Definition of Siegel Modular Forms

- Siegel Upper Half Space: $\mathcal{H}_n = \{Z \in M_{n\times n}^{\mathrm{sym}}(\mathbb{C}) : \operatorname{Im} Z > 0\}$.
- Symplectic group: $\sigma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{R})$ acts on $Z \in \mathcal{H}_n$ by $\sigma \cdot Z = (AZ+B)(CZ+D)^{-1}$.
- $\Gamma \subseteq \mathrm{Sp}_n(\mathbb{R})$ such that $\Gamma \cap \mathrm{Sp}_n(\mathbb{Z})$ has finite index in Γ and $\mathrm{Sp}_n(\mathbb{Z})$
- Slash action: For $f : \mathcal{H}_n \to \mathbb{C}$ and $\sigma \in \mathrm{Sp}_n(\mathbb{R})$, $(f|_k\sigma)(Z) = \det(CZ+D)^{-k} f(\sigma \cdot Z)$.
- Siegel Modular Forms: $M_k(\Gamma)$ is the $\mathbb{C}$-vector space of holomorphic $f : \mathcal{H}_n \to \mathbb{C}$ that are "bounded at the cusps" and that satisfy $f|_k\sigma = f$ for all $\sigma \in \Gamma$.
- Cusp Forms: $S_k(\Gamma) = \{f \in M_k(\Gamma)$ that "vanish at the cusps"$\}$

Definition of paramodular form

- A *paramodular form* is a Siegel modular form for a paramodular group. In degree 2, the paramodular group of level N, is

$$\Gamma = K(N) = \begin{pmatrix} * & N* & * & * \\ * & * & * & */N \\ * & N* & * & * \\ N* & N* & N* & * \end{pmatrix} \cap \mathrm{Sp}_2(\mathbb{Q}), \quad * \in \mathbb{Z},$$

- $K(N)$ is the stabilizer in $\mathrm{Sp}_2(\mathbb{Q})$ of $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus N\mathbb{Z}$.
- ${}^T K(N) \backslash \mathcal{H}_2$ is a moduli space for complex abelian surfaces with polarization type $(1, N)$. (T is "transpose" here.)
- The paramodular Fricke involution splits paramodular forms into plus and minus spaces.

$$S_k(K(N)) = S_k(K(N))^+ \oplus S_k(K(N))^-$$

Arithmetic spin L-function

Any errors are my own.

Roberts and Schmidt define the following Hecke operators in order to compute the Langlands L-function of an eigenform $f \in S_k(K(N))^\epsilon$.

- $T_{0,1}(p) = K(N)\,\mathrm{diag}(p, p, 1, 1)K(N)$
- $T_{1,0}(p) = K(N)\,\mathrm{diag}(p, p^2, p, 1)K(N)$
- $f|_k T_{0,1}(p) = \lambda_p f$, $f|_k T_{1,0}(p) = \mu_p f$, $\epsilon_p =$ Atkin-Lehner sign
- $p \nmid N$
  $Q_p(f,t) = 1 - \lambda_p t + (p\mu_p + p^{2k-3} + p^{2k-5})t^2 - p^{2k-3}\lambda_p t^3 + p^{4k-6}t^4$
- $p || N$
  $Q_p(f,t) = 1 - (\lambda_p + p^{k-3}\epsilon_p)t + (p\mu_p + p^{2k-3})t^2 + p^{3k-5}\epsilon_p t^3$
- $p^2 \mid N$
  $Q_p(f,t) = 1 - \lambda_p t + (p\mu_p + p^{2k-3})t^2$

Arithmetic spin L-function

Roberts and Schmidt compute the Langlands L-function of an eigenform $f \in S_k(K(N))^\epsilon$. (But I rewrote it in the arithmetic normalization.)

