Paramodular forms as orthogonal modular forms

A computational approach

Gonzalo Tornaría

Universidad de la República
Uruguay

Mainz, day 61

January 28, 2019

The main goal of this talk is to describe a method to compute eigenvalues of paramodular forms of weight 3.

In fact I will explain how to use neighbouring lattice methods to construct spaces of algebraic modular forms for orthogonal groups together with their Hecke operators.

For a presentation of orthogonal modular forms oriented towards computation see the Ph.D. thesis of Jeffery Hein (student of John Voight).

For the particular case of $\mathrm{SO}(5)$, see the Ph.D. thesis of Watson Ladd (student of Kenneth Ribet)

- For $\mathrm{SO}(3)$ orthogonal modular forms lift to classical modular forms of weight ≥ 2

  This is well known, by lifting to quaternionic modular forms (i.e. Brandt matrices) and using the Eichler correspondence (aka Jacquet-Langlands). This involves the fact that $\mathrm{Spin}(3)$ is a group of quaternions and an inner form of $\mathrm{SL}(2)$.

- For $\mathrm{SO}(5)$ orthogonal modular forms should lift to paramodular forms of weight ≥ 3

  This is conjectural: since $\mathrm{Spin}(5)$ is an inner form of $\mathrm{Sp}(2)$ it is fair to expect that quinary orthogonal modular forms correspond to Siegel modular forms of degree 2.

  In this way, I learned from Hein and Voight, one can easily construct eigenvalues of paramodular forms of weight 3

The objects of interest

- On the motivic side:
  Four-dimensional Galois representations of weight 3 of Calabi-Yau type (Hodge numbers 1,1,1,1)
- On the automorphic side:
  Automorphic forms for $\mathrm{GSp}(2)$, e.g.
  - Siegel modular forms of degree 2 and weight 3
  - Paramodular forms of weight 3
  - Algebraic modular forms for $\mathrm{SO}(5)$

Some computational problems:

1. Find such objects
2. Compute their L-functions
3. Do an exhaustive enumeration

Related theoretical problems

1. Understand the correspondence between the motivic and automorphic sides

For instance: given a paramodular form (of general type) there is a corresponding ℓ-adic Galois representation (due to Taylor, Laumon, Weissauer, Schmidt, Mok, ...).

Thus, if one were to find a matching CY 3-fold it would in principle be possible to prove its modularity by proving the two Galois representations are isomorphic using Faltings-Serre.

See BPPTVY where we make this idea feasible for $\mathrm{GSp}(2)$ and use it to prove the modularity of some abelian surfaces.

Related theoretical problems

2. Understand the relation between paramodular forms of weight 3 and algebraic modular forms for $\mathrm{SO}(5)$

See Rainer comments in the project pre-proposal. From a computational point of view on the automorphic side this is quite important since it seems much easier to compute algebraic modular forms for $\mathrm{SO}(5)$ than paramodular forms. (cf. Ibukiyama's Conjecture, see Ibukiyama-Kitayama)

However, I don't know how to recover the Fourier coefficients of the paramodular form from the orthogonal modular form.

It would be most interesting to find an explicit lifting, since the Fourier coefficients of paramodular forms are related to the central values of their twisted L-functions (similar to Waldspurger's formula, see my work with Ryan)

Let R be an order in a definite quaternion algebra over $\mathbb{Q}$.

A *quaternionic modular form* of level R is a function on the (finite) set of left R-ideal classes:

$$\mathcal{M}(R) = \{\varphi : \mathrm{Cl}(R) \to \mathbb{C}\}$$

There is an action of Hecke operators on this space given by

$$t_m\, \varphi([I]) = \sum_{\substack{J \subset I \\ N(J) = m\, N(I)}} \varphi([J])$$

This action is classically given by *Brandt matrices*, which are easy to compute, and can be expressed in terms of representation numbers of quaternary quadratic forms.

Eichler correspondence

Suppose $\varphi \in \mathcal{M}(R)$ is a quaternionic eigenform, then

$$\sum_{m \geq 0} \langle t_m \varphi, \varphi \rangle \, q^m$$

is a classical eigenform of weight 2 with the same eigenvalues.

In fact $(\varphi, \psi) \mapsto \sum_{m \geq 0} \langle t_m \varphi, \psi \rangle \, q^m$ defines a Hecke-bilinear pairing (*Eichler commuting relations*). In the standard basis $([I], [J])$ maps to the theta series of the quaternary lattice IJ^{-1}. Hence the expression is a linear combination of theta series.

Eichler correspondence

This provides a lifting from quaternionic modular forms to classical modular forms which was used by Eichler for his work on the basis problem.

- This is a (provable) way to construct classical eigenforms.
- A more difficult problem is to know which classical eigenforms will arise in this way.
  For this Eichler developed and used trace formulas.
- This is a precursor to Jacquet-Langlands correspondence.

