized by a continuous map, thus giving an affirmative answer to a problem posed by Steenrod. In its original formulation [1], Steenrod's question was about realizing integral homology classes by maps from *oriented* manifolds, and Thom also gave negative results in this direction, by constructing examples of non-realizable integral homology classes in dimensions 7 and above.

Thom's method was to reduce Steenrod's problem to the related question concerning realizability of homology classes by embeddings. The key insight which allowed him to solve this problem was that a homology class in the compact manifold N can be realized by a codimension k embedding if and only if its Poincaré dual cohomology class is induced from the Thom class by a map from $N/\partial N$ into the Thom space of the universal k-dimensional bundle. In other words, the Thom space of the universal k-dimensional bundle is the *classifying space for codimension k embeddings.* One can use this result to find homology classes which cannot be realized by embeddings, in two closely related ways.

The first is constructive, in that it gives specific obstructions to realizability. Namely, one shows that some expression P involving cup products and cohomology operations vanishes on the Thom class. If the dual of a cohomology class x is to be realizable, that same expression must also vanish on x (this approach was taken by Thom [6, Chapitre II]).

The second approach is less constructive, but equally valid. One compares the graded rank of the mod 2 cohomology of the Thom space of the universal k-dimensional bundle with that of the corresponding Eilenberg-Mac Lane space. In high degrees the latter is larger, and so this approach shows that in all dimensions $k > 1$ there exists a mod 2 cohomology class in some closed manifold of sufficiently high dimension which cannot be realized by an embedding (Thom says that this argument, outlined on page 46 of [6], was patterned after a remark of J.P. Serre).

We used the (generalization of the) first method to prove Theorem 1 and the second one for proving Theorem 4. In the latter case we had to use instead of the Thom space (which is the *classifying space for embeddings*) a *classifying space for τ-maps* (i.e. maps with a prescribed set of allowed multisingularities) constructed in papers [3], [4], [5], [2].

Finally I give a sketch of the proof of Claim 3 (that I failed to give in the talk).

Proof of Claim 3. Sketch. If a generic map $f : M^{n-k} \to N^n$ realizes a k-dimensional cohomology class x, then the cohomology class realized by the singularity locus of f can be identified with $f_!(W_{k+1}(\nu_f))$, where ν_f is the virtual normal bundle of f and W_{k+1} is the twisted integer Stiefel–Whitney class. (This is almost the definition of the class W_{k+1}.) It remained to show that $f_!(W_{k+1}(\nu_f)) = \beta(x^2) = \beta(Sq^k x)$. Composing f with the embedding $N^n \hookrightarrow N^n \times D^q$, where q is big, and D^q is a ball we reduce the statement to the case when the map is an embedding. Now we can consider the universal embedding $BO(m) \hookrightarrow MO(m)$. The mod 2 reductions of the two sides are equal in this case by the well-known formula $Sq^{k+1}U_m = w_{k+1}U_m$ for any $k+1 < m$, where U_m is the Thom class. The integer case follows from the fact that all the torsion part in $H^*(MO(m); Z)$ is 2-torsion, hence the reduction mod 2 is injective. The case of arbitrary embeddings follows by pulling back the obtained formula from the universal embedding. □

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Fundamental groups and Mordell-Weil groups

Anatoly Libgober

Let C be an irreducible algebraic curve in $\mathbf{C}^2$, transversal to the line at infinity. In this talk I discussed the following problem: *which polynomials can appear as the Alexander polynomials of the fundamental group* $\pi_1 = \pi_1(\mathbf{C}^2 - C)$.

The Alexander polynomial can be defined as the characteristic polynomial of the generator of the abelianization π_1/π_1' (which is a cyclic group) acting on the abelianization of the commutator: $\pi_1'/\pi_1'' \otimes \mathbf{Q}$ (cf. [1]). The roots of this polynomial $\Delta_C(t)$ are the roots of unity of degree $degC$ and it satisfies the following divisibility relation: if $\Delta_P(t)$ denotes the Alexander polynomial of the link of a singularity P of the curve C then Δ_C divides $\Pi_P \Delta_P$ where the product is taken over all singularities P of C (cf. [1]).

For curves with nodes and ordinary cusps as the only singularities this implies that $\Delta_C(t) = (t^2 - t + 1)^s, (s \geqslant 0)$ and $6|degC$. The largest known value of s at the moment is $s = 4$ (cf. Cogolludo-Libgober, [2]). The goal of this talk was to describe the relationship between the problem of finding a bound on s for curves with nodes and cusps and arbitrary degree and the problem of finding upper bounds for Mordell-Weil ranks of elliptic curves over the field of rational functions $\mathbf{C}(x, y)$.

Let $\mathbf{E}_f$ be (isotrivial) elliptic curve over $\mathbf{C}(x, y)$ given by the equation:

$$u^2 = v^3 + f(x, y)$$

where f is the equation of C and singularities as above.

Theorem 1. *(J.I.Cogolludo-A.Libgober, [2]) . The rank of Mordell-Weil group of* $\mathbf{E}_f$ *is equal to* $2s$.

This extends the results of Hulek-Kloosterman whose results imply Theorem 1 in the case when $degC = 6$.

The key step in the proof of this theorem (explaining the topological nature of the Mordell Weil rank in this case) is the following:

Theorem 2. *(J.I.Cogolludo-A.Libgober, [2]) Let* V_f *be a smooth projective model of the surface* $z^6 = f(x, y)$. *Then the Albanese variety of* V_f *is isogenous to* E_0^s *where* E_0 *is the elliptic curve with j-invariant zero.*

Theorems 1 and 2 can be extended to results connecting the Alexander polynomial of plane curves with singularities in certain class, called the singularities of CM type, and the Mordell Weil ranks of certain families of abelian varieties over $\mathbf{C}(x, y)$.

To define the singularities of CM type one first defines what we call the local Albanese variety of a plane curve singularity. This is the abelian part of the 1-motif (introduced by Deligne) of the limit mixed Hodge structure corresponding to a singularity of a plane curve.

