# Paramodular forms of weight three

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Workshop
Calabi-Yau Motives
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# An outline of this talk

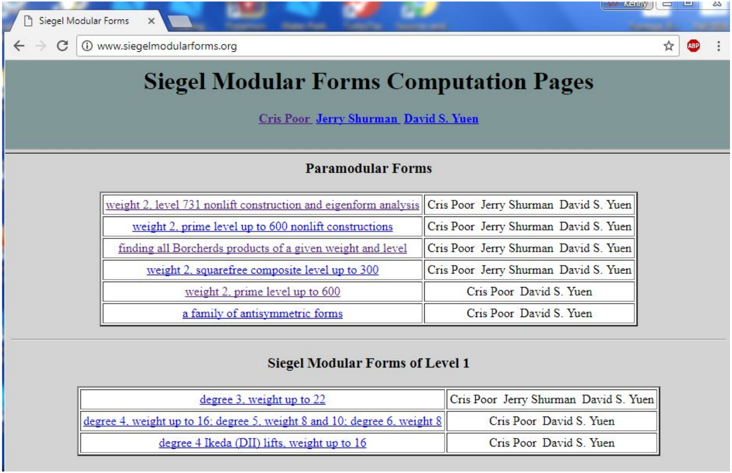
1. Part *I*. Locating nonlift paramodular newforms.
2. Part *II*. Definitions: paramodular forms, Jacobi forms, theta blocks.
3. Part *III*. Constructions of paramodular forms.

# Joint work

This video talk includes joint work with Jerry Shurman (Reed College, Portland, Oregon) and David S. Yuen (Univeristy of Hawaii, Manoa, Hawaii).

# siegelmodularforms.org

- Our paramodular website: www.siegelmodularforms.org
- (Joint with J. Shurman, D. S. Yuen.)


Siegel Modular Forms

www.siegelmodularforms.org

# Siegel Modular Forms Computation Pages

Cris Poor Jerry Shurman David S. Yuen

**Paramodular Forms**

| | |
|---|---|
| weight 2, level 731 nonlift construction and eigenform analysis | Cris Poor Jerry Shurman David S. Yuen |
| weight 2, prime level up to 600 nonlift constructions | Cris Poor Jerry Shurman David S. Yuen |
| finding all Borcherds products of a given weight and level | Cris Poor Jerry Shurman David S. Yuen |
| weight 2, squarefree composite level up to 300 | Cris Poor Jerry Shurman David S. Yuen |
| weight 2, prime level up to 600 | Cris Poor David S. Yuen |
| a family of antisymmetric forms | Cris Poor David S. Yuen |

**Siegel Modular Forms of Level 1**

| | |
|---|---|
| degree 3, weight up to 22 | Cris Poor Jerry Shurman David S. Yuen |
| degree 4, weight up to 16; degree 5, weight 8 and 10; degree 6, weight 8 | Cris Poor David S. Yuen |
| degree 4 Ikeda (DII) lifts, weight up to 16 | Cris Poor David S. Yuen |

# Objects of interest

- Paramodular cusp forms of weight $k$ and paramodular level $N$.

$$S_k(K(N))$$

- Jacobi cusp forms of weight $k$ and index $N$.

$$J_{k,N}^{\mathrm{cusp}}$$

- The Gritsenko lift from Jacobi cusp forms of index $N$ to paramodular cusp forms of level $N$ is an advanced version of the Maass lift.

$$\mathrm{Grit}: J_{k,N}^{\mathrm{cusp}} \to S_k(K(N))$$

- We are interested in the nonlift Hecke eigenforms, especially those of low weight.

$$\text{new eigenforms in } S_k(K(N)) \setminus \mathrm{Grit}\left(J_{k,N}^{\mathrm{cusp}}\right)$$

# Weight $k = 1$

- Paramodular cusp forms of weight 1 are trivial.

$$S_1\left(K(N)\right) = \{0\}$$

- This follows from the result of Skoruppa that Jacobi cusp forms of weight 1 (and trivial character) are trivial.

$$J_{1,N}^{\mathrm{cusp}} = \{0\}$$

- Not so relevant here but, if we allow a character, there are nontrivial weight 1 Jacobi cusp forms, for example the *theta quarks* of Gritsenko, Skoruppa, Zagier.