Let L be a positive definite lattice of dimension n.

An *orthogonal modular form* of level L is a function on the (finite) set of equivalence classes of lattices in the genus of L:

$$\mathcal{M}(L) = \{\varphi : \mathrm{gen}(L) \to \mathbb{C}\}$$

There is an action of Hecke operators on this space given by *neighbouring operators*, for instance:

$$t_{p,1} \, \varphi([\Lambda]) = \sum_{[\Lambda:\Lambda\cap\Pi]=[\Pi:\Lambda\cap\Pi]=p} \varphi([\Pi])$$

A key step in the computation of the neighbouring operators is testing for isometry of lattices. In some cases a reduction theory is available (e.g. Eisenstein reduction for $n = 3$).

$\mathrm{SO}(3)$

In the case of $n = 3$ orthogonal modular forms lift to classical modular forms of weight 2, and this is well-studied.

In fact given a quadratic space V of dimension 3 one can construct a quaternion algebra D (the *even Clifford algebra*) such that $\mathrm{SO}(V) \simeq D^\times/\mathbb{Q}^\times$. From this is not difficult to relate algebraic modular forms for $\mathrm{SO}(V)$ with algebraic modular forms for $D^\times$ (Ponomarev, Schulze-Pillot, Hein).

Birch used this in 1988 for computations. A limitation of this is that, in principle, it will compute only modular forms with sign $+1$ in the functional equation. To avoid this issue one needs to use characters of the spinor norm as I showed in my thesis (see also recent work with Hein and Voight, in progress).

SO(3): half-integral weight modular forms

Using theta series one can define a Hecke-linear map

$$\theta : \mathcal{M}(L) \to M_{3/2}(N_L)$$

This (almost) gives an explicit lifting. Moreover the Fourier coefficients of the weight 3/2 modular forms are related to the central values of twisted L-functions by Waldspurger.

However, theta series have linear relations which correspond to vanishing of the central value of L-functions, which means the lifting to weight 2 does *not* in general factor through this map.

One can avoid this issue by using generalized theta series (see my recent work with Sirolli)

Ibukiyama conjecture

In the case of $n = 5$ we have $\mathrm{SO}(5) \simeq \mathrm{USp}(2)/\{\pm 1\}$, where $\mathrm{USp}(2)$ is a compact twist of $\mathrm{Sp}(2)$.

A precise analogue for USp(2) of Eichler-Jacquet-Langlands correspondence was conjectured by Ibukiyama (for prime levels, and later by Ibukiyama-Kitayama for squarefree levels).

Thus it is expected that quinary orthogonal modular forms lift to paramodular forms of weight 3, but unlike the case of $n = 3$ no explicit lifting is known so far.

Half-integral weight modular forms

As before one can define a Hecke-linear map

$$\theta : \mathcal{M}(L) \to M_{5/2}(N_L)$$

to modular forms of weight 5/2. Those map to modular forms of weight 4 by Shimura correspondence, and then Gritsenko lifting provides a map to paramodular forms of weight 3.

This is *NOT* the lifting we want!

Indeed, by definition, this yields only Gritsenko lifts. However, the interesting part of the space of paramodular forms are the *non-lifts* (i.e. eigenforms outside the space of Gritsenko lifts).

Actually: because L is positive definite, θ yields forms with sign -1 in the functional equation, which correspond to Jacobi forms, which lift by Gritsenko to paramodular forms with sign $+1$ in the functional equation.

The subspace of non-lifts

Assuming Ibukiyama Conjecture we let $\mathcal{M}^0(L)$ be the subspace of $\mathcal{M}(L)$ spanned by eigenforms which correspond to non-lifts.

With Hein and Ladd we noted the following

Proposition

We have $\mathcal{M}^0(L) \subseteq \ker\theta$.

We expect equality to hold in general, and we do not know any example where it fails, but we cannot rule out the possibility of some linear relation between theta series for a different reason.

I am inclined to think if the latter happens it might correspond to classical modular forms of weight 4, with sign -1 in the functional equation, for which the central derivative vanishes.

Is there any such form? I could not find one.

I will discuss the paramodular non-lift of weight 3 and level 61.

This is the same as example 3.5.5 in Hein's thesis, although the order of the quadratic forms is different. We start with

$$Q_1 = x^2 + y^2 + z^2 + 8w^2 + t^2 + xy + xz + yw$$

a quinary quadratic form of discriminant 61.