Definition 3. *(cf. [3]) A singularity is called a singularity of CM-type if the local Albanese variety is an abelian variety with complex multiplication.*

Theorem 4. *(cf. [3]) 1. Uni-branched plane curve singularities have CM type.*

2. If the characteristic polynomial of the monodromy of a plane curve singularity does not have multiple roots then it has CM type.

On the other hand, ordinary multiple point with multiplicity greater than 3 generally does not have CM type.

Theorem 5. *(cf. [3]) Let C be a plane curve with equation $f(x,y)=0$ and with singularities of CM type. Let V_C be a smooth projective model of affine surface $z^{degC}=f(x,y)$. Then Albanese variety of V_C is isogenous to a product of abelian varieties of CM type corresponding to cyclotomic fields.*

This theorem leads to the relation between the factors of the Alexander polynomial of Δ_C and the Mordell-Weil ranks of abelian varieties over $\mathbf{C}(x,y)$.

Theorem 6. *Let $\mathbf{A}$ be an smooth projective model of an isotrivial abelian variety over field $\mathbf{C}(x,y)$, $\pi:\mathbf{A}\to\mathbf{P}^2$ and A be its generic fiber. Let $\Delta\subset\mathbf{P}^2$ be the discriminant of π and let $G\subset AutA$ be the holonomy group of $\mathbf{A}$. Assume that:*

a) the holonomy group G of isotrivial fibration over the complement to the discriminant Δ is a cyclic group of order d and has no fixed points outside of the zero of the generic fiber A.

b) The singularities of Δ have CM type and Δ is irreducible.

Then

1. the rank of the Mordell-Weil group of $\mathbf{A}$ is zero, unless the generic fiber of π is an abelian variety of CM-type with endomorphism algebra containing a cyclotomic field.

2. Assume that generic fiber A of π is a simple abelian variety of CM type corresponding to the field $\mathbf{Q}(\zeta_d)$ Let s be the multiplicity of the factor $\Phi_d(t)$ of the Alexander polynomial of $\pi_1(\mathbf{P}^2-\Delta)$ where $\Phi_d(t)$ is the cyclotomic polynomial of degree d. Then:

$$\mathrm{rk}MW(\mathbf{A},\mathbf{C}(x,y))\leqslant s\cdot\varphi(d)\qquad(*)$$

(here $\varphi(d)=deg\Phi_d(t)$ is the Euler function).

3. Let A be an abelian variety as in 2. If d is the order of the holonomy of $\mathbf{A}$ and the Albanese variety $Alb(X_d)$ of the d-fold cover X_d of X ramified over Δ has A is its direct summand with multiplicity s then one has equality in (*).

As example of this result we obtain that for the Jacobian of the curve over $\mathbf{C}(x,y)$ given in (u,v) plane by the equation

$$u^p=v^2+(x^p+y^p)^2+(y^2+1)^p$$

one has $rkMW=p-1$.

Details of these results appear in [3].

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Arrangements and Frobenius like structures

Alexander Varchenko

There are three places, where a flat connection depending on a parameter appears:
• KZ equations,

$$\kappa \frac{\partial I}{\partial z_i}(z) = K_i(z)I(z), \quad z = (z_1, \dots, z_n), \quad i = 1, \dots, n. \tag{1}$$

Here κ is a parameter, $I(z)$ a V-valued function, where V is a vector space from representation theory, $K_i(z) : V \to V$ are linear operators, depending on z. The connection is flat for all κ.
• Quantum differential equations,

$$\kappa \frac{\partial I}{\partial z_i}(z) = p_i *_z I(z), \quad z = (z_1, \dots, z_n), \quad i = 1, \dots, n. \tag{2}$$

Here $p_1, \dots, p_n$ are generators of some commutative algebra H with quantum multiplication depending on z. These equations are part of the Frobenius structure on the quantum cohomology of a variety.
• Differential equations for hypergeometric integrals associated with a family of weighted arrangements with parallelly translated hyperplanes,

$$\kappa \frac{\partial I}{\partial z_i}(z) = K_i(z)I(z), \quad z = (z_1, \dots, z_n), \quad i = 1, \dots, n. \tag{3}$$

It is well known that KZ equations are closely related with the differential equations for hypergeometric integrals. According to [6] the KZ equations can be presented as equations for hypergeometric integrals for suitable arrangements. Thus (1) and (3) are related. Recently it was realized that in some cases the KZ equations appear as quantum differential equations, see [1] and [4], and therefore the KZ equations are related to the Frobenius structures. On Frobenius structures see, for example, [2, 3, 5]. Hence (1) and (2) are related. The goal of this project is to explain how a Frobenius like structure may appear on the base of a family of weighted arrangements, in particular, the goal is to make equations (3) related to Frobenius structures.

The main ingredients of a Frobenius structure are a flat connection depending on a parameter, a constant metric, a multiplication on tangent spaces. In our case, the connection comes from the differential equations for the associated hypergeometric integrals, the flat metric comes from the contravariant form on the space of singular vectors and the multiplication comes from the multiplication in the algebra of functions on the critical set of the master function. To illustrate the constructions I consider the family of points on the line and a family of generic arrangements of lines on plane. I describe the associated Frobenius like structures. In particular, the potentials of second kind of these structures are

$$\tilde{P}(z_1,\dots,z_n) = -\frac{1}{2}\sum_{0<i<j<n+1} a_i a_j\,(z_i - z_j)^2 \log(z_i - z_j)$$

for the family of arrangements of n points on line and

$$\tilde{P}(z_1,\dots,z_n) = \frac{1}{4!}\sum_{0<i<j<k<n+1} \frac{a_i a_j a_k}{d_{i,j}^2 d_{j,k}^2 d_{k,i}^2} \times$$
$$\times (z_i d_{j,k} + z_j d_{k,i} + z_k d_{i,j})^4 \log(z_i d_{j,k} + z_j d_{k,i} + z_k d_{i,j})$$

for the family of arrangements of n generic lines on plane. The variables $z_1,\dots,z_n$ are parameters of the families, $a_1,\dots,a_n$ are weights, $|a| = a_1 + \cdots + a_n$, the number $d_{k,\ell}$ is the oriented area of the parallelogram generated by the normal vectors to the k-th and ℓ-th lines, see [7].