$$\frac{\vartheta_a \vartheta_b \vartheta_{a+b}}{\eta} \in J_{1,\, a^a+ab+b^2}^{\mathrm{cusp}}(\chi_3), \qquad a \not\equiv b \bmod 3$$

# Weight $k = 2$

- Paramodular Conjecture of Brumer and Kramer: the modularity of abelian surfaces defined over $\mathbb{Q}$ with minimal endomorphisms is shown by weight two nonlift paramodular newforms with rational eigenvalues.
- Poor, Shurman, Yuen have some (partly rigorous, partly heuristic) tables up to $N \leq 1000$

| $N$ | $\dim J_{2,N}^{\mathrm{cusp}}$ | $\dim S_2(K(N))$ | various comments |
|---|---|---|---|
| 249 | 5 | 6 | BP+Grit; Jac |
| 277 | 10 | 11 | modular! $Q/L$; Jac |
| 295 | 6 | 7 | BP+Grit; Jac |
| 349 | 11 | 12 | BP+Grit; Jac |
| 353 | 11 | 12 | modular! BP+Grit; Jac |

# Modularity proven for $N = 277, 353, 587^-$

(On arXiv— *On the paramodularity of typical abelian surfaces,* Brumer, Pacetti, Poor, Tornarìa, Voight, Yuen)

- Generalize method of Faltings-Serre to GSp(4).
- Need residual representation at $p = 2$ irreducible.
- Class field theory computer calculations classifying extensions with prescribed ramification and Galois group contained in $\mathrm{GSp}(4, \mathbb{F}_2)$.
- Hecke eigenvalue computer calculations for paramodular eigenforms written as rational functions of Gritsenko lifts and Borcherds products.
- Galois representations associated to automorphic representations whose archimedean component is a holomorphic limit of discrete series.

# Modularity proven for $N = 731$

Tobias Berger and Krzysztof Klosin: *Deformations of Saito-Kurokawa type and the Paramodular Conjecture.*

arXiv:1710.10228v2 (appendix by Poor, Shurman, Yuen)

- Use theory of Galois deformations.
- Use residual representation at $p$ *reducible* and find elliptic $E/\mathbb{Q}$ realizing a two dimensional piece.
- Possess a paramodular nonlift eigenform that is congruent to a Gritsenko lift modulo a prime $p$ for which the abelian surface $A/\mathbb{Q}$ has rational $p$-torsion. (For $N = 731$, this is $p = 5$.)
- Galois representations associated to automorphic representations whose archimedean component is a holomorphic limit of discrete series.

# Weight $k = 3$

- The modularity of some hypergeometric motives is conjecturally shown by weight three nonlift paramodular newforms with rational eigenvalues.
- Examples of candidate hypergeometric motives have been shared with us by Dave Roberts in 2014; for example, conductor $N = 257$ matches a few Euler factors of an eigenform in $S_3(K(257))$.
- Perhaps there are also other types of arithmetic objects or Galois representations that correspond to weight three paramodular newforms?

# Weight $k = 3$ should be more accessible than weight two

- Galois representations arising from weight three paramodular forms are better understood.
  Papers of Rainer Weissauer
- Dimension formulae for $\dim S_k\,(K(N))$ are known ($k \geq 3$) for $N$ prime by Ibukiyama and for $N$ squarefree by Ibukiyama and Kitayama.

Tomoyoshi Ibukiyama: *Dimension formulas of Siegel modular forms of weight 3 and supersingular abelian surfaces* Siegel Modular Forms and AbelianVarieties. Hamana Lake (2007).

Tomoyoshi Ibukiyama and Hidetaka Kitayama: *Dimension formulas of paramodular forms of squarefree level and comparison with inner twist* J. Math. Soc. Japan 69 (2017).

