Local structure of hyperkähler singularities in O'Grady's examples

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To Victor Ginzburg, on the occasion of his 50-th birthday

Contents

1 Introduction

An *irreducible hyperkähler* manifold is by definition a compact Riemannian manifold of dimension $4n$ with holonomy group $Sp(n)$. Manifolds of this type are known to possess many additional structures. In particular, an irreducible hyperkähler X is automatically complex and Kähler. Moreover, the space $H^{2,0}(X)$ of global holomorphic 2-forms is 1-dimensional and spanned by a single non-degenerate symplectic form Ω. Complex manifolds with this property are called *irreducible holomorphically symplectic*.

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Perhaps not unexpectedly for special holonomy manifolds, compact irreducible hyperkähler manifolds are very hard to construct – so much so that a large part of the structure theory of hyperkähler manifolds was developed by F. Bogomolov in the course of proving that none exist in complex dimensions $2n > 2$.

Fortunately for those who prefer positive statements, the last part of the paper [Bo] contained a mistake. This was noticed several years later by Bogomolov himself and by A. Beauville. In the meanwhile, the celebrated S.-T. Yau's proof of the Calabi conjecture has arrived, and special holonomy metrics became much more accessible. In particular, by a well-known argument Yau's Theorem implied that a compact Kähler holomorphically symplectic manifold admits a hyperkähler metric. This immediately gives two examples in complex dimension 2: a torus T and a surface S of type $K3$. However, in contrast to the related theory of Calabi-Yau manifolds, higher-dimensional examples are still not easy to come by – the purely algebro-geometric problem of finding compact holomorphically symplectic manifolds is itself very hard.

A. Beauville [B] has overcome the difficulties and constructed two series of examples of irreducible hyperkähler manifolds, each containing one manifold in each dimension $2n$, $n \geq 2$. To understand these examples, recall that for any complex surface S, the Hilbert scheme $S^{[n]}$ of 0-dimensional subschemes of length n is a smooth complex variety of complex dimension $2n$. Beauville's examples are:

(a) The Hilbert scheme $S^{[n]}$ of a $K3$-surface S, and

(b) The so-called *generalized Kummer variety* $K^{[n]}$ – namely, the preimage of $0 \in T$ under the natural summation map $\Sigma : T^{[n+1]} \to T$ from the Hilbert scheme $T^{[n]}$ of a 2-dimensional complex torus T to T itself.

Beauville has proved that for every $n \geq 2$, both these manifolds are Kähler and irreducible holomorphically symplectic, hence irreducible hyperkähler.

For a long time, the Beauville examples and their deformations were the only known examples of irreducible hyperkähler manifolds. The situation was rather frustrating and also a bit strange, because at least the example (a) admits an obvious generalization. Instead of the Hilbert scheme $S^{[n]}$ for a $K3$-surface S, one can consider the moduli space $\mathcal{M}_S(r, c_1, c_2)$ of torsion-free sheaves on S of rank r, with Chern classes c_1 and c_2. These spaces are naturally symplectic outside of their singular loci. Thus to produce a hyperkähler manifold, all one has to do is to choose the topological type (r, c_1, c_2) in such a way that the singular locus is empty, and apply Yau's Theorem. Unfortunately, all the hyperkähler manifolds obtained by this procedure are not new – it turns out that they are deformationally equivalent to an appropriately chosen Hilbert scheme.

The great breakthrough was achived by K. O'Grady about five years ago. His new approach consisted of taking a *singular* moduli space of sheaves on a $K3$ surface and resolving the singularities in an appropriate way. The moduli space he considered was $\mathcal{M}_S(2,0,4)$, the space of torsion-free sheaves $\mathcal{E}$ of rank 2 on a $K3$-surface S, with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = 4$. This is a singular complex variety of dimension 10, and in fact, this is the simplest topological type for which the moduli space is singular. In two papers [O1, part I, II] (see also the published version [O3]) O'Grady was able to construct a smooth symplectic desingularization of the space $\mathcal{M}_S(2,0,4)$ (part I) and to prove that the resulting hyperkähler manifold is irreducible and not deformationally equivalent to a manifold of Beauville's (part II). Later on, he also produced a generalization of the Kummer variety construction, thus obtaining a new irreducible hyperkähler manifold of complex dimension 6 ([O2]).

These examples are rather unique. In both cases, the singularity one has to resolve is the same. O'Grady also considered moduli spaces $\mathcal{M}(2,0,2k)$ of sheaves with $c_2 = 2k$ for $k > 2$; however, he was not able to resolve their singularities in a symplectic fashion, and conjectured that such a resolution does not exist.

The present paper arouse out of our attempts to understand the O'Grady construction and to study the spaces $\mathcal{M}_S(2,0,2k)$ for higher values of k. We focus on the local structure of the singularities. We give a relatively explicit description of the singularities in terms of the nilpotent coadjoint orbits of the group $Sp(k)$. Then we prove – using an idea of O'Grady himself – that he was right: the moduli space $\mathcal{M}_S(2,0,2k)$ does *not* admit a smooth projective symplectic resolution for $k \geq 3$. We also give some speculation as to what might be still possible to do at least in the case $k = 3$.

Our results are not really satisfactory because they are negative: we would much prefer to construct a symplectic resolution. Nevertheless, we believe that the present paper will be useful – firstly, because it settles definitely the case of moduli of rank-2 bundles on a $K3$-surface, for better or for worse, and secondly, because it might serve as an introduction to the rather technical papers [O1], [O3]. We hope that this will stimulate further research in the area. To us, it looks promising and important enough to merit wider attention.

Note added in proof. For technical reasons, this paper has spent a rather long time in the form of an electronic preprint. After it had been first posted to the web, we have learned that a completely different and very successful approach to the existence of resolutions had been developped by Jaeyoo Choy and Young-Hoon Kiem ([Ki], [CK]). Their method is based on motivic intergration, and allows to give a different proof of the non-existence of resolutions. Also, a general non-existence results for the singularities of the moduli spaces of sheaves on a K3 surface has been proved later in [KLS]. However, some of our results, such as the formality statement for the Hilbert scheme and an explicit description of the O'Grady singularity are still unavailable in the literature.

2 Main results

We will now state our results. Let S be a projective algebraic surface over $\mathbb{C}$ of type $K3$. Fix a polarization H on S. For every positive integer $k \geq 2$, let $\mathcal{M}_S(k) = \mathcal{M}_S(2, 0, 2k)$ be the Gieseker compactification of the moduli space of vector bundles $\mathcal{E}$ on S with $\operatorname{rk} \mathcal{E} = 2$, $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = 2k$. This is a projective algebraic variety of dimension $\dim \mathcal{M}_S(k) = 6k - 2$. In [O1], O'Grady has proved the following.

Lemma 2.1 ([O1, Lemma]). *Singular points $m \in \mathcal{M}_S(k)$ are in one-to one correspondence with isomorphism classes of sheaves of the form*

$$\mathcal{E}_m \cong \mathcal{I}_{\mathbf{z}} \oplus \mathcal{I}_{\mathbf{w}}, \tag{2.1}$$

where $\mathcal{I}_{\mathbf{z}}$ and $\mathcal{I}_{\mathbf{w}}$ are ideal sheaves of 0-dimensional subschemes $\mathbf{z}, \mathbf{w} \subset S$ of length k. □

Interchanging $\mathbf{z}$ and $\mathbf{w}$ in (2.1) of course gives an isomorphic sheaf. There are no other isomorphisms. Thus the singular locus $\mathcal{M}_S^{sing}(k)$ is naturally identified with the symmetric square $S^2(S^{[k]})$ of the Hilbert scheme of k points on S. This symmetric square is itself naturally stratified: we have the image of the diagonal in $S^{[k]} \times S^{[k]}$ and the open complement to this image. In all, the variety $\mathcal{M}_S(k)$ has a stratification with three strata,

$$\mathcal{M}_S^{bad}(k) \subset \mathcal{M}_S^{sing}(k) \subset \mathcal{M}_S(k), \tag{2.2}$$

with $\mathcal{M}_S^{bad}(k) \cong S^{[k]}$ and $\mathcal{M}_S^{sing}(k) \cong S^2(S^{[k]})$. The complements $\mathcal{M}_S^{sing}(k) \setminus \mathcal{M}_S^{bad}(k)$ and $\mathcal{M}_S(k) \setminus \mathcal{M}_S^{sing}(k)$ are non-singular.

One can characterize the stratification (2.2) by considering automorphisms of the corresponding sheaves. If a point $m \in \mathcal{M}_S(k)$ lies outside of the singular locus $\mathcal{M}_S^{sing}(k)$, the automorphism group $\operatorname{Aut}\mathcal{E}_m$ of the corresponding sheaf $\mathcal{E}_m$ consists only of scalars, $\operatorname{Aut}\mathcal{E}_m \cong \mathbb{C}^*$. For a point $m \in \mathcal{M}_S^{sing}(k) \setminus \mathcal{M}_S^{bad}(k)$, the group $\operatorname{Aut}\mathcal{E}_m$ of the sheaf $\mathcal{E}_m = \mathcal{I}_{\mathbf{z}} \oplus \mathcal{I}_w$ is a torus $\mathbb{C}^* \times \mathbb{C}^*$. Finally, for a point $m \in \mathcal{M}_S^{bad}(k)$ in the most singular stratum, we have $\operatorname{Aut}\mathcal{E}_m \cong GL(2,\mathbb{C})$.