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Normal crossings in codimension one

Mathias Schulze

(joint work with Michel Granger)

In [3], Kyoji Saito introduced logarithmic differential forms and their resiudes. If $D = \{h = 0\} \subset (\mathbb{C}^n, 0)$ is a reduced complex analytic hypersurface germ, then any logarithmic differential form $\omega \in \Omega^p(\log D) \subset \Omega^p(D)$ can be written as

$$g\omega = \frac{dh}{h} \wedge \xi + \eta$$

where $g \in \mathcal{O}_S$ induces a non-zero divisor in $\mathcal{O}_D$ and $\xi \in \Omega_S^{p-1}$ and $\eta \in \Omega_S^p$ are forms without pole. With this notation $\rho_D(\omega) := \frac{\xi}{g}$ is a well-defined meromorphic $(p-1)$-form on D, or on the normalization $\pi : \tilde{D} \to D$ of D. One can see easily that the image σ_D^0 of the 1st residue map ρ_D^1 contains the ring $\mathcal{O}_{\tilde{D}}$ of weekly holomorphic functions.

Saito proved the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) among the following conditions:

(1) The local fundamental groups of the complement $S\backslash D$ are abelian.
(2) Outside a codimension-2 subset of D, D has at most normal crossing singularities. We say that "D is normal crossing in codimension one".
(3) Every logarithmic one form has a weakly holomorphic residue, that is, $\sigma_D^0 = \mathcal{O}_{\tilde{D}}$.

The reverse implication (1) $\Leftarrow$ (2) is the Lê–Saito Theorem [1] which generalizes the Zariski conjecture for complex plane projective nodal curves proved by Fulton [4] and Deligne [5]. Saito proved (2) $\Leftarrow$ (3) for plane curves leaving the general case open.

Joint with Michel Granger [2], I introduce a dual logarithmic residue map. If D is a free divisor, this allows us to translate condition (3) into: equality

(4) The jacobian ideal J_D equals the conductor $C_{\tilde{D}/D}$.

Applying a result by Piene [6] (see also [7]), this leads to a proof of the missing implication (2) $\Leftarrow$ (3) for general D. For free D, we obtain another equivalent condition

(5) The Jacobian ideal J_D of D is reduced.

For free D with smooth normalization $\tilde{D}$, we can show that D must be normal crossing if it satisfies one/any of the above conditions (1) – (5). This is related to a conjecture of Eleonore Faber [8]: If D is free and the ideal $J_h \subset \mathcal{O}_S$ of partials of h is reduced then D must be normal crossing.

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Toric geometry, hypergeometric D-modules and mirror symmetry

Christian Sevenheck
(joint work with Thomas Reichelt)

In this talk, we describe a version of mirror symmetry for smooth toric varieties with numerically effective anticanonical bundle (e.g. toric Fano manifolds) and also for nef complete intersections in toric varieties. The correspondence is expressed as an equivalence of filtered $\mathcal{D}$-modules. On the A-side of the mirror picture, this is the so-called quantum $\mathcal{D}$-module of the variety X_Σ, that is, a family (parameterized by the space $H^*(X_\Sigma,\mathbb{C})$) of trivial vector bundles on $\mathbb{P}^1$ equipped with an integrable connection with poles along $\{0,\infty\}\times H^*(X_\Sigma,\mathbb{C})$. It is well-known that this object is basically equivalent to the quantum cohomology on $H^*(X_\Sigma,\mathbb{C})$. On the B-side, we consider the Landau-Ginzburg model in the sense of [1] and [3], that is, a family of Laurent polynomials parameterized by the Kähler moduli space of X_Σ. The precise definition is as follows.

Definition 1. *Let Σ be a smooth complete n-dimensional fan defining a smooth projective Fano variety X_Σ. Let $A=(\underline{a}_1|\ldots|\underline{a}_m)$ be the matrix with columns the primitive integral generators of the rays of Σ. Define*

$$\begin{array}{rcl}\varphi : S\times\Lambda := (\mathbb{C}^*)^n\times\mathbb{C}^m & \longrightarrow & \mathbb{C}_t\times\Lambda \\ (y_1,\ldots,y_n),(\lambda_1,\ldots,\lambda_m) & \longmapsto & \left(\sum_{i=1}^m \lambda_i \underline{y}^{\underline{a}_i},\lambda_1,\ldots,\lambda_m\right)\end{array}$$

*where $\underline{y}^{\underline{a}_i} := \prod_{k=1}^n y_k^{a_{ki}}$. This is called the generic family of Laurent polynomials associated to Σ (actually, it depends only on $\Sigma(1)$). On the other hand, there is an (non-canonical) embedding $g:\mathcal{K}_{X_\Sigma}\hookrightarrow\Lambda$, where $\mathcal{K}_{X_\Sigma}$ denotes the **complexified Kähler moduli space** of X_Σ. $\mathcal{K}_{X_\Sigma}$ is an $m-n$-dimensional torus, and a specific choice of a basis of $H^2(X_\Sigma,\mathbb{Z})$ (this choice depends on Σ, not only on $\Sigma(1)$) yields an identification $\mathcal{K}_{X_\Sigma}\cong(\mathbb{C}^*)^{m-n}$. Then we call the family of Laurent polynomials*

$$W := \varphi\circ(\mathrm{id}_S\times g): S\times\mathcal{K}_{X_\Sigma}\to\mathbb{C}_t\times\mathcal{K}_{X_\Sigma}$$

*the **Landau-Ginzburg model** of X_Σ.*

Consider the matrix $\widetilde{A}=(\underline{\widetilde{a}}_0,\underline{\widetilde{a}}_1,\ldots,\underline{\widetilde{a}}_m)\in\mathrm{Mat}((n+1)\times(m+1),\mathbb{Z})$ where $\underline{\widetilde{a}}_i := (1,\underline{a}_i)\in\mathbb{Z}^{n+1}$ for $i=1,\ldots,m$ and $\underline{\widetilde{a}}_0 := (1,\underline{0})$. Then for any $\beta\in\mathbb{Z}^{n+1}$, let $\mathcal{M}^\beta_{\widetilde{A}}$ be the Gelfand-Kapranov-Zelevinsky-hypergeometric $\mathcal{D}_{\mathbb{C}_t\times\Lambda}$-module (see, e.g., [2]).