# Weight $k = 3$

- Use Ibukiyama's formula for $\dim S_3\left(K(p)\right)$ and Skoruppa and Zagier's formula for $\dim J_{3,p}^{\text{cusp}}$.

| $p$ | $\dim J_{3,p}^{\text{cusp}}$ | $\dim S_3\left(K(p)\right)$ | various comments |
|---|---|---|---|
| 61 | 6 | 7 | BP+Grit and Q/L |
| 73 | 8 | 9 | BP+Grit and Q/L |
| 79 | 7 | 8 | BP+Grit and Q/L |
| 89 | 8 | 9 | |
| 97 | 11 | 13 | |
| 101 | 9 | 11 | |
| 103 | 10 | 12 | |

# Weight $k = 3$

- Ash, Gunnells, and McConnell saw eigensystems for $p \in \{61, 73, 79\}$ in $H^5(\Gamma_0(p), \mathbb{C})$ and predicted that these came from Siegel modular forms.
  Avner Ash, Paul E. Gunnells, and Mark McConnell: *Cohomology of congruence subgrous of* $\mathrm{SL}_4(\mathbb{Z})$*. II* J. Number Theory 128 (2008).
- $$\Gamma_0(p) = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ p* & p* & p* & * \end{pmatrix} \cap \mathrm{SL}(4, \mathbb{Z}), \qquad * \in \mathbb{Z}.$$
- It was Armand Brumer who noticed the connection to paramodular forms evident in the dimension formulae.

# Euler factors computed by Ash, Gunnells, and McConnell.

- Eigensystem from $H^5(\Gamma_0(61), \mathbb{C})$

$$Q_2(t) = 1 + 7t + 24t^2 + 56t^3 + 64t^4$$
$$Q_3(t) = 1 + 3t + 3t^2 + 81t^3 + 729t^4$$

- Eigensystem from $H^5(\Gamma_0(73), \mathbb{C})$

$$Q_2(t) = 1 + 6t + 22t^2 + 48t^3 + 64t^4$$
$$Q_3(t) = 1 + 2t + 3t^2 + 54t^3 + 729t^4$$

- Eigensystem from $H^5(\Gamma_0(79), \mathbb{C})$

$$Q_2(t) = 1 + 5t + 14t^2 + 40t^3 + 64t^4$$
$$Q_3(t) = 1 + 5t + 42t^2 + 135t^3 + 729t^4$$

# Weight $k = 3$

- Use Ibukiyama and Katsurada's formula for $\dim S_3(K(N))$, $N$ squarefree, or heuristic programs written by Jerry and David

| $p$ | $\dim J_{3,p}^{\text{cusp}}$ | $\dim S_3(K(p))$ | various comments |
|---|---|---|---|
| 61 | 6 | 7 | BP+Grit and Q/L |
| 69 | 5 | 6 | |
| 73 | 8 | 9 | BP+Grit and Q/L |
| 76 | 5 | 6 | not squarefree |
| 79 | 7 | 8 | BP+Grit and Q/L |
| 82 | 7 | 8 | |

# Part Two

Definitions: paramodular forms, Jacobi forms, theta blocks

# Definition of Siegel Modular Forms

- Siegel Upper Half Space: $\mathcal{H}_n = \{Z \in M_{n\times n}^{\mathrm{sym}}(\mathbb{C}) : \operatorname{Im} Z > 0\}$.
- Symplectic group: $\sigma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{R})$ acts on $Z \in \mathcal{H}_n$ by $\sigma \cdot Z = (AZ+B)(CZ+D)^{-1}$.
- $\Gamma \subseteq \mathrm{Sp}_n(\mathbb{R})$ such that $\Gamma \cap \mathrm{Sp}_n(\mathbb{Z})$ has finite index in $\Gamma$ and $\mathrm{Sp}_n(\mathbb{Z})$
- Slash action: For $f : \mathcal{H}_n \to \mathbb{C}$ and $\sigma \in \mathrm{Sp}_n(\mathbb{R})$, $(f|_k\sigma)(Z) = \det(CZ+D)^{-k} f(\sigma \cdot Z)$.
- Siegel Modular Forms: $M_k(\Gamma)$ is the $\mathbb{C}$-vector space of holomorphic $f : \mathcal{H}_n \to \mathbb{C}$ that are "bounded at the cusps" and that satisfy $f|_k\sigma = f$ for all $\sigma \in \Gamma$.
- Cusp Forms: $S_k(\Gamma) = \{f \in M_k(\Gamma)$ that "vanish at the cusps"$\}$