Take such a point $m \in \mathcal{M}_S^{bad}(k) \subset \mathcal{M}_S(k)$. Our main result describes the structure of the variety $\mathcal{M}_S(k)$ in a small neighborhood of the point m. To state it, consider a symplectic vector space $V = \mathbb{C}^{2k}$ of dimension $2k$, and let $Sp(V) = Sp(k)$ be the corresponding symplectic group, with its Lie algebra $\mathfrak{sp}(V)$. Identify $\mathfrak{sp}(V) \cong \mathfrak{sp}(V)^*$ by means of the Killing form. Denote by $\mathcal{N}_k \subset \mathfrak{sp}(V)$ the set of all symplectic endomorphisms $B : V \to V$ such that

$$B^2 = 0. \tag{2.3}$$

The set $\mathcal{N}_k$ is a closed $Sp(V)$-invariant subvariety in $\mathfrak{sp}(V)$. The only invariant of a symplectic endomorphism B satisfying (2.3) is its rank. Thus we have a decomposition

$$\mathcal{N}_k = \coprod \mathcal{N}_k^p$$

into coadjoint orbits

$$\mathcal{N}_k^p = \left\{B \in \mathfrak{sp}(V) \mid B^2 = 0 \text{ and } \operatorname{rk} B = p\right\} \subset \mathfrak{sp}(V) \cong \mathfrak{sp}(V)^*$$

of the group $Sp(V)$. The unions

$$\mathcal{N}_k^{\leq p} = \bigcup_{q \leq p} \mathcal{N}_k^q \subset \mathcal{N}_k$$

form a stratification of the variety $\mathcal{N}_k$ by closed $Sp(V)$-invariant strata. Since $B^2 = 0$, the image $B(V) \subset V$ of an endomorphism $B \in \mathcal{N}_k^p$ must be an isotropic subspace in V. Therefore $\mathcal{N}_k^p$ is empty unless $k \geq p$. When $p = 0$, the orbit $\mathcal{N}_k^0$ consists of a single point $0 \in \mathfrak{sp}(V)^*$.

Proposition 2.2. *Let $U_m \subset \mathcal{M}_S(k)$ be a small analytic neighborhood of $m \in \mathcal{M}_S^{bad}(k) \subset \mathcal{M}_S(k)$.*

Then there exists an open neighborhood $U \subset \mathcal{N}_k^{\leq 3} \times V$ of 0 in $\mathcal{N}_k^{\leq 3} \times V$, the product of the stratum $\mathcal{N}_k^{\leq 3} \subset \mathfrak{sp}(V)^*$ with the symplectic vector space V, and a map $\mu : U_m \to U$ such that

(i) *The map* $\mu : U_m \to U$ *is a two-to-one étale cover over* $U \cap \left(\mathcal{N}_k^3 \times V\right)$.

(ii) *Over* $U \cap \left(\mathcal{N}_k^{\leq 2} \times V\right)$, *the map* $\mu : U_m \to U$ *is one-to-one*

(iii) *The preimage* $\mu^{-1}\left(\mathcal{N}_k^1 \times V\right) \subset U_m$ *is the intersection* $U_m \cap \mathcal{M}_S^{sing}(k)$, *and the preimage* $\mu^{-1}(0 \times V) \subset U_m$ *is the intersection* $U_m \cap \mathcal{M}_S^{bad}(k)$.

Moreover, the map μ *sends the natural symplectic form on the smooth part on* U_m *to the natural symplectic from on* $\mathcal{N}_k \times V$ *obtained from the given form on* V *and the Kostant-Kirillov symplectic form on the coadjoint orbit.*

The statement of the Proposition hides an essential difference between the cases $k = 2$ and $k \geq 3$. Namely, as noted above, in the case $k = 2$ the orbit $\mathcal{N}_k^3 \subset \mathfrak{sp}(V)^*$ is empty. Therefore $\mathcal{N}_k^{\leq 2} \times V = \mathcal{N}_k^{\leq 3} \times V$, and we have a map $\mu : U_m \to \mathcal{N}_k^{\leq 2} \times V$ which is generically one-to-one. The variety $\mathcal{N}_k^{\leq 2}$ has a natural smooth symplectic resolution – namely, the total space $T^*\mathsf{G}(V)$ of the Grassmanian $\mathsf{G}(V)$ of Lagrangian subspaces $L \subset V$ in the symplectic vector space $V = \mathbb{C}^4$. This is essentially the resolution constructed by O'Grady, although he did not use the nilpotent orbit interpretation.

Recently this resolution also appeared in [F] as a part of a beautiful general theorem: roughly speaking, for any simple algebraic group G, *all* projective symplectic resolutions of nilpotent coadjoint G-orbits are of the form $T^*(G/P)$ for some parabolic subgroup $P \subset G$.

In the case $k \geq 3$, Proposition 2.2 only gives a map $\mu : U_m \to \mathcal{N}^{\leq 3} \times V$ which is generically two-to-one. In this case we have the following.

Theorem 2.3. *For* $k \geq 3$, *the two-fold cover* U_m *of the open neighborhood* $U \subset \mathcal{N}_k^{\leq 3} \times V$ *considered in Proposition* 2.2 *does not admit a smooth projective symplectic resolution compatible with the given symplectic form. Consequently, for* $k \geq 3$ *the moduli space* $\mathcal{M}_S(k)$ *does not admit a smooth symplectic desingularization* $\widetilde{\mathcal{M}_S(k)}$.

We note that in the case $k = 3$, the variety $\mathcal{N}_k^{\leq 3}$ itself does admit a smooth symplectic desingularization – again, it is the cotangent space $T^*\mathsf{G}(V)$ of the Lagrangian Grassmannian $\mathsf{G}(V)$. However, passing to a 2-fold cover introduces ramification, and the symplectic form is no longer non-degenerate. As we will see in the proof of Theorem 2.3, this cannot be cured.

We will prove Propostiton 2.2 in Section 4, after analyzing the deformation theory of singular sheaves on S in Section 3. In the course of the analysis, we need a technical result on formality (Propositon 3.1) whose proof we postpone so as not to interrupt the expostion. The proof of Theorem 2.3 is contained in the Section 5. The last Section 6 is taken up with the proof of Proposition 3.1.

3 Local deformations of the most degenerate sheaf

Fix a point $m \in \mathcal{M}_S^{bad}(k)$ in the most singular stratum in the moduli space $\mathcal{M}_S(k)$, and let $\mathcal{E}_m$ be the corresponding sheaf. Our first task is to analyze the deformation theory of the sheaf $\mathcal{E}_m$.

By Lemma 2.1, we have $\mathcal{E}_m \cong \mathcal{I}_{\mathbf{z}} \oplus \mathcal{I}_{\mathbf{z}} \cong \mathcal{I}_{\mathbf{z}} \otimes \mathbb{C}^2$, where $\mathcal{I}_{\mathbf{z}}$ is the ideal sheaf of some 0-dimensional subscheme $\mathbf{z} \subset S$ of length k. Throughout this section, we will work in wider generality and assume that $\mathcal{E}_m = \mathcal{I}_{\mathbf{z}} \otimes \mathbb{C}^r$ for some fixed integer $r \geq 2$. It will affect neither the results nor the proofs.

3.1 Generalities on deformations. We recall relevant general results from the deformation theory of sheaves. Every sheaf $\mathcal{E}$ on a smooth projective complex manifold admits a local deformation space $U_{\mathcal{E}}$ with a fixed origin point $o \in U_{\mathcal{E}}$; the pair $\langle U_{\mathcal{E}}, o\rangle$ parametrizes small deformations of the sheaf $\mathcal{E}$. To construct space $U_{\mathcal{E}}$, one starts with the complex vector space $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E})$. By a standard algebraic procedure, the DG Lie algebra structure on the complex $\mathrm{RHom}^\bullet(\mathcal{E}, \mathcal{E})$ defines a certain closed affine subvariety $\widetilde{U}_{\mathcal{E}} \subset U_0$ in a small neighborhood $U_0 \subset \mathrm{Ext}^1(\mathcal{E}, \mathcal{E})$ of $0 \in \mathrm{Ext}^1(\mathcal{E}, \mathcal{E})$. The variety $\widetilde{U}_{\mathcal{E}}$ is called *a versal deformation space* of the sheaf $\mathcal{E}$. The automorphisms group $G = \mathrm{Aut}(\mathcal{E})$ of the sheaf $\mathcal{E}$ acts on the Lie algebra $\mathrm{Ext}^\bullet(\mathcal{E}, \mathcal{E})$ preserving the versal deformation space $\widetilde{U}_{\mathcal{E}} \subset \mathrm{Ext}^1(\mathcal{E}, \mathcal{E})$. The local deformation space $U_{\mathcal{E}}$ is the affine algebraic variety obtained as the quotient $\widetilde{U}_{\mathcal{E}}/G$. The origin point $o \in U_{\mathcal{E}}$ is the image of $0 \in \mathrm{Ext}^1(\mathcal{E}, \mathcal{E})$.

Given a point $u \in U_{\mathcal{E}}$ in the local deformation space, one can also recover the automorphism group $\mathrm{Aut}\,\mathcal{E}_u$ of the corresponding deformed sheaf $\mathcal{E}_u$ from the general formalism. To do this, one notes that the $\mathrm{Aut}(E)$-action on the variety $\widetilde{U}_{\mathcal{E}}$ is not free. Choose a lifting $\widetilde{u} \in \widetilde{U}_{\mathcal{E}}$ of the point $u \in U_{\mathcal{E}}$. The stabilizer group $\mathrm{Stab}(\widetilde{u}) \subset \mathrm{Aut}(\mathcal{E})$ does not depend on the chosen lifting $\widetilde{u}$ and coincides with $\mathrm{Aut}(\mathcal{E}_u)$ (while the embedding $\mathrm{Aut}(\mathcal{E}_u) \cong \mathrm{Stab}(\widetilde{u}) \subset \mathrm{Aut}(\mathcal{E})$ does depend on the choice of $\widetilde{u}$ and is determined by u only up to a conjugation).

The algebraic procedure which gives a versal deformation space $\widetilde{U}_{\mathcal{E}}$ is particularly simple in the case when the DG Lie algebra $\mathrm{RHom}^\bullet(\mathcal{E}, \mathcal{E})$ is *formal* – in other words, quasiisomorphic to its homology Lie algebra $\mathrm{Ext}^\bullet(\mathcal{E}, \mathcal{E})$. In this case, the Yoneda commutator bracket in $\mathrm{Ext}^\bullet(\mathcal{E}, \mathcal{E})$ gives a quadratic map $Q : \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \to \mathrm{Ext}^2(\mathcal{E},\mathcal{E})$. The zero locus $Q^{-1}(0)$ of the map Q is a versal deformation space $\widetilde{U}_{\mathcal{E}}$ for the sheaf $\mathcal{E}$. In particular, $\widetilde{U}_{\mathcal{E}} \subset \mathrm{Ext}^1(\mathcal{E},\mathcal{E})$ is a conic subvariety in the whole space $\mathrm{Ext}^1(\mathcal{E},\mathcal{E})$, not only in a small neighborhood of 0. (However, the quotient $U_{\mathcal{E}} = \widetilde{U}_{\mathcal{E}}/G$ parametrizes deformations only in a small neighorhood of the origin point.)