Theorem 2 ([6])**.** *There is an exact sequence in $MHM_{\mathbb{C}_t\times\Lambda}$ (the abelian category of mixed Hodge modules on $\mathbb{C}_t\times\Lambda$)*

$$0\to H^{n-1}(S,\mathbb{C})\otimes\mathcal{O}_{\mathbb{C}_t\times\Lambda}\to\mathcal{H}^0\varphi_+\mathcal{O}_{S\times\Lambda}\to\mathcal{M}^{(0,\underline{0})}_{\widetilde{A}}\to H^n(S,\mathbb{C})\otimes\mathcal{O}_{\mathbb{C}_t\times\Lambda}\to 0$$

For any holonomic $\mathcal{D}_{\mathbb{C}\times\Lambda}$-module $\mathcal{M}$, we denote by $\mathrm{FL}(\mathcal{M})$ the $\mathcal{D}_{\mathbb{C}_z\times\Lambda}$-module obtained by applying a partial Fourier-Laplace transformation (sending t to $z^2\partial_z$ and ∂_t to z^{-1}) to $\mathcal{M}[\partial_t^{-1}] := \mathbb{C}[t,\lambda_1,\ldots,\lambda_m]\langle\partial_t,\partial_t^{-1},\partial_{\lambda_0},\ldots,\partial_{\lambda_m}\rangle\otimes_{\mathcal{D}_{\mathbb{C}\times\Lambda}}\mathcal{M}$.

Then we have the following corollary of the above result, which can actually be shown independently (and with a considerably simpler proof).

Corollary 3 ([7]). *There is an isomorphism of holonomic $\mathcal{D}_{\mathbb{C}_z\times\Lambda}$-modules*

$$\mathrm{FL}(\mathcal{H}^0\varphi_+\mathcal{O}_{S\times\Lambda}) \cong \mathrm{FL}(\mathcal{M}_{\widetilde{A}}^{(0,\underline{0})}) =: \widehat{\mathcal{M}}_{\widetilde{A}}^{(0,\underline{0})}.$$

From these results we can easily deduce a corresponding statement for the Landau-Ginzburg model.

Corollary 4. *There is an isomorphism of holonomic $\mathcal{D}_{\mathbb{C}_z\times\mathcal{K}_{X_\Sigma}}$-modules*

$$\mathrm{FL}(\mathcal{H}^0W_+\mathcal{O}_{S\times\mathcal{K}_{X_\Sigma}}) \cong (\mathrm{id}_{\mathbb{C}_z}\times g)^+\widehat{\mathcal{M}}_{\widetilde{A}}^{(0,\underline{0})},$$

and the latter module can be explicit described as a cyclic module (i.e., as quotient of $\mathcal{D}_{\mathbb{C}_z\times\mathcal{K}_{X_\Sigma}}$).

In order to lift these results into the category of filtered $\mathcal{D}$-modules, we consider the filtration $F_\bullet$ on $\mathcal{M}_{\widetilde{A}}^{\beta}$ induced by the order filtration on $\mathcal{D}$. This induces a filtration $G_\bullet$ on $\widehat{\mathcal{M}}_{\widetilde{A}}^{\beta}$, defined as $G_k\widehat{\mathcal{M}}_{\widetilde{A}}^{\beta} := \sum_{i\geqslant 0}\partial_t^{-i}F_{k+i}\mathcal{M}_{\widetilde{A}}^{\beta}$. In order to simplify the next statements, we restrict from now on to the case where X_Σ is Fano. For nef varieties, the results are basically the same, but slightly more complicated to state.

Theorem 5. (1) *There is an isomorphism of $\mathcal{O}_{\mathbb{C}_z\times\mathcal{K}_{X_\Sigma}}$-modules with connection*

$$G_0\widehat{\mathcal{M}}_{\widetilde{A}}^{(1,\underline{0})} \cong H^n(\Omega^\bullet[z], zd - dW_1) =: G_0,$$

where W_1 is the first component of the map W from above. Notice that the right hand side is usually called twisted de Rham cohomology*.*

(2) *The module $G_0\widehat{\mathcal{M}}_{\widetilde{A}}^{(1,\underline{0})}$ (and hence also the module $G_0 = H^n(\Omega^\bullet[z], zd - d\varphi_1)$) is $\mathcal{O}_{\mathbb{C}_z\times\mathcal{K}_{X_\Sigma}}$-free, and equipped with a connection operator with poles of Poincaré rank 1 along $\{0\}\times\mathcal{K}_{X_\Sigma}$ and no other singularities.*

In order to express the mirror correspondence as an isomorphism of *Frobenius manifolds*, one needs to extend the above objects to a family of trivial vector bundles over $\mathbb{P}^1_z$, such that the connection acquires a logarithmic pole at $z=\infty$. This is known as a *good basis* or a solution to the *Birkhoff problem* (see [10] and also [9]). The result in the present setup is as follows.