# Definition of paramodular form

- A *paramodular form* is a Siegel modular form for a paramodular group. In degree 2, the paramodular group of level $N$, is

$$\Gamma = K(N) = \begin{pmatrix} * & N* & * & * \\ * & * & * & */N \\ * & N* & * & * \\ N* & N* & N* & * \end{pmatrix} \cap \mathrm{Sp}_2(\mathbb{Q}), \quad * \in \mathbb{Z},$$

- $K(N)$ is the stabilizer in $\mathrm{Sp}_2(\mathbb{Q})$ of $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus N\mathbb{Z}$.
- ${}^{T}K(N)\backslash\mathcal{H}_2$ is a moduli space for complex abelian surfaces with polarization type $(1, N)$. ($T$ is "transpose" here.)
- The paramodular Fricke involution splits paramodular forms into plus and minus spaces.

$$S_k(K(N)) = S_k(K(N))^+ \oplus S_k(K(N))^-$$

# Arithmetic spin L-function

Any errors are my own.

Roberts and Schmidt define the following Hecke operators in order to compute the Langlands $L$-function of an eigenform $f \in S_k\left(K(N)\right)^{\epsilon}$.

- $T_{0,1}(p) = K(N)\,\mathrm{diag}(p, p, 1, 1)K(N)$
- $T_{1,0}(p) = K(N)\,\mathrm{diag}(p, p^2, p, 1)K(N)$
- $f|_k T_{0,1}(p) = \lambda_p f$, $f|_k T_{1,0}(p) = \mu_p f$, $\epsilon_p =$ Atkin-Lehner sign

- $p \nmid N$
  $Q_p(f, t) = 1 - \lambda_p t + (p\mu_p + p^{2k-3} + p^{2k-5})t^2 - p^{2k-3}\lambda_p t^3 + p^{4k-6}t^4$
- $p || N$
  $Q_p(f, t) = 1 - (\lambda_p + p^{k-3}\epsilon_p)t + (p\mu_p + p^{2k-3})t^2 + p^{3k-5}\epsilon_p t^3$
- $p^2 \mid N$
  $Q_p(f, t) = 1 - \lambda_p t + (p\mu_p + p^{2k-3})t^2$

# Arithmetic spin L-function

Roberts and Schmidt compute the Langlands $L$-function of an eigenform $f \in S_k(K(N))^{\epsilon}$. (But I rewrote it in the arithmetic normalization.)

- $$L^{\mathrm{arith}}(s, f, \mathrm{spin}) = \prod_p Q_p(f, p^{-s})^{-1}$$
- $$\Lambda^{\mathrm{arith}}(s, f, \mathrm{spin}) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s-k+2)L^{\mathrm{arith}}(s, f, \mathrm{spin})$$
- $\Gamma_{\mathbb{C}}(s) = 2\Gamma(s)(2\pi)^{-s}$
- Proven functional equation for the completed spin $L$-function

$$\Lambda^{\mathrm{arith}}(2k-2-s, f, \mathrm{spin}) = (-1)^k \epsilon\, N^{s-k+1}\Lambda^{\mathrm{arith}}(s, f, \mathrm{spin})$$

# Fourier-Jacobi expansions

- Fourier expansion of Siegel modular form:

$$f(Z) = \sum_{T \geq 0} a(T; f) e(\mathrm{tr}(ZT))$$

- Fourier expansion of paramodular form $f \in M_k(K(N))$ in coordinates:

$$f\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) = \sum_{\substack{n,r,m \in \mathbb{Z}: \\ n,m \geq 0,\, 4Nnm \geq r^2}} a\left(\begin{pmatrix} n & r/2 \\ r/2 & Nm \end{pmatrix}; f\right) e(n\tau + rz + Nm\omega)$$

- Fourier-Jacobi expansion of paramodular form $f \in M_k(K(N))$:

$$f\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) = \sum_{m \in \mathbb{Z}:\, m \geq 0} \phi_m(\tau, z) e(Nm\omega)$$

# Fourier-Jacobi expansion (FJE)

$$\text{FJE: } f\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) = \sum_{m \in \mathbb{Z}:\ m \geq 0} \phi_m(\tau, z) e(Nm\omega)$$