3.2 The Ext**-algebra of the most degenerate sheaf.** We apply the general theory to a sheaf $\mathcal{E} = \mathcal{E}_m$ of the form $\mathcal{E}_m \cong \mathcal{I}_{\mathbf{z}} \otimes \mathbb{C}^r$. Our first objective is to describe the DG Lie algebra $\mathrm{RHom}^{\bullet}(\mathcal{E}_m,\mathcal{E}_m)$. First, consider $\mathrm{RHom}^{\bullet}(\mathcal{E}_m,\mathcal{E}_m)$ as an associative DG algebra. We have

$$\mathrm{RHom}^{\bullet}(\mathcal{E}_m,\mathcal{E}_m) \cong \mathrm{End}(\mathbb{C}^r) \otimes \mathrm{RHom}^{\bullet}(\mathcal{I}_{\mathbf{z}},\mathcal{I}_{\mathbf{z}}).$$

It turns out that in order to study the deformations of $\mathcal{E}_m$, all we need to know is the Ext-algebras $\mathrm{Ext}^{\bullet}(\mathcal{E}_m,\mathcal{E}_m)$ and $\mathrm{Ext}^{\bullet}(\mathcal{I}_{\mathbf{z}},\mathcal{I}_{\mathbf{z}})$. This follows from the following general fact.

Proposition 3.1. *Let $\mathcal{I}_{\mathbf{z}}$ be the ideal sheaf of a subscheme $\mathbf{z} \subset S$ of length 0 in a $K3$-surface S. Then the associative DG algebra $\mathrm{RHom}^{\bullet}(\mathcal{I}_{\mathbf{z}},\mathcal{I}_{\mathbf{z}})$ is formal.*

The proof of this Proposition is slightly more technical than one would like. We will postpone it till the last Section, so as not to interrupt the exposition.

Granted Proposition 3.1, we wee that the DG algebra $\mathrm{RHom}^{\bullet}(\mathcal{E}_m\mathcal{E}_m)$ is also formal and quasiisomorphic to the product

$$\mathrm{Ext}^{\bullet}(\mathcal{E}_m,\mathcal{E}_m) \cong \mathrm{End}(\mathbb{C}^r) \otimes \mathrm{Ext}^{\bullet}(\mathcal{I}_{\mathbf{z}},\mathcal{I}_{\mathbf{z}}). \tag{3.1}$$

The Ext-algebra $\mathrm{Ext}^{\bullet}(\mathcal{I}_{\mathbf{z}},\mathcal{I}_{\mathbf{z}})$ is well-known. We have $\mathrm{Ext}^p(\mathcal{I}_{\mathbf{z}},\mathcal{I}_{\mathbf{z}}) = 0$ unless $p = 0,1,2$. For $p = 0$ and $p = 2$ we have natural isomorphisms

$$\mathrm{Ext}^0(\mathcal{I}_{\mathbf{z}},\mathcal{I}_{\mathbf{z}}) \cong \mathrm{Ext}^2(\mathcal{I}_{\mathbf{z}},\mathcal{I}_{\mathbf{z}}) \cong \mathbb{C}.$$

The space $\mathrm{Ext}^1(\mathcal{I}_{\mathbf{z}},\mathcal{I}_{\mathbf{z}})$ is a complex vector space of dimension $2k$. The Yoneda multiplication

$$\mathrm{Ext}^1(\mathcal{I}_{\mathbf{z}},\mathcal{I}_{\mathbf{z}}) \otimes \mathrm{Ext}^1(\mathcal{I}_{\mathbf{z}},\mathcal{I}_{\mathbf{z}}) \to \mathrm{Ext}^2(\mathcal{I}_{\mathbf{z}},\mathcal{I}_{\mathbf{z}}) \cong \mathbb{C}$$

is given by a non-degenerate symplectic form on the vector space V.

It will be convenient to introduce special notation for algebras of this type. Let V be a symplectic vector space. Define a supercommutative algebra $\mathcal{A}^\bullet(V)$ by setting

$$\mathcal{A}^p(V) \cong \begin{cases} \mathbb{C}, & p = 0, \\ V, & p = 1, \\ \mathbb{C}, & p = 2, \end{cases}$$

with multiplication $\mathcal{A}^1(V) \otimes \mathcal{A}^1(V) = V \otimes V \to \mathcal{A}^2(V) = \mathbb{C}$ obtained from the symplectic form on V.

With this notation, we have $\operatorname{Ext}^\bullet(\mathcal{I}_{\mathbf{z}}, \mathcal{I}_{\mathbf{z}}) \cong \mathcal{A}^\bullet(\operatorname{Ext}^1(\mathcal{I}_{\mathbf{z}}, \mathcal{I}_{\mathbf{z}}))$. Since this algebra is supercommutative, the supercommutator bracket in the algebra $\operatorname{Ext}^\bullet(\mathcal{E}_m, \mathcal{E}_m)$ is the product of the Lie bracket on the Lie algebra $\operatorname{End}(\mathbb{C}^r) \cong \mathfrak{gl}_r(\mathbb{C})$ and the Yoneda multiplication in $\operatorname{Ext}^\bullet(\mathcal{I}_{\mathbf{z}}, \mathcal{I}_{\mathbf{z}})$.

It is convenient and customary to use the trace map $\mathfrak{gl}_r(\mathbb{C}) \to \mathbb{C}$ to decompose the Lie algebra $\operatorname{Ext}^\bullet(\mathcal{E}_m, \mathcal{E}_m)$ into the sum of the abelian Lie algebra $\operatorname{Ext}^\bullet(\mathcal{I}_{\mathbf{z}}, \mathcal{I}_{\mathbf{z}})$ and the traceless part $\operatorname{Ext}^\bullet_0(\mathcal{E}_m, \mathcal{E}_m)$. The summand $\operatorname{Ext}^\bullet(\mathcal{I}_{\mathbf{z}}, \mathcal{I}_{\mathbf{z}})$ controls deformations $\widetilde{\mathcal{I}_{\mathbf{z}}}$ of the sheaf $\mathcal{I}_{\mathbf{z}}$ and deformations of the sheaf $\mathcal{E}_m \cong \mathcal{I}_{\mathbf{z}} \otimes \mathbb{C}^r$ which are of the form $\widetilde{\mathcal{E}_m} \cong \widetilde{\mathcal{I}_{\mathbf{z}}} \otimes \mathbb{C}^r$. The trace decomposition is compatible with the Lie bracket and induces a decompositon $U_m \cong U_0 \times U_m^0$, where U_0 is the local deformation space of the sheaf $\mathcal{I}_{\mathbf{z}}$, and U_m^0 is the deformation "in transversal directions" to $U_0 \subset U_m$. Since the Hilbert scheme $S^{[k]}$ is smooth, the space U_0 is simply a neighborhood of 0 in the vector space $\operatorname{Ext}^1(\mathcal{I}_{\mathbf{z}}, \mathcal{I}_{\mathbf{z}})$. All the information about singularities is contained in the transversal deformation space U_m^0, obtained from the Lie algebra $\operatorname{Ext}^\bullet_0(\mathcal{E}_m, \mathcal{E}_m)$.

3.3 Hamiltonian interpretation. By virtue of Proposition 3.1, in order to study the deformation theory of the sheaf $\mathcal{E}_m$, it suffices to describe the Yoneda square map in the traceless Lie algebra $\operatorname{Ext}^\bullet_0(\mathcal{E}_m, \mathcal{E}_m)$. To do this, it is convenient to use the language of Hamiltonian group actions.

Let G be a reductive Lie group with Lie algebra $\mathfrak{g}$, and let V be a symplectic vector space. Identify $\mathfrak{g} \cong \mathfrak{g}^*$ by means of the Killing form. Let G act on the vector space $\mathfrak{g} \otimes V$ by the product of the adjoint representation and the trivial one. Consider the symplectic form on $\mathfrak{g} \otimes V$ obtained as the product of the Killing form on $\mathfrak{g}$ and the given symplectic form on V. Then the action of G on $\mathfrak{g} \otimes V$ is Hamiltonian, and we have a quadratic *moment map* $Q : \mathfrak{g} \otimes V \to \mathfrak{g} \cong \mathfrak{g}^*$.

On the other hand, define a graded Lie algebra $\mathcal{L}^\bullet(\mathfrak{g}, V)$ by setting

$$\mathcal{L}^\bullet(\mathfrak{g}, V) = \mathfrak{g} \otimes \mathcal{A}^\bullet(V). \tag{3.2}$$

Lemma 3.2. *The moment map $Q : \mathfrak{g} \otimes V \to \mathfrak{g}$ for the G-action on $\mathfrak{g} \otimes V$ coincides with the Yoneda square map in the Lie algebra $\mathcal{L}^\bullet(\mathfrak{g}, V)$.*

Proof. Clear. □

Recall that whenever one has a Hamiltonian action of a reductive Lie group G on an affine symplectic manifold X, one defines the *Hamiltonian reduction* $X/\!\!/G$ as the quotient of the preimage $Q^{-1}(0) \subset X$ of 0 under the moment map $Q : X \to \mathfrak{g}^*$ by the action of G. Denote by $P(G, V) = \mathfrak{g} \otimes V /\!\!/ G$ the Hamiltonian reduction of the space $\mathfrak{g} \otimes V$ by the G-action.