Proposition 6. *Let X_Σ smooth toric and Fano. Consider the Landau-Ginzburg model $W: S\times\mathcal{K}_{X_\Sigma}\to\mathbb{C}_t\times\mathcal{K}_{X_\Sigma}$ and the $\mathcal{O}_{\mathbb{C}_z\times\mathcal{K}_{X_\Sigma}}$-locally free module G_0 from above. Let $\overline{\mathcal{KM}}_{X_\Sigma} = \mathbb{C}^{m-n}$ be the natural partial compactification of $\mathcal{K}_{X_\Sigma}$ induced by the choice of coordinates (i.e., by the identification $\mathcal{K}_{X_\Sigma}\cong(\mathbb{C}^*)^{m-n}$ defined by the choice of a basis of $H^2(X_\Sigma,\mathbb{Z})$). There is an extension $\overline{G}_0\to\mathbb{P}^1_z\times U$ of $(G_0)_{|\mathbb{C}_z\times U}$, where $U\subset\overline{\mathcal{KM}}_{X_\Sigma}$ is Zariski open and contains the origin. $\overline{G}_0$ has the following properties:*

(1) *It is fibrewise trivial, i.e. $p^*p_*\overline{G}_0\cong\overline{G}_0$ if $p:\mathbb{P}^1_z\times U\to U$ is the projection.*

(2) *The connection extends with a logarithmic pole along the normal crossing divisor* $(\{\infty\} \times U) \cup \big(\mathbb{P}^1 \times (U \backslash \mathcal{K}_{X_\Sigma})\big)$.

From this, we deduce the following construction theorem of Frobenius manifolds.

Theorem 7. *Put* $\mu := \dim_{\mathbb{C}} H^*(X_\Sigma, \mathbb{C})$. *There is a germ of a canonical Frobenius structure on* $\mathbb{C}^{\mu-(m-n)} \times U$ *associated to* W, *which has logarithmic poles (in the sense of* [5]*) along* $\mathbb{C}^{\mu-(m-n)} \times (\overline{\mathcal{KM}}_{X_\Sigma} \backslash U)$. *It is isomorphic to the big quantum cohomology of* X_Σ.

In the case of a nef complete intersection $Y \subset X_\Sigma$ (i.e., X_Σ is toric smooth projective as before and Y is the zero locus of a generic section of a split vector bundle $\mathcal{E} = \oplus_{j=1}^{c} \mathcal{L}_j \to X_\Sigma$ where $\mathcal{L}_j \in \mathrm{Pic}(X_\Sigma)$ are ample and such that $-K_{X_\Sigma} - \sum_{j=1}^{c} c_1(\mathcal{L}_j)$ is nef), we can construct a **non-affine** Landau-Ginzburg model, which is a projective morphism $\Pi : Z \to \mathbb{C}_z \times \mathcal{K}_{X_\Sigma}$ from a quasi-projective variety Z (which is not smooth in general). Then the result is as follows.

Theorem 8 ([8]). *Let* $(X_\Sigma, \mathcal{L}_1, \ldots, \mathcal{L}_c)$ *define a nef complete intersection* Y *in* X_Σ. *Consider the ambient (or reduced) quantum* $\mathcal{D}$*-module* $\mathrm{QDM}(X_\Sigma, \mathcal{E} := \oplus_{j=1}^{c} \mathcal{L}_j)$ *of* Y, *as defined in* [4]. *Then we have*

$$\big(\mathrm{FL}\left(\mathcal{H}^0 DR^{-1}(R\Pi_* IC_Z)\right)\big)_{|\mathbb{C}_z \times B_\varepsilon} \cong (id_{\mathbb{C}_z} \times \mathrm{Mir})^* \mathrm{QDM}(X_\Sigma, \mathcal{E})(*\{0\} \times \mathcal{K}_{X_\Sigma})_{|\mathbb{C}_z \times B_\varepsilon},$$

where IC_Z *is the intersection complex of* Z, DR^{-1} *denotes a complex of* $\mathcal{D}$*-modules corresponding to a given constructible complex via the Riemann-Hilbert correspondence,* B_ε *is a small ball in* $\mathcal{K}_{X_\Sigma}$ *around the origin in* $\overline{\mathcal{KM}}_{X_\Sigma}$ *and* Mir *is Givental's* ***mirror map***.

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Strange duality of orbifold Landau-Ginzburg models

Wolfgang Ebeling
(joint work with A. Takahashi, S. M. Gusein-Zade)

Arnold discovered a strange duality $X \leftrightarrow X^\vee$ between the 14 exceptional unimodal singularities. There are two main features of this duality (for functions in three variables):

1. The Dolgachev numbers of X are the Gabrielov numbers of $X^\vee$ and vice versa.
2. K. Saito observed that there is a duality between the characteristic polynomials (reduced zeta functions) of the monodromy such that, if d is the (quasi)degree of a weighted homogeneous equation of X and

$$\varphi(t) = \prod_{m|d} (1 - t^m)^{s_m}, \quad s_m \in \mathbb{Z}, \tag{1}$$

   is the characteristic polynomial of X, then

$$\varphi^\vee(t) = \prod_{m|d} (1 - t^{d/m})^{-s_m} \tag{2}$$

   is the corresponding polynomial of $X^\vee$.

The object of this talk is to show that these features generalize to a mirror symmetry between certain orbifold Landau-Ginzburg models. An *orbifold Landau-Ginzburg model* is a pair (f, G) where f is a non-degenerate invertible polynomial, i.e. a weighted homogeneous polynomial

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i \prod_{j=1}^{n} x_j^{E_{ij}}, \quad a_i \in \mathbb{C}^*, \quad E := (E_{ij}) \text{ invertible over } \mathbb{Q},$$

with an isolated singularity at the origin, and

$$G \subset G_f = \{(\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n : f(\lambda_1 x_1, \ldots, \lambda_n x_n) = f(x_1, \ldots, x_n)\}$$

is a subgroup of its (finite) maximal group of diagonal symmetries. The *Berglund-Hübsch-Henningson transpose* (f^T, G^T) with

$$f^T(x_1, \ldots, x_n) := \sum_{i=1}^{n} a_i \prod_{j=1}^{n} x_j^{E_{ji}}, \quad G^T := \mathrm{Hom}(G_f/G, \mathbb{C}^*),$$

is conjectured to define a mirror dual model. Let G_0 be the subgroup of G_f generated by the exponential grading operator $g_0 := (e^{2\pi\sqrt{-1}w_1}, \ldots, e^{2\pi\sqrt{-1}w_n})$, where $w_1, \ldots, w_n$ are the (rational) weights of f.