The Fourier-Jacobi expansion of a paramodular form is fixed *term-by-term* by the following subgroup of the paramodular group $K(N)$:

$$P_{2,1}(\mathbb{Z}) = \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \cap \mathrm{Sp}_2(\mathbb{Z}), \qquad * \in \mathbb{Z},$$

- $P_{2,1}(\mathbb{Z})/\{\pm I\} \cong \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathrm{Heisenberg}(\mathbb{Z})$

Thus the coefficients $\phi_m$ are automorphic forms in their own right and easier to compute than Siegel modular forms. This is one motivation for the introduction of Jacobi forms.

# Definition of Jacobi Forms: Automorphicity

Level one

- Assume $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ is holomorphic.

$$\begin{aligned} E_m\phi : \mathcal{H}_2 &\to \mathbb{C} \\ \left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) &\mapsto \phi(\tau, z) e(m\omega) \end{aligned}$$

- Assume that $E_m\phi$ transforms by $\chi \det(CZ + D)^k$ for

$$P_{2,1}(\mathbb{Z}) = \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \cap \mathrm{Sp}_2(\mathbb{Z}), \qquad * \in \mathbb{Z},$$

# Definition of Jacobi Forms: Support

- Jacobi forms are tagged with additional adjectives to reflect the support $\operatorname{supp}(\phi) = \{(n, r) \in \mathbb{Q}^2 : c(n, r; \phi) \neq 0\}$ of the Fourier expansion

$$\phi(\tau, z) = \sum_{n,r \in \mathbb{Q}} c(n, r; \phi) q^n \zeta^r, \qquad q = e(\tau), \zeta = e(z).$$

- $\phi \in J_{k,m}^{\mathrm{cusp}}$: automorphic and $c(n, r; \phi) \neq 0 \implies 4mn - r^2 > 0$
- $\phi \in J_{k,m}$: automorphic and $c(n, r; \phi) \neq 0 \implies 4mn - r^2 \geq 0$
- $\phi \in J_{k,m}^{\mathrm{weak}}$: automorphic and $c(n, r; \phi) \neq 0 \implies n \geq 0$
- $\phi \in J_{k,m}^{\mathrm{wh}}$: automorphic and $c(n, r; \phi) \neq 0 \implies n \gg -\infty$ ("wh" stands for *weakly holomorphic*)

The Jacobi $V(\ell)$ are images of the elliptic $T(\ell)$.

Elliptic Hecke Algebra $\longrightarrow$ Jacobi Hecke Algebra

$$\sum \mathrm{SL}_2(\mathbb{Z}) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \sum P_{2,1}(\mathbb{Z}) \begin{pmatrix} a & 0 & b & 0 \\ 0 & ad-bc & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\sum_{\substack{ad=\ell \\ b \bmod d}} \mathrm{SL}_2(\mathbb{Z}) \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \sum_{\substack{ad=\ell \\ b \bmod d}} P_{2,1}(\mathbb{Z}) \begin{pmatrix} a & 0 & b & 0 \\ 0 & \ell & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T(\ell) \mapsto V(\ell)$$

# Fourier-Jacobi expansion of the Gritsenko lift

Any Jacobi cusp form can be the leading Fourier-Jacobi coefficient of a paramodular form.

**Theorem (Gritsenko)**

*For* $\phi \in J_{k,m}^{\mathrm{cusp}}$ *the series* $\mathrm{Grit}(\phi)$ *converges and defines a map*

$$\mathrm{Grit} : J_{k,m}^{\mathrm{cusp}} \to S_k\left(K(m)\right)^{\epsilon}, \quad \epsilon = (-1)^k.$$

$$\mathrm{Grit}(\phi)\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) = \sum_{\ell \in \mathbb{N}} (\phi | V_\ell)(\tau, z) e(\ell m \omega).$$

$$c(n, r; \phi | V_\ell) = \sum_{\substack{a \in \mathbb{N}: \\ a | \gcd(n, r, \ell)}} a^{k-1} c\left(\frac{n\ell}{a^2}, \frac{r}{a}; \phi\right)$$