By (3.1), the traceless part $\mathrm{Ext}_0^\bullet(\mathcal{E}_m, \mathcal{E}_m)$ of the Lie algebra $\mathrm{Ext}^\bullet(\mathcal{E}_m, \mathcal{E}_m)$ is isomorphic to $\mathcal{L}^\bullet(\mathfrak{sl}_r, V)$, $V = \mathrm{Ext}^1(\mathcal{I}_{\mathbf{z}}, \mathcal{I}_{\mathbf{z}})$. The automorphism group $\mathrm{Aut}(\mathcal{E}_m) \cong GL(n)$ acts on $\mathcal{L}^\bullet(\mathfrak{sl}_r, V)$ through its quotient $G = PGL(n)$. Thus we are in the situation described by Lemma 3.2. We conclude that the Yoneda bracket map Q for the Lie algebra $\mathcal{L}^\bullet(\mathfrak{sl}_r, V)$ coincides with the moment map for the G-action on $\mathfrak{sl}_r \otimes V$. Since the DG Lie algebra $\mathrm{RHom}^\bullet(\mathcal{E}_m, \mathcal{E}_m)$ is formal, this implies that the transversal local deformation space U_m^0 of the sheaf $\mathcal{E}_m$ is canonically an open neighborhood $U_n^0 \subset P(G, V)$ of 0 in the Hamiltonian reduction $P(G, V)$. To sum up, we have proved the following.

Proposition 3.3. *The local deformation space U_m of the sheaf $\mathcal{E}_m \cong \mathcal{I}_{\mathbf{z}} \otimes \mathbb{C}^r$ is isomorphic to an open neighborhood of* 0 *is the product $P(PGL(r), V) \times V$ of the Hamiltonian reduction $P(PGL(r), V)$ with the symplectic vector space V.* □

We note that the product $P(PGL(r), V) \times V$ is itself a Hamiltonian reduction: we have $P(PGL(r), V) \times V \cong P(GL(r), V)$. It is also possible to prove that the embedding $U_m \subset P(PGL(r), V) \times V \cong P(GL(r), V)$ is symplectic outside of singularities. To do this, one has to unwind the definition of the symplectic form on the deformation space U_m and see that it is the same form as the one obtained from the reduction. We leave the details to the reader.

Remark 3.4. When $\dim V \geq 4$, one can also find the dimension of the space $P(PGL(r), V)$ – hence also of the deformation space U_m^0. It is equal to

$$\dim U_m^0 = \dim P(PGL(r), V) = \dim \mathfrak{sl}_r \cdot (\dim V - 2).$$

Indeed, decompose symplectically $V = \mathbb{C}^2 \oplus V'$. Since the Lie algebra $\mathfrak{sl}_r$ is semisimple, we have $[\mathfrak{sl}_r, \mathfrak{sl}_r] = \mathfrak{sl}_r$. Thus for a generic vector $v' \in$

$V' \otimes \mathfrak{sl}_r$, one can find $a, b \in \mathfrak{sl}_r$ with $[a, b] = Q(v') \in \mathfrak{sl}_r$. Then $\langle a, b\rangle \oplus v' \in V \otimes \mathfrak{sl}_r$ satisfies $Q(v) = 0$. On the other hand, since v' is generic, the stabilizer $\mathrm{Stab}(v) \subset \mathrm{Stab}(v') \subset PGL(r)$ of the vector v is the trivial subgroup. Therefore the differential of the moment map $Q : \mathfrak{sl}_r \otimes V \to \mathfrak{sl}_r$ is surjective in the generic point of the zero locus $Q^{-1}(0)$, the $PGL(r)$-action is generically free, and one has

$$\begin{aligned} \dim P(PGL(r), V) &= \dim \mathfrak{sl}_r \otimes V - \dim PGL(r) - \dim \mathfrak{sl}_r \\ &= \dim \mathfrak{sl}_r \otimes V - 2 \dim \mathfrak{sl}_r. \end{aligned}$$

This coincides with the expected dimension of the moduli space computed by Riemann-Roch. It is interesting to note that when $\dim V = 2$, the dimension of the space $P(PGL(r), V)$ is actually strictly bigger than the right-hand side of this equation (which in this case is equal to 0). Geometrically, this means that the sheaf $\mathcal{E}_m$ for $k = 1$ cannot be deformed to a non-singular vector bundle.

4 Hamiltonian action of the symplectic group

By virtue of Proposition 3.3, in order to study the local deformation space of a sheaf $\mathcal{E}_m \cong \mathcal{I}_{\mathbf{z}} \otimes \mathbb{C}^r$ and to prove Proposition 2.2, it is sufficient to study the Hamiltonian reduction space $P(G, V)$ with $V = \mathrm{Ext}^1(\mathcal{I}_{\mathbf{z}}, \mathcal{I}_{\mathbf{z}})$ and $G = PGL(r)$. This is the goal of this section.

The symplectic group $Sp(V)$ acts naturally on the vector space V preserving the symplectic form Ω. This action is Hamiltonian: the moment map $\mu_0 : V \to \mathfrak{sp}(V)^*$ is quadratic and comes from the standard identification $S^2V \cong \mathfrak{sp}(V) \cong \mathfrak{sp}(V)^*$. Tensoring with the Lie algebra $\mathfrak{sl}_r$ gives a Hamiltonian action of the group $Sp(V)$ on the symplectic vector space $\mathfrak{sl}_r \otimes V$. The corresponding moment map $\mu : \mathfrak{sl}_r \otimes V \to \mathfrak{sp}(V)^*$ is also quadratic and coincides with the product of the moment map $\mu_0 : V \to \mathfrak{sp}(V)^*$ with the Killing form on $\mathfrak{sl}_r$. Moreover, the $Sp(V)$-action on $\mathfrak{sl}_r \otimes V$ commutes with the $PGL(r)$-action – $Sp(V)$ acts on the second factor, while $PGL(r)$ acts on the first one.

Thus we have a symplectic vector space $\mathfrak{sl}_r \otimes V$ with two commuting Hamiltonian group actions – firstly, a $PGL(r)$-action with the moment map $Q : \mathfrak{sl}_r \otimes V \to \mathfrak{sl}_r$, secondly, a $Sp(V)$-action with the moment map $\mu : \mathfrak{sl}_r \otimes V \to \mathfrak{sp}(V)^*$. Our strategy is to study the reduction $P(G, V) = \mathfrak{sl}_r \otimes V /\!\!/ PGL(r)$ by means of the second moment map $\mu : \mathfrak{sl}_r \otimes V \to \mathfrak{sp}(V)$.

From now on, we restrict our attention to the case of rank $r = 2$. Denote $G = PGL(2)$, $\mathfrak{g} = \mathfrak{sl}_2$. We will use the identification $PGL(2) \cong SO(3)$

and think of G as the group of automorphisms of a 3-dimensional complex vector space W which preserve a non-degenerate symmetric form $h \in S^2W^*$ on W and an orientation – or, equivalently, an isomorphism $\chi : \Lambda^2 W \cong W$. One can think of W as the Lie algebra $\mathfrak{g}$, with its Killing form and the commutator map $[-,-] : \Lambda^2\mathfrak{g} \cong \mathfrak{g}$.

Identify the symplectic vector space $\mathfrak{g} \otimes V \cong W \otimes V$ with the space

$$\mathrm{Hom}(W^*, V)$$

of linear maps from the dual space W^* to V. In this interpretation, it is very easy to describe the moment maps Q and μ.

Lemma 4.1. *Let $a \in \mathrm{Hom}(W^*, V) \cong \mathfrak{g} \otimes V$ be an arbitrary map. Then*

$$Q(a) = \chi^*(a^*\Omega) \in W \cong \mathfrak{g},$$

while

$$\mu(a) = a(h) \in S^2V \cong \mathfrak{sp}(V). \tag{4.1}$$

Proof. Clear. □

We see that a map $a \in \mathrm{Hom}(W^*, V)$ lies in the zero set of the moment map Q if and only if the image $a(W^*) \subset V$ is an isotropic subspace. By definition, the reduction space $P(G, V) = \mathfrak{sl}_r \otimes V /\!\!/ PGL(r)$ is the geometric invariant theory quotient of the zero set $Q^{-1}(0)$ by the group G. In particular, this means that points in $P(G, V)$ are in one-to-one correspondence with the closed G-orbits in $Q^{-1}(0)$. In order to analyze these orbits, we need a preliminary linear algebra result.

Lemma 4.2. *Let U, U' be two finite-dimensional complex vector spaces that are equipped with symmetric pairings. Assume that $\mathsf{dim}\, U \leq \mathsf{dim}\, U'$ and that the pairing on U' is non-degenerate. Denote by $U_0 \subset U$ the radical of the pairing on U. Consider the space $O(U, U')$ of maps $f : U \to U'$ compatible with the pairings, and let the orthogonal group $O(U')$ act on $O(U, U')$ by multiplication on the right.*

Then $O(U, U')$ contains exactly one closed $O(U')$-orbit, and it coincides with the subset $O(U/U_0, U') \subset O(U, U')$ of maps $f \in O(U, U')$ which vanish on $U_0 \subset U$.

Proof. This is a reasonably standard fact about parabolic subgroups in the orthogonal group; however, we include a direct linear-algebraic proof for the convenience of the reader. First of all, it is obvious that $O(U)$ acts transitively on the set $O(U/U_0, U')$, so that it indeed forms a single $O(U)$-orbit. A map $f \in O(U, U')$ lies in $O(U/U_0, U')$ if and only if $\operatorname{dim}\operatorname{Im} f = \operatorname{dim}(U/U_0)$. Since for any $f \in O(U, U')$ we must have $\operatorname{dim}\operatorname{Im} f \geq \operatorname{dim}(U/U_0)$, the subset $O(U/U_0, U') \subset O(U, U')$ is indeed closed.