- $$L^{\mathrm{arith}}(s, f, \mathrm{spin}) = \prod_p Q_p(f, p^{-s})^{-1}$$
- $$\Lambda^{\mathrm{arith}}(s, f, \mathrm{spin}) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s-k+2)L^{\mathrm{arith}}(s, f, \mathrm{spin})$$
- $\Gamma_{\mathbb{C}}(s) = 2\Gamma(s)(2\pi)^{-s}$
- Proven functional equation for the completed spin L-function

$$\Lambda^{\mathrm{arith}}(2k-2-s, f, \mathrm{spin}) = (-1)^k \epsilon\, N^{s-k+1}\Lambda^{\mathrm{arith}}(s, f, \mathrm{spin})$$

Fourier-Jacobi expansions

- Fourier expansion of Siegel modular form:

$$f(Z) = \sum_{T \geq 0} a(T; f) e(\mathrm{tr}(ZT))$$

- Fourier expansion of paramodular form $f \in M_k(K(N))$ in coordinates:

$$f\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) = \sum_{\substack{n,r,m \in \mathbb{Z}: \\ n,m \geq 0,\, 4Nnm \geq r^2}} a\left(\begin{pmatrix} n & r/2 \\ r/2 & Nm \end{pmatrix}; f\right) e(n\tau + rz + Nm\omega)$$

- Fourier-Jacobi expansion of paramodular form $f \in M_k(K(N))$:

$$f\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) = \sum_{m \in \mathbb{Z}:\, m \geq 0} \phi_m(\tau, z) e(Nm\omega)$$

Fourier-Jacobi expansion (FJE)

$$\text{FJE: } f\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) = \sum_{m \in \mathbb{Z}:\, m \geq 0} \phi_m(\tau, z) e(Nm\omega)$$

The Fourier-Jacobi expansion of a paramodular form is fixed *term-by-term* by the following subgroup of the paramodular group $K(N)$:

$$P_{2,1}(\mathbb{Z}) = \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \cap \mathrm{Sp}_2(\mathbb{Z}), \qquad * \in \mathbb{Z},$$

- $P_{2,1}(\mathbb{Z})/\{\pm I\} \cong \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathrm{Heisenberg}(\mathbb{Z})$

Thus the coefficients ϕ_m are automorphic forms in their own right and easier to compute than Siegel modular forms. This is one motivation for the introduction of Jacobi forms.

Definition of Jacobi Forms: Automorphicity

Level one

- Assume $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ is holomorphic.

$$\begin{aligned} E_m\phi : \mathcal{H}_2 &\to \mathbb{C} \\ \left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) &\mapsto \phi(\tau, z) e(m\omega) \end{aligned}$$

- Assume that $E_m\phi$ transforms by $\chi \det(CZ + D)^k$ for

$$P_{2,1}(\mathbb{Z}) = \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \cap \mathrm{Sp}_2(\mathbb{Z}), \qquad * \in \mathbb{Z},$$

Definition of Jacobi Forms: Support

- Jacobi forms are tagged with additional adjectives to reflect the support $\mathrm{supp}(\phi) = \{(n, r) \in \mathbb{Q}^2 : c(n, r; \phi) \neq 0\}$ of the Fourier expansion

$$\phi(\tau, z) = \sum_{n,r \in \mathbb{Q}} c(n, r; \phi) q^n \zeta^r, \qquad q = e(\tau), \zeta = e(z).$$

- $\phi \in J_{k,m}^{\mathrm{cusp}}$: automorphic and $c(n, r; \phi) \neq 0 \implies 4mn - r^2 > 0$
- $\phi \in J_{k,m}$: automorphic and $c(n, r; \phi) \neq 0 \implies 4mn - r^2 \geq 0$
- $\phi \in J_{k,m}^{\mathrm{weak}}$: automorphic and $c(n, r; \phi) \neq 0 \implies n \geq 0$
- $\phi \in J_{k,m}^{\mathrm{wh}}$: automorphic and $c(n, r; \phi) \neq 0 \implies n \gg -\infty$ ("wh" stands for *weakly holomorphic*)