# Borchherds Product Summary

**Theorem (Borcherds, Gritsenko, Nikulin)**

*Given* $\psi \in J^{\mathrm{wh}}_{0,N}(\mathbb{Z})$*, a weakly holomorphic weight zero, index* $N$ *Jacobi form with integral coefficients*

$$\psi(\tau, z) = \sum_{n,r\in\mathbb{Z}:\, n\geq -N_o} c\,(n, r)\, q^n \zeta^r$$

*there is a weight* $k' \in \mathbb{Z}$*, a character* $\chi$*, and a* meromorphic *paramodular form* $\mathrm{Borch}(\psi) \in M^{\mathrm{mero}}_{k'}(K(N))(\chi)$

$$\mathrm{Borch}(\psi)(Z) = q^A \zeta^B \xi^C \prod_{n,m,r\in\mathbb{Z}} \left(1 - q^n \zeta^r \xi^{Nm}\right)^{c(nm,r)}$$

*converging in a nghd of infinity and defined by analytic continuation.*

# Theta Blocks: a great way to make Jacobi forms

due to Gritsenko, Skoruppa, and Zagier

- Dedekind Eta function $\eta \in J^{\mathrm{cusp}}_{1/2,0}(\epsilon)$

$$\eta(\tau) = q^{1/24} \prod_{n \in \mathbb{N}} (1 - q^n)$$

- Odd Jacobi Theta function $\vartheta \in J^{\mathrm{cusp}}_{1/2,1/2}(\epsilon^3 v_H)$

$$\vartheta(\tau, z) = q^{1/8} \left(\zeta^{1/2} - \zeta^{-1/2}\right) \prod_{n \in \mathbb{N}} (1 - q^n)(1 - q^n\zeta)(1 - q^n\zeta^{-1})$$

- $\mathrm{TB}_k[d_1, d_2, \ldots, d_\ell](\tau, z) = \eta(\tau)^{2k-\ell} \prod_{j=1}^{\ell} \vartheta(\tau, d_j z) \in J^{\mathrm{mero}}_{k,m}(\epsilon^{2k+2\ell})$
where $2m = d_1^2 + d_2^2 + \cdots + d_\ell^2$ and $d_i \in \mathbb{N}$.

# Part Three

Constructions of paramodular forms

# Integral Closure (PY 2009)

Pick your two favorite Gritsenko lifts $G_1, G_2 \in S_k(K(N))$ and define a map

$$M : S_k(K(N)) \to \{(H_1, H_2) \in S_{2k}(K(N)) \times S_{2k}(K(N)) : H_1 G_2 = H_2 G_1\}$$
$$f \mapsto (G_1 f, G_2 f)$$

If $\dim\{(H_1, H_2) \in S_{2k}(K(N)) \times S_{2k}(K(N)) : H_1 G_2 = H_2 G_1\} \leq \dim S_k(K(N))$ then various meromorphic $\frac{H_1}{G_1} = \frac{H_2}{G_2}$ are proven holomorphic.

- Useful when it is easier to span spaces of higher weight. (Hecke spreading on products of Gritsenko lifts of theta blocks.)

# Construction $N = 61$

PY 2009

## Theorem

*We have* $\dim S_3(K(61)) = 7$ *and* $\dim J_{3,61}^{\text{cusp}} = 6$. *There is a nonlift Hecke eigenform* $f \in S_3(K(61))^-$ *which has integral Fourier coefficients of content one. Such an* $f$ *is defined by*

$$f = -9G_1 - 2G_2 + 22G_3 + 9G_4 - 10G_5 + 19G_6 - 43\frac{G_1 G_6}{G_2}.$$

*This* $f$ *is congruent to a Gritsenko lift from* $\mathrm{Grit}\left(J_{3,61}^{\text{cusp}}(\mathbb{Z})\right)$ *modulo* 43 *and this is the only such congruence. Each* $G_i = \mathrm{Grit}(\mathrm{TB}_3(\mathbf{d}_i))$ *is the Gritsenko lift of a theta block given by* $\mathbf{d}_1 = [2, 2, 2, 3, 3, 3, 3, 5, 7]$, $\mathbf{d}_2 = [2, 2, 2, 2, 3, 4, 4, 4, 7]$, $\mathbf{d}_3 = [2, 2, 2, 2, 3, 3, 4, 6, 6]$, $\mathbf{d}_4 = [1, 2, 3, 3, 3, 3, 4, 4, 7]$, $\mathbf{d}_5 = [1, 2, 3, 3, 3, 3, 3, 6, 6]$, $\mathbf{d}_6 = [1, 2, 2, 2, 4, 4, 4, 5, 6]$.