1. Mirror symmetry between orbifold curves and cusp singularities with group action

Let $n = 3$ and (f, G) a pair with $G_0 \subset G \subset G_f$. Let $\widehat{G}$ be the extension of $\mathbb{C}^*$ by G and consider the orbifold curve (Deligne-Mumford stack)

$$\mathcal{C}_{(f,G)} := \left[f^{-1}(0)\backslash\{0\} \Big/ \widehat{G} \right].$$

The *Dolgachev numbers* $A_{(f,G)} = (\alpha_1, \ldots, \alpha_r)$ of the pair (f, G) are defined to be the orders of the isotropy groups of G. The genus of the underlying smooth projective curve $\mathcal{C}_{(f,G)}$ is denoted by $g_{(f,G)}$. Let

$$e_{\rm st}(\mathcal{C}_{(f,G)}) := \sum_{p,q \in \mathbb{Q}_{\geqslant 0}} (-1)^{p-q} \dim_{\mathbb{C}} H^{p,q}_{\rm st}(\mathcal{C}_{(f,G)})$$

be the *stringy Euler number* of the orbifold curve $\mathcal{C}_{(f,G)}$, where $H^{p,q}_{\rm st}(\mathcal{C}_{(f,G)})$ denotes the (p,q)th Chen-Ruan orbifold cohomology group of $\mathcal{C}_{(f,G)}$.

The dual pair (f^T, G^T) satisfies $G^T_f = \{1\} \subset G^T \subset G^T_0 = G_{f^T} \cap \mathrm{SL}_3(\mathbb{C})$. If f^T is not simple or simple elliptic then $f^T(x,y,z) - xyz$ is right equivalent to a cusp singularity

$$F(x,y,z) = x^{\gamma_1'} + y^{\gamma_2'} + z^{\gamma_3'} - axyz \quad \text{for some } a \in \mathbb{C}^*$$

which is G^T-invariant. We use this to define *Gabrielov numbers* $\Gamma_{(f^T,G^T)} = (\gamma_1, \ldots, \gamma_s)$ for the pair (f^T, G^T). Let $\mu_{(F,G^T)}$ be the G^T-equivariant Milnor number of F defined by Wall. For an element $g \in \mathrm{SL}_3(\mathbb{C})$ of order r, there is a basis of eigenvectors such that $g = \mathrm{diag}(e^{2\pi\sqrt{-1}a_1/r}, e^{2\pi\sqrt{-1}a_2/r}, e^{2\pi\sqrt{-1}a_3/r})$ with $0 \leqslant a_i < r$. Following Ito and Reid, the number $\frac{1}{r}(a_1 + a_2 + a_3)$ is called the *age* of g. Let j_{G^T} be the number of elements $g \in G^T$ of age 1 which only fix the origin.

We have the following results:

Theorem 1 (—, Takahashi [2])**.** *We have*

$$A_{(f,G_f)} = \Gamma_{(f^T,\{1\})}, \quad A_{(f^T,G_{f^T})} = \Gamma_{(f,\{1\})}.$$

The 14 exceptional unimodal singularities can be defined by suitable non-degenerate invertible polynomials with $G_0 = G_f$. Therefore, Arnold's strange duality is a special case of Theorem 1.

Theorem 2 (—, Takahashi [3])**.** *Let $G_0 \subset G \subset G_f$. Then we have*

$$A_{(f,G)} = \Gamma_{(f^T,G^T)}, \quad e_{\rm st}(\mathcal{C}_{(f,G)}) = \mu_{(F,G^T)}, \quad g_{(f,G)} = j_{G^T}.$$

As a special case, we obtain the extension of Arnold's strange duality by the author and Wall.

2. Equivariant Saito duality

Now let n be arbitrary. We introduce some general notions. Let G be a finite group. A G-set is a set with an action of the group G. The *Grothendieck ring* $K_0(\text{f.}G\text{-sets})$ *of finite G-sets* (also called the *Burnside ring* of G) is the (abelian) group generated by the isomorphism classes of finite G-sets modulo the relation

$[A \amalg B] = [A] + [B]$ for finite G-sets A and B. The multiplication in $K_0(\text{f.}G\text{-sets})$ is defined by the cartesian product.

Let f be a weighted homogeneous polynomial of degree d with an isolated singularity at the origin. The Milnor fibre $V_f = f^{-1}(1)$ of f can be regarded as a $\mathbb{Z}_d$-set. A function of the form (1) corresponds to the element $\sum_{m|d} s_m[\mathbb{Z}_d/\mathbb{Z}_{d/m}]$ of the Burnside ring $K_0(\text{f.}\mathbb{Z}_d\text{-sets})$. Let G be a subgroup of the symmetry group G_f of f containing the monodromy transformation. The *G-equivariant zeta function* of f is the element

$$\zeta_f^G = \sum_{H \subset G} \chi(V_f^{(H)}/G)[G/H]$$

of the Burnside ring $K_0(\text{f.}G\text{-sets})$, where $V_f^{(H)}$ denotes the set of points of the Milnor fibre V_f with isotropy group H. The *reduced G-equivariant zeta function* of f is $\overline{\zeta}_f^G = \zeta_f^G - 1$.