It remains to show that all the other $O(U)$-orbits in $O(U, U')$ are *not* closed. Fix an element $f \in O(U, U')$ which does not lie i $O(U/U_0, U') \subset O(U, U')$, and choose a complement $U_1 \subset U$ to the subspace $U_0 \subset U$. Then the pairing is non-degenerate on U_1, so that the map $f : U_1 \to U'$ is injective. Let $U_0' \subset U'$ be the orthogonal complement to $f(U_1) \subset U'$. Then f induces a non-trivial map $f_0 : U_0 \to U_0'$, we have a natural group embedding $O(U_0') \subset O(U')$, and it suffices to check that $O(U_0') \cdot f_0$ is not closed in $O(U_0, U_0')$. In other words, we can assume that the pairing on U is trivial (and $f \neq 0$). Then the claim becomes obvious: indeed, for any $f \in O(U, U')$ the group $O(U)$ contains a one-parameter subgroup $e : \mathbb{C}^* \to O(U)$ such that $e(\lambda)f = \lambda f$ for any $\lambda \in \mathbb{C}^*$. □

Lemma 4.3. *Let $a \in \operatorname{Hom}(W^*, V)$ lie in the zero level of the moment map Q, and assume that the orbit $O(W) \cdot a$ is closed in $Q^{-1}(0)$. Then*

$$\mu(a) \in \mathcal{N}_k^p \subset \mathfrak{sp}(V),$$

where $p = \operatorname{rk} W'$ is the dimension of the image $W' = \operatorname{Im} a \subset V$. Moreover, $\operatorname{Im} \mu(a) = W'$ and the associated symmetric form $h(\mu(a)) \in S^2 W'$ coincides with $a(h)$.

Proof. For every $a \in \operatorname{Hom}(W^*, V)$ with image $W' = \operatorname{Im} a \subset V$, (4.1) shows that the moment value $\mu(a) \in \mathfrak{sp}(V)$ considered as a symplectic automorphism $\mu(a) : V \to V$ satisfies $\operatorname{Im} \mu(a) \subset W'$. Equip W' with the symmetric form $h(\mu(a)) \in S^2 W'$. By Lemma 4.2 (or rather, by the dual statement), the orbit $O(W) \cdot a$ is closed in $O(W, W')$ if and only if we have an equality, $\operatorname{Im} \mu(a) \subset W'$. This implies $\operatorname{Ker} \mu(a) = W'^{\perp}$, the symplectic orthogonal to the subspace $W' \subset V$. If we have $Q(a) = 0$, then W' is isotropic; therefore $W' \subset W'^{\perp}$ and $\mu(a)^2 = 0$, so that $\mu(a) \in \mathcal{N}_k$. Conversely, every symplectic endomorphism $B \in \mathcal{N}_k$ with $W' = \operatorname{Im} B \subset V$ corresponds to a non-degenerate symmetric $h(B) \in S^2 W' \subset S^2 V$ on the dual space W'^* under the identification $S^2 V \cong \mathfrak{sp}(V)$. □

Thus the moment map μ restricts to a map $\mu : Q^{-1}(0) \to \mathcal{N}_k \subset \mathfrak{sp}(V)$. Since it is G-invariant, it descends to the reduction $P(G, V)$.

Lemma 4.4. *Let $B \in \mathcal{N}_k \subset \mathfrak{sp}(V)$ be a symplectic endomorphims $B : V \to V$ satisfying $B^2 = 0$, let $W' = \mathsf{Im}\, B \subset V$ be its image, and let $h(B) \in S^2W'$ be the non-degenerate symmetric form on W'^* associated to B.*

Then the fiber $\mu^{-1}(B) \subset P(G, V)$ is isomorphic to the set $O(W', W)$ of embeddings $W'^* \hookrightarrow W$ with $a^*(h) = h(B)$, modulo the natural action of $G = SO(W)$.

Proof. By (4.1), every $a \in Q^{-1}(0) \subset \mathrm{Hom}(W^*, V)$ with $\mu(a) = B$ must send $h \in S^2W^*$ to $h(B)$ on $W' = \mathsf{Im}\, a = \mathsf{Im}\, B$. In other words, the transposed map $a^* : W'^* \hookrightarrow W$ satisfies $a^*(h) = h(B)$. Conversely, since $B^2 = 0$, W' is isotropic. Therefore by Lemma 4.1 every $a \in \mathrm{Hom}(W^*, V)$ with $\mathsf{Im}\, a = W'$ and $a^*(h) = h(B)$ satisfies $Q(a) = 0$ and $\mu(a) = B$. □

We can now prove Proposition 2.2

Proof of Proposition 2.2. Embed U_m into $P(G, V) \times V$ by Proposition 3.3. Take $B \in \mathcal{N}_k \subset \mathfrak{sp}(V)$ with $W' = \mathsf{Im}\, B$. By Lemma 4.3, the fiber of the map $\mu : P(G, V) \to \mathfrak{sp}(V)$ over the point B coincides with the set $O(W', W)$ of Euclidean embeddings $W'^* \hookrightarrow W$ modulo the natural $SO(W)$-action.

If $B \in \mathcal{N}_k^3$, so that $\mathsf{dim}\, W' = 3$, the quotient set $O(W', W)/SO(W)$ consists of two points (corresponding to the choice of the orientation induced on W'). This proves (i). If $B \in \mathcal{N}_k^{\leq 2}$, so that $\mathsf{dim}\, W' \leq 2$, the action of $SO(W)$ on $O(W', W)$ is transitive, and this proves (ii). However, when $B \in \mathcal{N}_k^{\leq 1}$, any corresponding point $a^* \in O(W', W)$ – hence also any point $a \in Q^{-1}(0) \subset \mathrm{Hom}(W^*, V)$ with $\mu(a) = B$ – acquires a non-trivial stabilizer in $G = PGL(2)$. If $\mathsf{rk}\, B = 1$, this stabilizer is $\mathbb{C}^* = \mathbb{C}^* \times \mathbb{C}^*/\mathbb{C}^* \subset PGL(2) = GL(2)/\mathbb{C}^*$, while for $B = 0$ the stabilizer is the whole $PGL(2)$. By Subsection 3.1, this stabilizer $\mathrm{Stab}(a)$ is the image in $PGL(2)$ of the automorphism group $\mathrm{Aut}(\mathcal{E}_{a,v}) \subset GL(2)$ of the sheaf $\mathcal{E}_{a,v}$ parametrized by the point $a \times v \in U_m \subset P(G, V) \times V$ (no matter what is the vector $v \in V$). Hence $\mu^{-1}(\mathcal{N}_k^1)$ parametrizes sheaves $\mathcal{E}$ with $\mathrm{Aut}\,\mathcal{E} = \mathbb{C}^* \times \mathbb{C}^*$, and $\mu^{-1}(0)$ parametrizes sheaves with $\mathrm{Aut}\,\mathcal{E} = GL(2)$. This proves (iii). □

As a corollary, we see that the $Sp(V)$-action on $P(G, V)$ is transitive. The two-to-one ramification in the moment map $\mu : P(G, V) \to \mathcal{N}_k^{\leq 3}$ comes from the action of the element $\iota = -\,\mathsf{id} : V \to V$, $\iota \in Sp(V)$. This element acts trivially on the orbit $\mathcal{N}_k^3$ and non-trivially on the variety $P(G, V)$.

Remark 4.5. As the reader can see, our proofs rely heavily on the identification $PGL(2) \cong SO(3)$ and the isomorphism $\Lambda^2\mathfrak{sl}_2 \cong \mathfrak{sl}_2$. Thus they do not generalize to the case of higher rank r. However, the main result,

– namely, the fact that the $Sp(V)$-action on $P(G,V)$ is transitive, and the moment map $\mu : P(G,V) \to \mathfrak{sp}(V)$ is generically finite onto its image, – may generalize to the higher rank case, at least when the dimension of the vector space V is comparable to the dimension of the Lie algebra $\mathfrak{sl}_r$. We conjecture that it is enough to require

$$\operatorname{dim} V \geq 2 \operatorname{dim} \mathfrak{sl}_r = 2(r^2 - 1).$$

This would give a cubic lower bound on the second Chern class $c_2 = \frac{1}{2} r \operatorname{dim} V$.

To be more precise, we note that the first part of our approach, namely, Lemma 4.1, holds in full generality modulo an appropriate change of notation. However, for $r \geq 3$ the commutator map $\chi : \Lambda^2 \mathfrak{sl}_r \to \mathfrak{sl}_r$ is no longer an isomorphism. Therefore the condition $Q(a) = 0$ no longer implies that $\operatorname{Im} a \subset V$ is an isotropic subspace, but only imposes some weaker condition on the subspace $\operatorname{Im} a \subset V$. Consequently, the orbits one encounters in the image of the moment map μ, while still nilpotent, do not contain isomorphisms with $B^2 = 0$ but something more complicated. Whether these orbits or their covers admit symplectic resolutions should be the topic of further research.

Remark 4.6. Proposition 3.3 is purely local. However, we note that the O'Grady singularity $\mathcal{N} = \mathcal{N}_2^{\leq 2} \subset \mathfrak{sp}(\mathbb{C}^4)$ is $\mathbb{Q}$-factorial. Using this fact, it is easy to check that the argument in [K1, Proposition 5.6] applies to $\mathcal{N}$ and proves that the resolution $T^*\mathsf{G}(\mathbb{C}^4)$ is isomorphic

$$T^*\mathsf{G}(\mathbb{C}^4) \cong \mathrm{Bl}(\mathcal{N}, \mathcal{E}_l)$$

to the blow-up of the variety $\mathcal{N}$ in a certain sheaf of ideals $\mathcal{E}_l \subset \mathcal{O}_{\mathcal{N}}$. The sheaf $\mathcal{E}_l$ is obtained by taking an appropriate positive integer l and extending, by push-forward, the l-th power of the ideal sheaf of the subvariety $\mathcal{N}_2^{\leq 1} \subset \mathcal{N}$ from $\mathcal{N} \setminus \{0\}$ to $\mathcal{N}$. This shows how to construct the global symplectic resolution of the moduli space $\mathcal{M}_S(4)$: it suffices to take the l-th power of the ideal sheaf of the subvariety

$$\left(\mathcal{M}_S^{sing}(2) \setminus \mathcal{M}_S^{bad}(2)\right) \subset \left(\mathcal{M}_S(2) \setminus \mathcal{M}_S^{bad}(2)\right),$$

extend it by push-forward from $\mathcal{M}_S(2) \setminus \mathcal{M}_S^{bad}(2)$ to the whole $\mathcal{M}_S(2)$, and take the blow-up of the resulting ideal sheaf. In this way one can avoid a reference to the Mori Cone Theorem in [O1].