# Euler factors computed by PY 2009.

- $f \in S_3\left(K(61)\right)^{-}(\mathbb{Z})$

$$\begin{aligned} Q_2(f,t) =& 1 + 7t + 24t^2 + 56t^3 + 64t^4 \\ Q_3(f,t) =& 1 + 3t + 3t^2 + 81t^3 + 729t^4 \\ Q_5(f,t) =& 1 - 3t + 85t^2 - 375t^3 + 15625t^4 \end{aligned}$$

# Borchers products

- Each $G_i$ that occurs in the expression for $f$

$$f = -9G_1 - 2G_2 + 22G_3 + 9G_4 - 10G_5 + 19G_6 - 43\frac{G_1 G_6}{G_2}.$$

  is actually also a Borcherds product!

- This follows from:
  Gritsenko, Poor, and Yuen: *Borcherds products everywhere,* Journal of Number Theory 148 (2015).

# Construction $N = 73$

PY 2009

## Theorem

*We have* $\dim S_3\left(K(73)\right) = 9$ *and* $\dim J_{3,73}^{\text{cusp}} = 8$. *There is a nonlift Hecke eigenform* $f \in S_3\left(K(73)\right)^-$ *which has integral Fourier coefficients of content one. Such an* $f$ *is defined by*

$$f = 9G_1 + 19G_2 + 2G_3 - 13G_4 + 34G_5 - 15G_6 - 12G_7 - 10G_8 - 39\frac{G_2 G_6}{G_4}.$$

*This* $f$ *is congruent to a Gritsenko lift from* $\mathrm{Grit}\left(J_{3,73}^{\text{cusp}}(\mathbb{Z})\right)$ *modulo* 3 *and* 13, *and these are the only such congruences. Each* $G_i = \mathrm{Grit}\left(\mathrm{TB}_3(\mathbf{d}_i)\right)$ *is the Gritsenko lift of theta blocks of the form* 9 *thetas over three etas.*

# Euler factors computed by PY 2009.

- $f \in S_3\left(K(73)\right)^-(\mathbb{Z})$

$$\begin{aligned}
Q_2(f,t) =& 1 + 6t + 22t^2 + 48t^3 + 64t^4 \\
Q_3(f,t) =& 1 + 2t + 3t^2 + 54t^3 + 729t^4 \\
Q_5(f,t) =& 1 + 130t^2 + 15625t^4
\end{aligned}$$

# Construction $N = 79$

PY 2009

## Theorem

*We have* $\dim S_3\left(K(79)\right) = 8$ *and* $\dim J_{3,73}^{\mathrm{cusp}} = 7$. *There is a nonlift Hecke eigenform* $f \in S_3\left(K(79)\right)^-$ *which has integral Fourier coefficients of content one. Such an* $f$ *is defined by*

$$
\begin{aligned}
f &= 26G_1 - 38G_2 + 19G_3 + 3G_4 - 17G_5 + 27G_6 - 68G_7 \\
&- 32\frac{G_1^2 - G_2^2 - G_1G_3 + 2G_2G_3 - G_3^2 - G_1G_5 + G_2G_6}{G_4} \\
&- 32\frac{-G_3G_6 - 2G_1G_7 - 2G_2G_7 + 3G_3G_7 + G_5G_7 + G_6G_7}{G_4}.
\end{aligned}
$$

*This* $f$ *is congruent to a Gritsenko lift from* $\mathrm{Grit}\left(J_{3,79}^{\mathrm{cusp}}(\mathbb{Z})\right)$ *modulo* $2^5$, *and any other such congruence is a reduction of this one. Each* $G_i$ *is the Gritsenko lift of theta blocks of the form* 9 *thetas over three etas.*