For a finite abelian group G, denote by $G^* = \mathrm{Hom}\,(G, \mathbb{C}^*)$ its group of characters. The reason for the minus sign in formula (2) is connected with the fact that it was originally formulated only for functions in $n = 3$ variables. If we neglect the sign then Saito's duality can be expressed in terms of Burnside rings as follows:

$$a = \sum_{H \subset \mathbb{Z}_d} s_H[\mathbb{Z}_d/H] \in K_0(\text{f.}\mathbb{Z}_d\text{-sets}) \mapsto \widehat{a} = \sum_{H \subset \mathbb{Z}_d} s_H[\mathbb{Z}_d^*/H^T] \in K_0(\text{f.}\mathbb{Z}_d^*\text{-sets}).$$

This leads to the following definition of an *equivariant Saito duality*:

$$\begin{array}{cccc} D_G & : & K_0(\text{f.}G\text{-sets}) & \rightarrow & K_0(\text{f.}G^*\text{-sets}) \\ & & a = \sum_{H \subset G} s_H[G/H] & \mapsto & \widehat{a} = D_G a = \sum_{H \subset G} s_H[G^*/H^T] \end{array}.$$

The isomorphism D_G can be regarded as a Fourier transformation from $K_0(\text{f.}G\text{-sets})$ to $K_0(\text{f.}G^*\text{-sets})$.

Now let f be an invertible polynomial in n variables and $G = G_f$ be its maximal group of symmetries. Then $G^* = G_f^* = G_{f^T}$ is the maximal group of symmetries of the transpose f^T. We have the following result:

Theorem 3 (—, Gusein-Zade [1]). *The reduced equivariant zeta functions $\overline{\zeta}_f^G$ and $\overline{\zeta}_{f^T}^{G^*}$ of the polynomials f and f^T respectively are (up to the sign $(-1)^n$) Saito dual to each other:*

$$\overline{\zeta}_{f^T}^{G^*} = (-1)^n D_G \overline{\zeta}_f^G \,.$$

Since for the 14 exceptional unimodal singularities $n = 3$ and $G_f = G_0 = \mathbb{Z}_d$, we obtain Saito's original duality as a special case.

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Characteristic classes of Hilbert schemes of points via symmetric products

Jörg Schürmann

(joint work with S. Cappell, L. Maxim, T. Ohmoto and S. Yokura)

We are considering the complex algebraic context with X a smooth quasi-projective variety of pure dimension d as in the following cartesian diagram:

$$\begin{array}{ccc} (Hilb^n_X)_{red} =: X^{[n]} & \longleftarrow & Hilb^n_{X,x} \simeq Hilb^n_{\mathbb{C}^d,0} \\ {\scriptstyle\pi_n}\downarrow & & \downarrow \\ X^n/S_n =: X^{(n)} & \overset{d^n}{\longleftarrow} & X \supset \{x\}, \end{array}$$

with π_n the proper Hilbert-Chow morphism from the (reduced) Hilbert scheme $X^{[n]}$ of n points on X to the symmetric product $X^{(n)}$. Then π_n is a nice stratified map, whose fiber over a point x in the deepest stratum X (diagonally embedded by d^n) is given by the punctual Hilbert scheme $Hilb^n_{X,x}$. This fiber is independent of the choice of the smooth manifold X of dimension d so that $Hilb^n_{X,x} \simeq Hilb^n_{\mathbb{C}^d,0}$.

S.M.Gusein-Zade, I.Luengo and A.Melle-Hernández [3] introduced the notion of a power structure on a (semi-)ring R giving sense to an expression f^m for $m \in R$ and $f \in 1 + tR[[t]]$ a normalized formal power series with coefficients from R, satisfying seven expected rules like

$$(iii)\ (f \cdot g)^m = f^m \cdot g^m, \quad (v)\ f^{n \cdot m} = (f^n)^m \quad \text{and} \quad (vii)\ f(t^k)^m = f^m|_{t \mapsto t^k}.$$

Moreover, they proved the formula

$$(1) \qquad 1 + \sum_{n \geqslant 1} [X^{[n]}] \cdot t^n = \left(1 + \sum_{n \geqslant 1} \left[Hilb^n_{\mathbb{C}^d,0} \right] \cdot t^n \right)^{[X]} \in S_0(var/\mathbb{C})[[t]]$$

in the motivic semi-ring $S_0(var/\mathbb{C})$ of complex algebraic varieties. Here for

$$A(t) = 1 + \sum_{i=1}^{\infty} [A_i] t^i \in S_0(var/\mathbb{C})[[t]] \quad \text{and} \quad [X] \in S_0(var/\mathbb{C}) :,$$

the following expression defines *geometrically* a power structure on $S_0(var/\mathbb{C})$:

$$(2) \quad (A(t))^{[X]} := 1 + \sum_{n=1}^{\infty} \left\{ \sum_{\sum ik_i = n} \left[\left((\prod_i X^{k_i}) \setminus \Delta \right) \times \prod_i A_i^{k_i} / \prod_i S_{k_i} \right] \right\} \cdot t^n,$$

where $k_i \in \mathbb{N}_0$, Δ is the large diagonal in $X^{\sum_i k_i}$, and the symmetric group S_{k_i} acts by permuting the corresponding k_i factors in $\prod_i X^{k_i} \supset (\prod_i X^{k_i}) \setminus \Delta$ and the spaces A_i simultaneously. The Kapranov zeta function

$$\lambda_t([X]) := (1-t)^{-[X]} := (1 + t + t^2 + \cdots)^{[X]} = 1 + \sum_{n=1}^{\infty} \left[X^{(n)} \right] \cdot t^n$$

defines a pre-lambda structure on the associated Grothendieck ring $K_0(var/\mathbb{C})$ of complex algebraic varieties. And a pre-lambda structure $\lambda_t(\cdot) =: (1-t)^{-(\cdot)}$ on a ring R determines *algebraically* a power structure on R, since a power series $A(t) = 1 + \sum_{i=1}^{\infty} a_i t^i \in 1 + tR[[t]]$ admits a unique *Euler product decomposition*

$$A(t) = \prod_{k=1}^{\infty} (1-t^k)^{-b_k} = \prod_{k=1}^{\infty} \left((1-t)^{-b_k}|_{t\mapsto t^k} \right) \tag{3}$$

with $b_k \in R$ (see also [2]). Then a power structure on R can be uniquely defined by using (iii) and (vii).