5 Existence of resolutions

Consider the Hamiltonian reduction space

$$U = P(GL(2), V) = P(PGL(2), V) \times V$$

which, by Proposition 2.2, gives a local model for the singularity of the moduli space $\mathcal{M}(2, 0, k)$. We will now prove Theorem 2.3 claiming that for $k \geq 3$, the space U does not admit a smooth projective resolution compatible with the given symplectic form.

We will need the following general facts on the geometry of a projective symplectic resolution.

Lemma 5.1. *Let X be a smooth symplectic algebraic variety equipped with a projective birational map $X \to Y$ to an irreducible normal affine algebraic variety Y.*

(i) *The map $X \to Y$ is semismall – in other words, we have $\operatorname{dim} X \times_Y X = \operatorname{dim} X$.*

(ii) *Every vector field ξ on Y lifts to a (unique) vector field on X.*

Proof. (i) is [K2, Proposition 1.2], and (ii) is [GK, Lemma 5.3]. □

This immediately implies the following. Assume given an algebraic variety Y, and a non-degenerate symplectic form Ω on the smooth locus $Y^{sm} \subset Y$; then we will say that a smooth resolution $\pi : X \to Y$ is *compatible with the form* Ω if $\pi^*\Omega$ extends to a non-degenerate sympletic form on the whole smooth algebraic variety X.

Lemma 5.2. *Let $Y = Y' \times Z$ be the product of a smooth affine symplectic variety Z and an affine variety Y' equipped with a symplectic form on the smooth locus $Y'_{sm} \subset Y'$. Assume that Y admits a projective smooth resolution $f : X \to Y$ compatible with the product symplectic form on $Y'_{sm} \times Z$.*

Then for an arbitrary point $z \in Z$, the preimage

$$X' = f^{-1}(Y' \times \{z\}) \subset X$$

is a smooth projective resolution of the variety Y' compatible with the given symplectic form on $Y'_{sm} \subset Y'$.

Proof. The only thing to prove is the smoothness of X'. In other words, we have to prove that the composition $X \to Y \to Z$ of the map $f : X \to Y$ with the natural projection $Y \to Z$ is a smooth map over every point $z \in Z$. To prove it, choose a set of vector fields $\xi_1, \dots, \xi_l$ on Y such that the vectors $\xi_i(z)$ form a basis of the tangent space $T_z Z$. By Lemma 5.1, all the vector fields ξ_i lift to vector fields on X. Therefore the differential of the projection $X \to Z$ is surjective for every point $x \in X$ lying over $z \in Z$. □

Lemma 5.2 immediately implies that to prove that $U = P(PGL(2), V) \times V$ does not admit a symplectic resolution, it suffices to prove that the first factor $P(PGL(2), V)$ does not admit such a resolution.

However, it turns out that we can do more – namely, apply Lemma 5.2 to the space $P(PGL(2), V)$ itself. To do this, denote $P = P(PGL(2), V)$ and let $P_1 = \mu^{-1}(\mathcal{N}_k^1) \subset P$ be the singular stratum – that is, the preimage of the stratum $\mathcal{N}_k^1 \subset \mathcal{N}_k^{\leq 3}$ under the $Sp(V)$-moment map $\mu : P \to \mathcal{N}_k^{\leq 3} \subset \mathfrak{sp}(V)$. We want to show that locally near a point $p \in P_1$, the space $P = P(PGL(2), V)$ admits a product decomposition of the type required to apply Lemma 5.2.

Recall that in the language of Section 4, points in the space P correspond to maps $a : W^* \to V$ from the Euclidean 3-dimensional vector space W^* to the symplectic space V with isotropic image and Euclidean kernel, considered modulo the action of the group $SO(W) \cong PGL(2)$. Then a point $p \in P$ lies in the stratum P_1 if and only if the corresponding subspace $a(W^*) \subset V$ is of dimension exactly 1.

Fix once and for all a vector $w \in W^*$. We have the orthogonal decomposition $W^* = \mathbb{C} \cdot w \oplus W^\perp$. For every point $p \in P_1$, we can apply a suitable element $g \in SO(W)$ and arrange so that the kernel $\operatorname{Ker} a$ of the corresponding map $a : W^* \to V$ coincides with the 2-dimensional subspace $W^\perp \subset W^*$. Having done this, we are left with only one invariant of the point $p \in P_1$ –namely, the vector $v_p = a(w) \in V$. The stabilizer of the subspace $W^\perp$ in $SO(W)$ is the group $O(W^\perp) \subset SO(W)$. It acts on the line $\mathbb{C} \cdot w \subset W^*$ by multiplication by ± 1, so that the vector $v_p \subset V$ is well-defined up to a sign. Thus we have an unramified two-fold cover $\widetilde{P}_1 \to P_1$ and an isomorphism $\widetilde{P}_1 \cong V \setminus \{0\}$, $p \mapsto v_p$.

Let now V' be a symplectic vector space of dimension $\operatorname{dim} V - 2 = 2(k-1)$, and consider the vector space $\operatorname{Hom}(W^\perp, V')$. Since $W^\perp$ is Euclidean, $\operatorname{Hom}(W^\perp, V')$ is naturally a symplectic vector space equipped with a Hamiltonian action of the group $O(W^\perp)$. In particular, we have a Hamiltonian action of the subgroup $\mathbb{C}^* \cong SO(W^\perp) \subset O(W^\perp)$. Denote by $P' = \operatorname{Hom}(W^\perp, V') /\!\!/ \mathbb{C}^*$ the variety obtained by Hamiltonian reduction. As in Section 4, points $p' \in P'$ correspond to maps $a' : W^\perp \to V'$ with isotropic image.

Lemma 5.3. *There exists a dense open subset $U \subset P$, a dense open subset $U_1 \subset \widetilde{P}_1$ and an étale map*

$$\rho : U_1 \times P' \to U$$

compatible with the natural symplectic forms on the smooth parts of both sides.

Proof. Fix a subspace $V_1 \subset V$ of codimension 1. Its symplectic orthogonal $V_1^\perp \subset V$ is a line in V_1, and the quotient $V_1/V_1^\perp$ is a symplectic vector space of dimension $\operatorname{dim} V - 2$. Fix an identification $V_1/V_1^\perp \cong V'$. Let $U \subset P$ be the set of points $p \in P$ such that the corresponding map $a : W^* \to V$ is transversal to V_1 – in other words, $V = a(W^*) + V_1$. This is true generically, so that $U \subset P$ is a dense open subset. Set $U_1 = V \setminus V_1 \subset V \setminus \{0\} \cong \widetilde{P}_1$.

To define the map $\rho : U_1 \times P' \to U$, note that since for every point $v_p \times p' \in U_1 \times P'$ the intersection $V_1 \cap v_p^\perp$ is transversal, so the projection induces a canonical symplectic isomorphism $V_1 \cap v_p^\perp \cong V'$. In other words, we have a canonical symplectic embedding $\chi_p : V' \to v_p^\perp \subset V$ associated to the vector $v_p \in V \setminus V_1$. If $a' : W^\perp \to V'$ is the map associated to a point $p' \in P'$, then the composition $\chi_p \circ a' : W^\perp \to V_1$ is isotropic embedding into $v_p^\perp \subset V$. We define $\rho(v_p \times p')$ as the point $p \in P$ corresponding to the direct sum map $(\chi_p \circ a') \oplus v_p : W^\perp \oplus \mathbb{C} \cdot w \to V$; since $(\chi_p \circ a')(W^\perp)$ is isotropic and orthogonal to v_p, the point $p \in P$ is well-defined.

We claim that the map ρ is étale, moreover, it is an unramified two-fold cover. Indeed, take a point $p \in P$, and let $a : W^* \to V$ be the associated map. The point p lies in the image of the map ρ if and only if $a(W^\perp)$ lies in the subspace $V_1 \subset V$. But if $p \in U$, the preimage $a^{-1}(V_1) \subset W^*$ is by definition a 2-dimensional subspace. Therefore applying a suitable $g \in SO(W)$, we can arrange so that $a^{-1}(V_1) = W^\perp \subset W$, and modulo $SO(W^\perp)$, there are exactly two choices of such a g. $\square$

By Lemma 5.2, this Lemma implies that if the space $P = P(PGL(2), V)$ were to admit a symplectic resolution, then so would the space P'. Indeed, then the subset $U \subset P$ would also have a symplectic resolution X, and the étale cover $\widetilde{X} = X \times_U (U_1 \times P')$ would be a symplectic resolution for $U_1 \times P'$.

By itself, this is not enough to derive a contradiction and prove Theorem 2.3 – we need to throw in one more ingredient. Namely, consider the element $\iota \in Sp(V)$ given by $-\mathsf{id} : V \to V$. This element acts on P interchanging the leaves of the two-fold covering $\mu : P \to \mathcal{N}_k^{\leq 3}$. In particular, it acts trivially on the points lying over $\mathcal{N}_k^{\leq 2} \subset \mathcal{N}_k^{\leq 3}$, such as points $p \in P_1$ in the stratum $P_1 = \mu^{-1}(\mathcal{N}_k^1) \subset P$. Moreover, it is easy to see that ι preserves $U \subset P$ and is compatible with the two-fold étale cover $\rho : U_1 \times P \to U$ – namely, there exists an involution $\iota : P \ \to P'$ such that $\rho \circ (\mathsf{id} \times \iota) = \iota \circ \rho$. To prove this, notice that ι acts on the space

$$P = (W \otimes V) /\!\!/ PGL(2)$$

by the endomorphism $-\mathsf{id}$ on V or, which is equivalent, by the endomorphism $-\mathsf{id}$ on W. Moreover, since we take a quotient by the group $PGL(2) = SO(3)$, we can replace $-\mathsf{id}$ with an arbitrary orthogonal map $\tau : W \to W$ with determinant -1. To get a model compatible with the étale map ρ, we need to choose τ so that it preserves $W^\perp \subset W^*$ and the orthogonal vector $w \in W$. It suffices to take an arbitrary orthogonal map $W^\perp \to W^\perp$ with determinant -1. The transposition τ interchanging w_1 and w_2 is a good choice. The involution ι on $P' = \mathrm{Hom}(W^\perp, V') /\!\!/ SO(W^\perp)$ is then induced by composition with τ on $\mathrm{Hom}(W^\perp, V')$.