# Euler factors computed by PY 2009.

- $f \in S_3\left(K(79)\right)^-(\mathbb{Z})$

$$
\begin{aligned}
Q_2(f,t) =& 1+5t+14t^2+40t^3+64t^4 \\
Q_3(f,t) =& 1+5t+42t^2+135t^3+729t^4 \\
Q_5(f,t) =& 1-3t+80t^2-375t^3+15625t^4
\end{aligned}
$$

# Euler factors via Fourier coefficients.

- $f \in S_3\left(K(61)\right)^-(\mathbb{Z})$

$$
\begin{aligned}
Q_2(f,t) =& 1 + 7t + 24t^2 + 56t^3 + 64t^4 \\
Q_3(f,t) =& 1 + 3t + 3t^2 + 81t^3 + 729t^4 \\
Q_5(f,t) =& 1 - 3t + 85t^2 - 375t^3 + 15625t^4
\end{aligned}
$$

In 2009, these Euler factors were computed by computing Fourier coefficients. The Fourier coefficients were obtained by field operations on three variable Fourier series.

# Euler factors via specialization to $q$-series.

The modularity proofs in *On the paramodularity of typical abelian surfaces* required many more Euler factors.

Instead of computing Hecke eigenvalues from Fourier coefficients, we computed eigenvalues from specializations of Siegel modular forms to the $q$-series of elliptic modular forms.

Let $s$ be an integral positive definite 2-by-2 matrix.

$$\begin{aligned} \phi_s \colon \mathcal{H}_1 &\to \mathcal{H}_2 \qquad \text{(Eichler's trick)} \\ \tau &\mapsto s\tau. \end{aligned}$$

$$\phi_s^* : M_k(K(N)) \to M_{2k}(\Gamma_0(\det(s)N))$$

# Euler factors via specialization to $q$-series.

### Lemma

*Let $R \subseteq \mathbb{C}$ be a subring. Let $s = \begin{pmatrix} a & b \\ b & c/N \end{pmatrix} \in \mathrm{Mat}_2^{\mathrm{sym}}(\mathbb{Q})_{>0}$ with $a, b, c \in \mathbb{Z}$. Then the pullback under $\phi_s$ defines a ring homomorphism*

$$\phi_s^* : M(K(N), R) \to M(\Gamma_0(\det(s)N), R)$$

*from the graded ring of Siegel paramodular forms of level $N$ with coefficients in $R$ to the graded ring of elliptic modular forms of level $\det(s)N$ with coefficients in $R$. The map $\phi_s^*$ multiplies weights by 2 and maps cusp forms to cusp forms.*

# Euler factors via specialization to $q$-series.

The eigenvalue $a_p(f)$ for $T_p$ can be computed by performing field operations on $q$-series via

- $f \mid T_p = \sum_j f \mid M_j = a_p(f) f$
- $\phi_s^*(f \mid T_p) = \sum_j \phi_s^*(f \mid M_j) = a_p(f) \phi_s^*(f)$
- $a_p(f) = \frac{1}{\phi_s^*(f)} \sum_j \phi_s^*(f \mid M_j) \in \mathbb{Z}[1/p, \zeta_p][[q^{1/p}]]$
- In this way, we avoid computing Fourier coefficients and need only work with expansions in one variable. This technique allows other technical speed-ups as well.

# Eigenvalues via specialization to $q$-series: PSY 2019

- $f \in S_3\left(K(61)\right)^-(\mathbb{Z})$ is a rational function of Gritsenko lifts.

$$Q_2(f,t) = 1 + 7t + 24t^2 + 56t^3 + 64t^4$$
$$Q_3(f,t) = 1 + 3t + 3t^2 + 81t^3 + 729t^4$$
$$Q_5(f,t) = 1 - 3t + 85t^2 - 375t^3 + 15625t^4$$
$$\lambda_2 = -7, \lambda_3 = -3, \lambda_5 = 3, \lambda_7 = -9, \lambda_{11} = -4, \lambda_{13} = -3, \ldots$$

- We look forward to computing many more Euler factors for $f \in S_3\left(K(61)\right)^-(\mathbb{Z})$ !

*Thank you!*