We compute its 2-neighbours, there are $15 = 2^3 + 2^2 + 2 + 1$, of which 7 are equivalent to Q_1, 4 to Q_2 and 4 to Q_3. Here

$$Q_2 = x^2 + y^2 + z^2 + 3w^2 + 3t^2 + xy + xw + yw + zw - xt - yt - 3wt$$
$$Q_3 = x^2 + y^2 + z^2 + 2w^2 + 3t^2 + xy + xt - wt$$

We keep computing 2-neighbours of Q_2, Q_3, etc. until we "close the graph". Note that this will span the whole genus of Q_1 because there is only one spinor genus in this case.

This procedure gives explicit representatives for the 8 classes in the genus of Q_1 and the matrix of the Hecke operator $t_{2,1}$.

By taking theta series we obtain 8 classical modular forms of weight $5/2$ and level $4 \cdot 61$. We verify $\ker\theta$ has dimension 1, with eigenvalue -7 for $t_{2,1}$.

The characteristic polynomial of $t_{2,1}$ factors as

$$(x-15) \cdot (x+7) \cdot (\text{degree 6 irreducible})$$

The first factor corresponds to the Eisenstein eigenform, the second factor corresponds to $\ker\theta$, and the third factor corresponds to a classical modular form of weight 4.

Since there are no modular forms of weight 4, level 61, with rational eigenvalues, we confirm that $\ker\theta$ is a non-lift.

Our second problem is computing the L-function.
Suppose we want the Dirichlet coefficients up to a certain bound X. For this we need to compute the eigenvalues of $t_{p,1}$ for p up to X and those of $t_{p,2}$ for p up to $\sqrt{X}$.

Since the number of p-neighbours needed for $t_{p,1}$ is $O(p^3)$, while the number of p-neighbours needed for $t_{p,2}$ is $O(p^4)$, the former will dominate the computation.

For our form, the eigenvalues of $t_{p,2}$ for $p < 100$ are listed in Hein's thesis, enough for Dirichlet coefficients up to 10000.

We want the eigenvalues of $t_{p,1}$ for all p up to some X.

Two tricks

Computing $t_{p,1}$ requires iterating over p-neighbours, of which there are $O(p^3)$. For each one the expensive step is to determine in which class it is. A priori this requires testing for isometry with (on average) half the classes in the genus.

For our case we can check in advance that the 8 classes in the genus of Q_1 can be distinguished by the first 3 coefficients of their theta series. Hence determining the class of a neighbour is a simple matter of computing 3 coefficients of its theta series and a table lookup.

Second: a linear algebra trick means that we don't need to compute the p-neighbours for every class. Since we already know we are dealing with a (multiplicity one) eigenform, it is enough to compute the p-neighbours for *one* class (properly chosen), then project onto the eigenform.

Computation

Using those two tricks I wrote a PARI/GP program which computes the p-neighbours for Q_3, and then it recovers from this the eigenvalue of $t_{p,1}$ for our eigenform.

It took $\sim$ 1100 cpu-hours to compute for all p up to 1000, which took about a week using a workstation.

it took $\sim$ 500 cpu-days to compute for all p in (1000,1700), which took a few days using a small cluster.

For example: the eigenvalue for $t_{1699,1}$ is 8495 ($\sim$ 10 cpu-days).

Some estimates:

- For all p in (1700,2000) $\sim$ 500 cpu-days.
- For all p in (2000,4000) $\sim$ 50 cpu-years.

Open problems

- Construct forms with sign -1 in the functional equation

  As explained here, the method seems to construct only forms with sign $+1$ in the functional equation.

- Exhaustive enumeration

  Requires theory (e.g. proving Ibukiyama's Conjecture).

- Explicit lifting

  Can we obtain the Fourier coefficients of the paramodular form given a corresponding orthogonal modular form?

Epilogue: congruences

The characteristic polynomial of $t_{2,1}$ factors as

$$(x-15)\cdot(x+7)\cdot f(x)$$

Let v_0 be a (primitive) eigenvector with eigenvalue -7, which corresponds to our non-lift eigenform and it spans $\ker\theta$.

Now let $K=\mathbb{Q}(a)$ where a is a root of f. Then we can choose, by suitable normalization, an eigenvector v_a with eigenvalue a such that $v_0 \equiv v_a \pmod{\mathfrak{p}_{43}}$, where $\mathfrak{p}_{43}$ is a prime in K of norm 43, namely the one generated by 43 and $a+7$.

Note that $a \equiv -7 \pmod{\mathfrak{p}_{43}}$. In fact it follows from $v_0 \equiv v_a \pmod{\mathfrak{p}_{43}}$ that *all* the eigenvalues of v_0 and v_a are congruent modulo $\mathfrak{p}_{43}$. In this way we can find and prove congruences of the first type mentioned by Neil in the project pre-proposal.

Proving congruences of the second type seems more difficult.

Paramodular forms as orthogonal modular forms

A computational approach

Gonzalo Tornaría

Universidad de la República
Uruguay

Mainz, day 61
January 28, 2019