Lemma 5.4. *An arbitrarily small ι-invariant neighborhood $U' \subset P'$ of $0 \subset P'$ does not admit a ι-equivariant smooth projective resolution $X' \to U'$ compatible with the given symplectic form on the smooth locus $U'_{sm} \subset U'$.*

Proof. The space $W^\perp$ is a 2-dimensional Euclidean vector space over $\mathbb{C}$. The group $\mathbb{C}^* = SO(2)$ acts on $W^\perp$ with two eigenvalues of opposite signs. Fix corresponding eigenvectors $w_1, w_2 \in W$. The vectors w_1, w_2 have length 0 and form a basis of the vector space $W^\perp$. Therefore we have a decomposition

$$\mathrm{Hom}(W^\perp, V) \cong W^\perp \otimes V \cong w_1 V \oplus w_2 V,$$

and both $w_1 V$ and $w_2 V$ are Lagrangian subspaces. Identifying $V \cong V^*$ by means of the symplectic form, we obtain a decomposition $W^\perp \otimes V \cong V \oplus V^*$. The symplectic form in this interpretation is the standard form induced by the duality between V and V^*. It no longer depends on the symplectic form on V.

The Hamiltonian reduction $P' \cong (V \oplus V^*)/\mathbb{C}^*$ is well-known – it is a quadratic cone obtained by contracting the zero section in the total space $T^*P(V)$ of the cotangent bundle to the projectivization $P(V)$ of the complex vector space V. In particular, the complement

$$P' \setminus \{0\} \cong T^*P(V) \setminus P(V)$$

is smooth. Since $\operatorname{codim} P(V) \subset T^*P(V)$ is equal to $\dim P(V) = 2k-1 \geq 2$, the Picard group of the complement $P' \setminus \{0\}$ coincides with the Picard group of the projective space $P(V)$. It is freely generated by the class of a single line bundle, say L.

Under the splitting $W^\perp \otimes V \cong V \oplus V^*$, the transposition τ interchanges V and V^* (and induces the given identification $V \cong V^*$). It is well-known that the action of such map $\iota = \tau$ on the Picard group of the complement $P' \setminus \{0\}$ is non-trivial: we have $\iota^* L \cong L^{-1}$.

Assume that a small ι-invariant neighborhood $U' \subset P'$ of $0 \subset P'$ admits a smooth projective resolution X' compatible with the symplectic form. By Lemma 5.1 the map $f : X' \to U'$ is semismall. In particular, $\operatorname{codim} f^{-1}(0) \subset X'$ is at least $\frac{1}{2} \dim U' = 2k-1 \geq 2$. Therefore the Picard group of the variety X' is the same as the Picard group of the complement $U' \setminus \{0\}$. Thus the line bundle L extends to X', and either L or L^{-1} is an ample line bundle on X'. But ι interchanges these two bundles. If it were to extend to an involution $\iota : X' \to X'$, we would get that both L and L^{-1} are ample line bundles on X'. Therefore $\mathcal{O}_{X'}$ must be ample, which means that X' is affine, and the projective map $f : X' \to U'$ from X' to the normal variety U' must be one-to-one. This contradicts the smoothness of X'. □

Remark 5.5. The space P' of course does admit a smooth symplectic resolution, namely, the total space $T^*P(V)$. The point is that the involution ι induces a flop of this resolution, not an automorphism. The idea of using this involution to prove that the moduli space $\mathcal{M}(2,0,k)$, $k \geq 3$ does not admit a good resolution is due to K. O'Grady. We would like to thank him for explaining it to us.

Proof of Theorem 2.3. Assume given a smooth projective resolution $X \to U_m$ of a small neighborhood U_m of $0 \subset P(PGL(2), V) \times V$, and assume that the resolution X is compatible with the symplectic form. Shrinking U_m if necessary, we can assume that $U_m = U_a \times U_b$, where $U_a \subset V$, $U_b \subset P = P(PGL(2), V)$ are small neighborhoods of 0.

By Lemma 5.1, the natural action of the Lie algebra $\mathfrak{sp}(V)$ on the space P lifts to a $\mathfrak{sp}(V)$-action on the symplectic resolution $X \to U_b$. Shrinking U_b further, if still necessary, we can assume that it is $Sp(V)$-invariant. Since $Sp(V)$ is a simply-connected semisimple algebraic group, the action of the Lie algebra $\mathfrak{sp}(V)$ integrates to an $Sp(V)$-action on the resolution X. In particular, the element $\iota \in Sp(V)$ must act on the variety X.

Consider the open subset $U \subset P$ defined in Lemma 5.3. Since $U \subset P$ is dense, the intersection $U \cap U_b$ is dense in U_b. Replacing U_b with $U_b \cap U$, shrinking it further if necessity persists, and applying Lemma 5.3, we can assume that $\rho^{-1}(U_b) = U_c \times U'$, where $U_c \subset U_1 \subset \widetilde{P}_1$ is an open subset, and $U' \subset P'$ is an open ι-invariant neighborhood of $0 \in P'$. Applying Lemma 5.2, we conclude that the neighborhood $U' \subset P'$ must have a ι-equivariant smooth projective resolution X'/U' compatible with the given symplectic form. This contradicts Lemma 5.4. $\square$

6 Formality

Our remaining task is to prove Proposition 3.1. We will need one general result on formality in families proved in [K3]. Namely, recall that for any associative algebra B, or, more generally, for a flat sheaf $\mathcal{B}$ of associative algebras on a scheme X, the *Hochschild cohomology sheaves* $\mathcal{HH}^{\bullet}(\mathcal{B})$ are defined. Explicitly, they can be computed by the standard complex with terms

$$\mathcal{H}om(\mathcal{B}^{\otimes \bullet}, \mathcal{B})$$

and a certain differential, whose precise form we will not need. If the algebra $\mathcal{B}^{\bullet}$ is graded, the Hochschild cohomology sheaves inherit the grading; we denote the component of degree l by $\mathcal{HH}^{\bullet}_l(\mathcal{B}^{\bullet})$.

Lemma 6.1 ([K3, Theorem 4.2]). *Let $\mathcal{A}^{\bullet}$ be a DG algebra of flat quasi-coherent sheaves on a reduced irreducible scheme X. Let $\mathcal{B}^{\bullet}$ be the homology algebra of the DG algebra $\mathcal{A}^{\bullet}$. Assume that the sheaves $\mathcal{B}^{\bullet}$ are coherent and flat on X, and that for any integers l, i, the degree-l component $\mathcal{HH}^i_l(\mathcal{B}^{\bullet})$ of the i-th Hochschild cohomology sheaf $\mathcal{HH}^i(\mathcal{B}^{\bullet})$ is also coherent and flat.*

(i) *Assume that X is affine. If the fiber $\mathcal{A}^{\bullet}_x$ is formal for a generic point $x \in X$, then it is formal for an arbitrary point $x \in X$.*

(ii) *Assume that $\mathcal{HH}^2_l(\mathcal{B}^{\bullet})$ has no global sections for all $l \leq -1$. Then the DG algebra $\mathcal{A}^{\bullet}_x$ is formal for every point $x \in X$.* $\square$

We will apply this result to the study of the ideal sheaf $\mathcal{I}_{\mathbf{z}}$ of a 0-dimensional subscheme $\mathbf{z} \subset S$ of some length p in a $K3$ surface S. We want to prove that the DG algebra $\operatorname{RHom}^{\bullet}(\mathcal{I}_{\mathbf{z}}, \mathcal{I}_{\mathbf{z}})$ is formal.

Our first observation is that by Lemma 6.1 we can assume that $\mathbf{z} \subset S$ is a union of several points with reduced scheme structure. Indeed, every subscheme $\mathbf{z} \subset S$ can be deformed to such a subscheme, and the algebra structure on $\operatorname{Ext}^{\bullet}(\mathcal{I}_{\mathbf{z}}, \mathcal{I}_{\mathbf{z}})$ does not depend on the choice of $\mathbf{z}$, so that the

Hochschild cohomology sheaves in the assumptions of Lemma 6.1 are locally trivial. Thus Lemma 6.1 (i) applies and shows that formality is stable under specialization.

Assume therefore that $\mathbf{z} = \bigcup_{1 \leq i \leq p} \{z_i\} \subset S$ is the union of points. Choose a hyperkähler metric on the $K3$ surface S. Recall that to a hyperkähler metric on a holomorphically symplectic manifold M, one associates a set of integrable complex structures on M, one for each imaginary quaternion h with $h^2 = -1$. The set of such quaternions is naturally parametrized by the complex projective line $\mathbb{C}P^1$. One can collect all these complex structures into a single integrable complex structure on the product $M \times \mathbb{C}P^1$. In this way, one obtains the so-called holomorphic *twistor space* X of the hyperkähler manifold M. The product decomposition $X \cong M \times \mathbb{C}P^1$ is not holomorphic. However, the natural projection $\pi : X \to \mathbb{C}P^1$ is holomorphic. Moreover, for every point $m \in M$ the product $\{m\} \times \mathbb{C}P^1 \subset X$ is a holomorphic submanifold, called *horizontal section* of the twistor projection $\pi : X \to \mathbb{C}P^1$ corresponding to m. The fiber X_0 over the point $0 \in \mathbb{C}P^1$ does not depend on the choice of hyperkähler metric on M, it is canonically isomorphic to the original complex manifold M.