In [4] we extended the geometric definition of the motivic power structure to a *motivic exponentation* with values in the motivic Pontrjagin semi-ring. Let F be a functor to the category of abelian (semi-)groups defined on complex quasi-projective varieties, covariantly functorial for all (proper) morphisms. Assume F is also endowed with a commutative, associative and bilinear cross-product $\boxtimes$ commuting with (proper) push-forwards $(-)_*$ (or $(-)_!$), with a unit $1 \in F(pt)$. Our main examples for $F(X)$ are the relative motivic Grothendieck (semi-)group $K_0(var/X)$ or $S_0(var/X)$ and the Borel-Moore homology $H_*(X) := H^{BM}_{even}(X) \otimes \mathbb{Q}[y]$. We define the commutative Pontrjagin (semi-)ring $(PF(X), \odot)$ by

$$PF(X) := \sum_{n=0}^{\infty} F(X^{(n)}) \cdot t^n := \prod_{n=0}^{\infty} F(X^{(n)}),$$

with product $\odot$ induced via

$$\odot : F(X^{(n)}) \times F(X^{(m)}) \overset{\boxtimes}{\to} F(X^{(n)} \times X^{(m)}) \overset{(-)_*}{\to} F(X^{(n+m)}),$$

and unit $1 \in F(X^{(0)}) = F(pt)$. It is easy to see that, if $f : X \to Y$ is a (proper) morphism, then we get an induced (semi-)ring homomorphism

$$f_* := (f_*^{(n)})_n : PF(X) \to PF(Y),$$

with $f^{(n)} : X^{(n)} \to Y^{(n)}$ the corresponding (proper) morphism on the n-th symmetric products. The *k-th power operation* $P_k = \left(p_{k*}^{(n)}\right) : PF(X) \to PF(X)$ for $k \geqslant 1$ is the (semi-)ring homomorphism defined by the push forwards $p_{k*}^{(n)}$ for the natural maps $p_k^{(n)} : X^{(n)} \to X^{(nk)}$ induced by the diagonal embeddings $X^n \to (X^n)^k \cong X^{nk}$. Viewing the coefficients of t^n in (2) as elements of $S_0(var/X^{(n)})$, one gets for X fixed a *motivic exponentation*:

$$(-)^X : 1 + tS_0(var/(\mathbb{C})[[t]] \to PS_0(var/X) := \sum_{n\geqslant 0} S_0(var/X^{(n)}) \cdot t^n. \tag{4}$$

This satisfies rules like (iii') $(A(t)\cdot B(t))^X = (A(t))^X \odot (B(t))^X$,

(v') $\pi_! \left((A(t))^{X'\times X} \right) = \left((A(t))^{[X']} \right)^X$, for $\pi : X' \times X \to X$ the projection,

(vii') $\left(A(t^k)\right)^X = P_k((A(t))^X)$, with P_k the k-th power operation.

Let X be a smooth and pure d-dimensional complex quasi-projective variety. Then

$$1+\sum_{n\geqslant 1}[X^{[n]}\stackrel{\pi_n}{\to}X^{(n)}]\cdot t^n=\left(1+\sum_{n\geqslant 1}\left[\mathrm{Hilb}^n_{\mathbb{C}^d,0}\right]\cdot t^n\right)^X. \tag{5}$$

The un-normalized Hirzebruch class of J.-P.Brasselet, J.Schürmann and S.Yokura [1] is the unique natural transformation

$$T_{y*}:K_0(var/X)\to H_*(X):=H^{BM}_{even}(X)\otimes\mathbb{Q}[y]$$

commuting with push-forward for proper morphisms, satisfying for X smooth the *normalization*:

$$T_{y*}(X):=T_{y*}([id_X])=\sum_{i\geqslant 0}:\big(ch^*(\Omega^i_X)\cdot td^*(X)\cap[X]\big)\cdot y^i:.$$

Here ch^* is the Chern character and td^* the Todd-class. For $X=pt$ one gets the χ_y-genus. Moreover, T_{y*} also commutes with the cross-products $\boxtimes$, so that one gets a functorial ring homomorphism of Pontrjagin rings $T_{y*}(-):PK_0(var/X)\to PH_*(X)$ commuting with the power operations P_k. In this way one finally gets from (5) the following (see [4]):

Theorem 1. *Let X be a smooth complex quasi-projective variety of pure dimension d. Then the following generating series formula for the push-forwards under the Hilbert-Chow morphisms of the un-normalized Hirzebruch classes $T_{(-y)_*}(X^{[n]})$ of Hilbert schemes holds in the Pontrjagin ring $PH_*(X)$:*

$$\sum_{n=0}^{\infty}\pi_{n*}T_{(-y)_*}(X^{[n]})\cdot t^n=\prod_{k=1}^{\infty}(1-t^k\cdot d^k_*)^{-\chi_{-y}(\alpha_k)\cdot T_{(-y)_*}(X)}, \tag{6}$$

where the $\alpha_k\in K_0(var/\mathbb{C})$ are the coefficients appearing in the Euler product of (1) for the geometric power structure on the pre-lambda ring $K_0(var/\mathbb{C})$.

Here we use the group homomorphisms

$$(1-t\cdot d_*)^{-(\cdot)}:=\exp\left(\sum_{r=1}^{\infty}\Psi_r d^r_*(\cdot)\frac{t^r}{r}\right):\big(H^{BM}_{even}(X)\otimes\mathbb{Q}[y],+\big)\to(PH_*(X),\odot)$$

and $(1-t^k\cdot d^k_*)^{-(\cdot)}:=P_k\circ\big((1-t\cdot d_*)^{-(\cdot)}\big)$, with P_k the k-th power operation and $\Psi_r:H^{BM}_{2k}(-)\otimes\mathbb{Q}[y]\to H^{BM}_{2k}(-)\otimes\mathbb{Q}[y]$ the *r-th homological Adams operation* defined by multiplying with $1/r^k$ on $H^{BM}_{2k}(-;\mathbb{Q})$ together with $\Psi_r(y)=y^r$.

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