Thus, having chosen a hyperkähler metric on our $K3$ surface S, we obtain the associated twistor deformation $X/\mathbb{C}P^1$ – the fiber X_0 over $0 \in \mathbb{C}P^1$ is by definition the $K3$ surface S itself, and the fibers over other points in $\mathbb{C}P^1$ correspond to S with complex structures coming from different quaternions. Lift each of the points z_i to the corresponding horizontal section $\mathcal{Z}_i \subset X$, $\mathcal{Z}_i \cong \mathbb{C}P^1$ of the twistor projection $\pi : X \to \mathbb{C}P^1$. Let $\mathcal{I}_{\mathcal{Z}}$ be the ideal sheaf of the union $\bigcup \mathcal{Z}_i \subset X$ of these sections on the complex manifold X.

The DG algebra $\mathrm{RHom}^{\bullet}(\mathcal{I}_{\mathbf{z}}, \mathcal{I}_{\mathbf{z}})$ naturally deforms to a flat DG algebra $\mathbf{R}\mathcal{H}om^{\bullet}(\mathcal{I}_{\mathcal{Z}}, \mathcal{I}_{\mathcal{Z}})$ of coherent sheaves on $\mathbb{C}P^1$ – to obtain $\mathbf{R}\mathcal{H}om^{\bullet}(\mathcal{I}_{\mathcal{Z}}, \mathcal{I}_{\mathcal{Z}})$, one considers the relative $\mathbf{R}\mathcal{H}om^{\bullet}$ on X and applies the direct image $R^{\bullet}\pi_*$ with respect to the twistor projection $\pi : X \to \mathbb{C}P^1$. The cohomology algebra $\mathcal{E}xt^{\bullet}(\mathcal{I}_{\mathcal{Z}}, \mathcal{I}_{\mathcal{Z}})$ is a flat deformation of the algebra $\mathrm{Ext}^{\bullet}(\mathcal{I}_{\mathbf{z}}, \mathcal{I}_{\mathbf{z}})$.

Definition 6.2. A coherent sheaf on $\mathbb{C}P^1$ is *of weight* l if it a sum of several copies of the sheaf $\mathcal{O}(l)$.

Lemma 6.3. *For every $k \geq 0$, the sheaf $\mathcal{E}xt^k(\mathcal{I}_{\mathcal{Z}}, \mathcal{I}_{\mathcal{Z}})$ on $\mathbb{C}P^1$ is of weight k.*

Proof. The only k with non-trivial $\mathcal{E}xt^k(\mathcal{I}_{\mathcal{Z}}, \mathcal{I}_{\mathcal{Z}})$ are $k = 0, 1, 2$. For $k = 0$, the sheaf $\mathcal{E}xt^0(\mathcal{I}_{\mathcal{Z}}, \mathcal{I}_{\mathcal{Z}})$ is obviously the trivial sheaf $\mathcal{O} = \mathcal{O}(0)$. For $k = 2$, one applies the relative Serre duality to $X/\mathbb{C}P^1$ and concludes that

$$\mathcal{E}xt^2(\mathcal{I}_\mathcal{Z}, K_{X/\mathbb{C}P^1} \otimes \mathcal{I}_\mathcal{Z})^* \cong \mathcal{E}xt^0(\mathcal{I}_\mathcal{Z}, \mathcal{I}_\mathcal{Z}).$$

This proves the claim, since the relative canonical bundle $K_{X/\mathbb{C}P^1}$ is canonically isomorphic to $\pi^*\mathcal{O}(2)$ – the isomorphism is given by the relative symplectic form $\Omega \in H^0(X, K_{X/\mathbb{C}P^1})$.

It remains to consider the case $k = 1$. It is not diffcult to compute the sheaf $\mathcal{E}xt^1(\mathcal{I}_\mathcal{Z}, \mathcal{I}_\mathcal{Z})$ directly. However, we prefer to use the following geometric argument. The ideal sheaf $\mathcal{I}_\mathbf{z}$ corresponds to a point $z \in S^{[p]}$ in the Hilbert scheme parametrizing 0-dimensional subschemes in S of some length p (if S is not algebraic, replace the Hilbert scheme with the Douady space). Consider the moduli space $X^{[p]}$ of pairs

$$\Big\langle \text{ a point } I \in \mathbb{C}P^1, \text{ a 0-dimensional subscheme } \mathbf{z} \subset X_I \text{ of length } p \Big\rangle.$$

The space $X^{[p]}$ projects onto $\mathbb{C}P^1$, and the fiber over a point $I \in \mathbb{C}P^1$ is the Hilbert scheme $X_I^{[p]}$ (in particular, the fiber $\big(X^{[p]}\big)_0$ is the Hilbert scheme $S^{[p]}$). The union $\bigcup \mathcal{Z}_i \subset X$ gives a section $Z \subset \mathfrak{X}^{[p]}$, $Z \cong \mathbb{C}P^1$ of the projection $X^{[p]} \to \mathbb{C}P^1$. By the usual deformation theory, the sheaf $\mathcal{E}xt^1(\mathcal{I}_\mathcal{Z}, \mathcal{I}_\mathcal{Z})$ is isomorphic to the normal bundle $\mathcal{N}_Z(X^{[p]})$.

Now, since the points $z_i \in S$ are distinct by assumption, the Hilbert scheme $S^{[p]}$ near the point $z \in S^{[p]}$ is isomorphic to the p-fold product S^p. Analogously, in a small neighborhood of $Z \subset X^{[p]}$, the space $X^{[p]}$ is isomorphic to the p-fold product X^p taken over $\mathbb{C}P^1$. But this product is the twistor space of the product S^p, with the product hyperkähler metric. Therefore, by the general twistor theory, the normal bundle $\mathcal{N}_Z(X^{[p]})$ is a sum of several copies of the sheaf $\mathcal{O}(1)$. □

Lemma 6.3 allows to finish the proof of Proposition 3.1 very quickly.

Proof of Proposition 3.1. Consider the sheaf of DG algebra $\mathbf{R}\mathcal{H}om^\bullet(\mathcal{I}_\mathcal{Z}, \mathcal{I}_\mathcal{Z})$ on the scheme $X = \mathbb{C}P^1$. Its cohomology sheaf $\mathcal{B}^\bullet = \mathcal{E}xt^\bullet(\mathcal{I}_\mathcal{Z}, \mathcal{I}_\mathcal{Z})$ is locally constant as a sheaf of algebras, so that the Hochschild cohomology sheaves $\mathcal{HH}^\bullet(\mathcal{B}^\bullet)$ are locally trivial, and we can apply Lemma 6.1. By Lemma 6.3, the degree-l component of the sheaf

$$\mathcal{H}om^\bullet(\mathcal{B}^{\bullet\otimes k}, \mathcal{B}^\bullet)$$

is of weight l for any integer l and any $k \geq 0$. But for any map of sheaves of weight l, its kernel and cokernel are obviously also of weight l. Therefore the Hochschild cohomology sheaves $\mathcal{HH}^k_l(\mathcal{B}^\bullet)$ are also of weight l for any integers l and k. Since sheaves of negative weight have no global sections, the condition of Lemma 6.1 (ii) is also satisfied, and we conclude that indeed, the DG algebra $\mathrm{RHom}^{\bullet}(\mathcal{I}_{\mathbf{z}}, \mathcal{I}_{\mathbf{z}})$ is formal. □

We conclude this Section with a sort of an extended remark. The idea behind our proof of formality was discovered by M. Verbitsky ([V], see also an exposition in [KV]). He considered deformations of vector bundles on a hyperkähler manifold, and proved formality by a version of our Lemma 6.3. We essentially extend his approach to degenerating sheaves, in the simplest case of a hyperkähler surface. However, we do need to assume that the degeneration locus $\mathbf{z} \subset S$ is a reduced subscheme, and use a roundabout argument to handle the general case. The reason for this is very simple: for a non-reduced subscheme $\mathbf{z} \subset S$, the statement of Lemma 6.3 is *false*.

Our proof would work just as well if the deformation $X^{[p]}$ were the twistor space for some hyperkähler structure on the Hilbert scheme $S^{[p]}$. This is not the case though. The simplest way to see this is to note that the fiber $X^{[p]}$ over an arbitrary point $I \in \mathbb{C}P^1$, being a Hilbert scheme in its own right, contains a non-trivial divisor – namely, the locus of non-reduced subschemes $\mathbf{z} \in X_I$. On the other hand, the generic fiber of the twistor deformation of an arbitrary hyperkähler manifold has no rational cohomology classes of Hodge type $(1,1)$.

Algebraically, the sheaf $\mathcal{E}xt^1(\mathcal{I}_{\mathcal{Z}}, \mathcal{I}_{\mathcal{Z}})$ for a non-reduced scheme $\mathcal{Z} \in X^{[p]}$ acquires several summands of type $\mathcal{O}(2)$ (and the same number of summands of type $\mathcal{O}(0)$).

A natural thing to do would be to replace $X^{[p]}$ with the twistor space for some hyperkähler structure on $S^{[p]}$. However, we do not have a modular interpretation of these twistor spaces extending the modular interpretation of the Hilbert scheme $S^{[p]}$.

An analogous problem for the Hilbert scheme of $\mathbb{C}^2$ has been studied recently in [KKO]. In that paper the authors managed to find a modular interpretation of an actual twistor deformation of the Hilbert scheme of $\mathbb{C}^2$. To do this, however, they had to consider sheaves on a *non-commutative* deformation of the underlying manifold $\mathbb{C}^2$.

An analogy with [KKO] suggests that in order to obtain a modular interpretation of the twistor space for the Hilbert scheme $S^{[p]}$ – in particular, in order to obtain a natural proof of formality – one has to consider quantizations of the underlying $K3$ surface S.

Of course, the case of a $K3$ surface is much more difficult than the case of $\mathbb{C}^2$ – where an explicit quantization is given by the algebra of differential operators on an affine line. Still, even the $K3$ case might not be completely beyond modern techniques. D. Huybrechts informs us that the line l in the period space of the hyperkähler manifold $S^{[p]}$ defined by the family $X^{[p]}/\mathbb{C}P^1$, while not being a twistor line, is very close to a twistor line – there is a twistor line in an arbitrarily small neighborhood of l. Thus the amount of non-commutativity one has to introduce into the problem can arbitrarily small. We hope to return to this in the future.

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